# A COMPARISON BETWEEN REAL AND COMPLEX SCHOTT SPHERICAL SYMMETRY TEST FOR POLSAR DATA ANALYSIS

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## ABSTRACT

Most of the tests proposed in the literature to verify if a given random multivariate dataset fits a spherical or elliptical distribution are designed for real valued data and rely on the estimation of high order moment matrices. Recently, a test that considers complex random vectors, derived based on the Schott spherical symmetry test was proposed aiming in a more proper analysis of PolSAR data. Results showed its effectiveness in discriminating data that fits or not the complex spherically invariant random vector model (product model), inherent to high resolution heterogeneous PolSAR systems. Within this context, this paper further extends the assessment of the referred test efficiency, verifying its performance under different stochastic model assumptions and comparing the results with the ones achieved when the Schott test derived for real random vectors is employed.

***Index Terms***— Polarimetric Synthetic Aperture Radar, Multiplicative model (SIRV), Spherical symmetry test

## 1. INTRODUCTION

Testing for spherical or elliptical distributions in multivariate random data is not a relative new subject in neither signal processing nor statistics community [8, 9, 10]. Nevertheless, most of the tests described in the literature are designed for real valued data and rely on the estimation of high order moment matrices [11, 12]. When complex random variables are under analysis, which is the case of PolSAR data, the performance of the aforementioned tests can be compromised. Extending an $m \times 1$ complex random vector into a $2m \times 1$ real vector is not always straightforward. Furthermore, such mapping ($\mathbb{C} \rightarrow \mathbb{R}$) doubles the dimensionality of the problem, increasing significantly the complexity of the algorithms.

According to [5], one of the most powerful spherical symmetry tests was proposed in [6] for real random vectors. Recently, a general framework which allows quantitative evaluation of fitting complex spherically invariant random vectors, based on the latter, was proposed to better analyse a given multidimensional PolSAR dataset [3]. Polarimetric Synthetic Aperture Radar (PolSAR) data describes the interaction between the electromagnetic waves and the scatters inside a resolution cell, for each polarimetric state of the former. High heterogeneity scenes (inherent to high resolution systems) may eventually lead to non-Gaussian clutter modelling. SIRVs (Spherically Invariant Random Vectors), have then been constantly employed for modelling high-resolution POLSAR data [1, 2]. The SIRV is a multiplicative model expressed as a product between the square root of a scalar positive quantity (texture) and the description of an equivalent homogeneous surface (speckle). It is important to highlight that in the SIRV definition, the texture probability density function is not explicitly specified. As a consequence, SIRVs describe a whole class of stochastic processes.

Results showed that, for this given application, the referred test presents a good performance when the texture is assumed to have a Gamma distribution. In the present work, the performance of the test is further extended considering different texture distributions. Furthermore, a comparison with the results achieved considering the Schott test for real random vectors is also performed.

In Section II we introduce the SIRV stochastic model of PolSAR data [2]. Section III presents the Schott test for assessing the spherical symmetry properties of real random vectors and the recently proposed alternative for complex random vectors. In Section IV, the performance of the aforementioned tests are verified taken into consideration a synthetic dataset. Finally, in Section V some conclusions are drawn.

## 2. POLSAR DATA PRODUCT MODEL

For an $m$-dimensional PolSAR system ($m \leq 4$), in each $i$th azimuth / range location, $\mathbf{k_i}$ is the $m \times 1$ complex target vector corresponding to the same area on the ground. For distributed targets, the corresponding $\mathbf{k}$ vector is considered non-deterministic and may be written, under the SIRV model assumption. The SIRV is a class of non-homogeneous Gaussian processes with random variance [2], also referred in the literature as product or multiplicative model. It is an important subclass of Complex Elliptically Symmetric Distributions (CES), or simply compound-Gaussian [4], where each $m$-dimensional observation vector $\mathbf{k}$ is defined as

$$\mathbf{k} = \sqrt{\boldsymbol{\tau}} \cdot \mathbf{z} \tag{1}$$

where $\mathbf{z}$ is an independent complex circular Gaussian vector, characterising the speckle, with zero mean and covariance matrix of the form $[T] = \sigma_0 \cdot [M]$, such that Tr$\{|M|\}$ = 1 and $\sigma_0$ is the total power (span). In (1), $\boldsymbol{\tau}$ represents the texture, a positive random variable characterising the spatial variations in the radar backscattering, statistically independent of the speckle. The probability density function of the texture random variable is not explicitly specified by the mode, therefore, the SIRV model describe a wide range of well known specific models. For a more detailed description of such class of random vectors, the reader is advised to go to [4].

## 3. SHOTT'S APPROACH FOR TESTING SPHERICAL SYMMETRY OF RANDOM VECTORS

The test proposed by Schott [6] consists in verifying if the structure of the sampled data fourth order moment (quadricovariance matrix) matches the one of Elliptical Symmetric Distributions (CES). Assuming that the sample $m \times n$ dataset is extracted from a finite second order moment elliptical distribution with zero mean vector and covariance matrix $[M]$. The fourth order moment matrix $[M]_4 = E\left[kk^H \otimes kk^H\right]$ is given by

$$[M]_4 = (1+\omega)\left[(I_{m^2} + K_{mm})\left([M] \otimes [M]\right)\right] \tag{2}$$

where $K_{mm}$ is a commutation matrix, $[\cdot]^H$ is the complex transpose operator and $\otimes$ is the Kronecker product operator. Note that 2 is valid for both complex and real random vectors, when $[\cdot]^H$ can be simply replaced by the transpose operation ($[\cdot]^T$).

The sample quadricovariance estimator can be expressed in terms of the Kronecker product as

$$\widehat{[M]}_4 = \frac{1}{n}\sum_{i=1}^{n} \mathbf{k}_i\mathbf{k}_i^H \otimes \mathbf{k}_i\mathbf{k}_i^H \tag{3}$$

Its corresponding standardized form is given by

$$\widehat{[M]}_{4*} = \left(\widehat{[M]}^{-\frac{1}{2}^H} \otimes \widehat{[M]}^{-\frac{1}{2}^H}\right)\widehat{[M]}_4\left(\widehat{[M]}^{-\frac{1}{2}} \otimes \widehat{[M]}^{-\frac{1}{2}}\right) \tag{4}$$

where $\widehat{[M]}^{-\frac{1}{2}} \cdot \widehat{[M]}^{-\frac{1}{2}^H} = \widehat{[M]}^{-1}$.

At this point, the test derivation of real and complex random vectors present different particularities, which are properly addressed in the next couple of sections.

### 3.1. Real random vectors spherical symmetry test

According to the Schott's theorem, the Wald test statistic for spherical symmetry can be expressed as:

$$\begin{aligned} T_{real} = N\{&\beta_1 \mathrm{tr}\left(\widehat{[M]}^2_{4*}\right) + \\ &\beta_2 \mathrm{vec}\left(\widehat{[I]}_m\right)^H \widehat{[M]}^2_{4*} \mathrm{vec}\left(\widehat{[I]}_m\right) - \\ &\left[3\beta_1 + (m+2)\beta_2\right] m(m+2)(1+\widehat{\kappa})^2\} \end{aligned} \tag{5}$$

where $vec(\cdot)$ is the operator that transforms a matrix into a column vector and

$$\beta_1 = (1+\widehat{\theta})^{-1}/24, \tag{6}$$

$$\beta_2 = -3a[24(1+\widehat{\theta})^2 + 12(m+4)a(1+\widehat{\theta})]^{-1}, \tag{7}$$

$$a = (1+\widehat{\theta}) + (1+\widehat{\kappa})^3 - 2(1+\widehat{\kappa})(1+\widehat{\eta}) \tag{8}$$

with the Mardia's kurtosis $\widehat{\kappa}$ and the generalized higher order scalar moments $\widehat{\theta}$, $\widehat{\eta}$ given by:

$$(1+\widehat{\kappa}) = \frac{1}{m(m+2)n}\sum_{i=1}^{n}\left[\mathbf{k}_i^H \widehat{[M]}^{-1}\mathbf{k}_i\right]^2, \tag{9}$$

$$(1+\widehat{\theta}) = \frac{1}{nm(m+2)(m+4)}\sum_{i=1}^{n}\left[\mathbf{k}_i^H \widehat{[M]}^{-1}\mathbf{k}_i\right]^3, \tag{10}$$

$$(1+\widehat{\eta}) = \frac{1}{nm(m+2)(m+4)(m+6)}\sum_{i=1}^{n}\left[\mathbf{k}_i^H \widehat{[M]}^{-1}\mathbf{k}_i\right]^4. \tag{11}$$

Asymptotically, $T_{Schott} \to \chi^2_{f_{real}}$ with $f_{real} = m^2 + \frac{m(m-1)(m^2+7m-6)}{24} - 1$ as in [6]. It has been proven in [7] that the Wald test and the LRT are asymptotically equivalent.

### 3.2. Complex random vectors spherical symmetry test

In [3], a new procedure to test the spherical symmetry of complex random vectors data was proposed, based on what had been previously done by Schott with real valued random vectors [6]. Considering 4 and assuming that $A = \widehat{[M]} - [M]$ and $C = \widehat{[M]}_4 - [M]_4$, it is shown in [3] that

$$\begin{aligned} vec(\widehat{[M]}_{4*}) \;&=\; (1+\omega)vec([N]_4) + vec(C) \\ &\quad -(1+\omega)H vec(A) + O_p(n^{-1/2}) \end{aligned} \tag{12}$$

where $[N]_4$ is what $[M]_4$ simplifies to when $k_i \sim CN_m(0, I_m)$ and $H$ is an operator given by

$$\begin{aligned} H \;=\; &[I_{m^2} \otimes (I_{m^2} + K_{mm})] \cdot \\ &\{I_m \otimes [(K_{mm} \otimes I_m) \cdot (I_m \otimes vec(I_m))] \\ &+[(I_m \otimes K_{mm}) \cdot (vec(I_m) \otimes I_m)] \otimes I_m\} \end{aligned} \tag{13}$$

Note that (12) is asymptotically equals to $vec(\widehat{[M]}_{4*}) = (1+\omega)vec([N]_4) + O_p(n^{-1/2})$. Therefore, defining $G = \rho^{-1} vec(N_4) vec(N_4)^T$, with $\rho = vec(N_4)^T vec(N_4)$, it is possible to state that $G vec(\widehat{[M]}_{4*})$ is a consistent estimator of $[M_4]$ if and only if $[M_4]$ has the structure defined in (2). Hence, assuming that the latter is true, it is of common knowledge that

$$n^{1/2} v = n^{1/2} (I_{m^4} - G) vec\left(\widehat{[M]}_{4*}\right) \quad (14)$$

is asymptotically normal with zero mean and cov. matrix

$$\Phi = (I_{m^4} - G)\Xi(I_{m^4} - G) \quad (15)$$

where $\Xi$ denotes the asymptotic cov. matrix of $n^{1/2} vec(\widehat{[M]}_{4*})$.

From (12), $\Xi$ can be written as

$$\begin{aligned}\Xi &= [M]_C - (1+\omega)[M]_{C,A} H^H - (1+\omega) H [M]_{A,C} \\ &\quad + (1+\omega)^2 H [M]_A H^H \end{aligned} \quad (16)$$

where $[M]_C$ is the covariance matrix of $vec(C)$, $[M]_A$ is the covariance matrix of $vec(A)$, and $[M]_{Q,R}$ is the cross-covariance matrix between $Q$ and $R$. In [3], it is shown that

$$[M]_A = (1+\omega)(I_m \otimes I_m) + \omega\left(vec(I_m) vec(I_m)^T\right) \quad (17)$$

$$\begin{aligned}[M]_C &= (1+\theta)(1+\omega)^2 (N_4^T \otimes N_4) \\ &\quad + \theta(1+\omega)^2 vec(N_4) vec(N_4)^T \end{aligned} \quad (18)$$

$$\begin{aligned}[M]_{C,A} &= (1+\eta)\sum_i (e_i \otimes I_{m^3}) N_6 (e_i \otimes I_{m^2}) \\ &\quad - (1+\omega) vec(N_4) vec(I_m)^T \end{aligned} \quad (19)$$

where $e_i$ denotes the $i$th column of the identity matrix $I_m$.

The Wald test for complex-valued signals states that

$$T = n v^H \Gamma v \quad (20)$$

has an asymptotic chi-squared distribution with degrees of freedom $f$ equal to the rank of $\Phi$ if $\Gamma$ is a consistent estimator of a generalised inverse of the latter. Note that the latter is obtained by specifying $\hat{\omega}$, $\hat{\eta}$, $\hat{\theta}$, respectively consistent estimators of $\omega$, $\eta$ and $\theta$. They are given by [3]

$$\hat{\omega} = \frac{1}{nm(m+1)} \sum_{i=1}^{n} \left[k_i^H \widehat{[M]}^{-1} k_i\right]^2 \quad (21)$$

$$\hat{\eta} = \frac{1}{nm(m+1)(m+2)} \sum_{i=1}^{n} \left[k_i^H \widehat{[M]}^{-1} k_i\right]^3 \quad (22)$$

$$\hat{\theta} = \frac{1}{nm(m+1)(m+2)(m+3)} \sum_{i=1}^{n} \left[k_i^H \widehat{[M]}^{-1} k_i\right]^4 \quad (23)$$

Summarising, the proposed framework for the complex elliptical symmetry starts with the estimation of (21), (22) and (23). Next, (18), (19) and (17) are calculated and consequently, (16) is derived. Then (15) is used along with (14) into (20) and the test is finally finished. The degrees of freedom of the test is equal to the rank of $\Phi$ and is given by [3]

$$f_{complex} = m^2 + \frac{m(m-1)(m^2+19m+6)}{24} - 1 \quad (24)$$

## 4. SYNTHETIC DATA ANALYSIS

The synthetic data used in the present analysis is divided into 6 regions, each containing 100 x 100 samples of a specific type of heterogeneous clutter. Three different probability distributions for the texture were taken into consideration in the analysis: Gamma, Inverse Gamma and Weibull. These distributions were chosen given their high correspondence to natural phenomenons [1]. For each distribution, two datasets are generated. One considering the texture polarization independent (fitting the SIRV model) and the other considering the texture polarization dependent (not fitting the SIRV model). It is important to highlight that the ability to correctly discriminate between these two types of datasets is crucial for proposing more efficient detection, classification and geophysical parameters inversion algorithms in PolSAR data analysis.

The shape and scale parameters that characterises the texture random variables for each region are such that their mean are fixed and set to 1. With no loss of generality, the speckle covariance matrix, $[M]$, was kept the same for all regions and given by the identity matrix $I_3$. Figure 1 presents the output of the tests described in the previous section, where, in green, are the non spherical symmetric pixels, backgrounded by the data span.

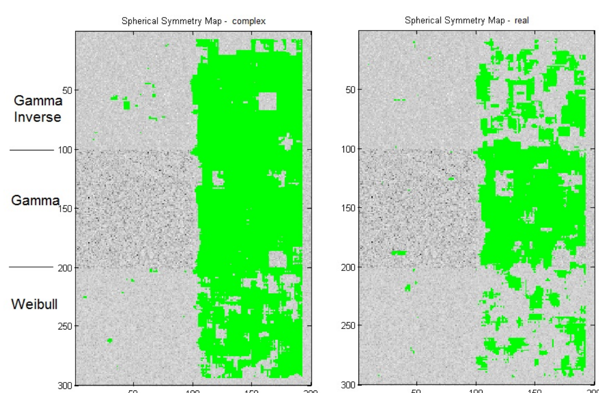


**Fig. 1**: Spherical symmetry tests map.

It is important to mention that a key point in the analysis of statistical tests performance is the choice of the set of samples (size and location within the data) used in the derivation of their stochastic properties. Since the synthetic data used in the present study is composed by homogeneous regions, a sliding window approach is sufficient for the definition of the set of samples used. The size of the sliding window was chosen to be $15$, a proper choice for the given application [3]. Finally, the tests confidence levels were set to $0.99$.

Note that the test designed specifically for complex random variables outperforms the test designed for real random vectors considering a $2m \times 1$ vector. Furthermore, note that both tests performance are dependent on the texture probabilistic distribution. The percentage of pixels correctly identified as not fitting the SIRV model in the polarization dependent texture datasets are presented in Table 1.

**Table 1**: Percentage of pixels correctly identified as not fitting the SIRV model in the polarization dependent texture datasets.

| Texture Distribution | Complex test | Real test |
|---|---|---|
| Inverse Gamma | $73\%$ | $33\%$ |
| Gamma | $83\%$ | $73\%$ |
| Weibull | $61\%$ | $21\%$ |
| | | |

## 5. CONCLUSION

This paper addressed a comparison of performance between the recently proposed spherical symmetry test for complex random vectors [3] and the Schott test derived for real random vectors in PolSAR data analysis. The analysis showed that for specific types of PolSAR clutter and taking into consideration design parameters better suited for such application (window size and confidence level) the performance of the latter is not as good as the performance of the test specially suited for such type of data. Furthermore, it was shown that, the performance of both tests are dependent of the texture probabilistic distribution. Finally, it is important to highlight that when the latter is assumed to have a Gamma distribution (a good representative of many natural phenomenons), both tests (complex and real) present a good performance. Nevertheless, the same behaviour is not verified for other distributions, such as Weibull.

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