Noise Enhanced Distributed Bayesian Estimation

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***Abstract*—In this paper we consider distributed estimation of an unknown Gaussian random variable with known mean and variance, where each sensor observation is affected by both multiplicative and additive Gaussian observation noises. We derive the corresponding Cramer Rao Lower Bound (CRLB) for both quantized and full precision observations. In sequel we provide some closed-form approximations for both CRLB expressions which provide us with better understanding of behavior of CRLBs. Afterwards through analytic and simulation results we report some scenarios that multiplicative observation noise can play an enhancive role in terms of estimation accuracy. We call this phenomena *enhancement mode* of multiplicative noise.**

I. Introduction

The problem of distributed parameter estimation has been studied in numerous literature. Most of the literature have assumed that the observation model is completely specified and the only statistical uncertainty is due to the additive noise [1]–[15]. On the other hand some literature considered the problem where some type of uncertainty is involved in observation gains in addition to the presence of additive noise [16]–[24] [1]. For example [18] proposed approximate Maximum Likelihood (ML) estimators for localizing a source by means of a sensor array when the received signal is corrupted by multiplicative noise. The authors in [19], [20] studied the MinMax estimation of a deterministic unknown parameter where the model matrix is not exactly known, and the uncertainty is modeled as an additive noise and a bounded perturbation in model matrix. In [21]–[23] the authors studied the MinMax and ML estimation of a deterministic unknown vector, where the elements of the model matrix are modeled as random variables with known second order statistics. The authors in [23] derived the Cramer Rao Lower Bound (CRLB) and demonstrated that for some specific values of deterministic unknown vector, randomness in the model matrix improves the performance in terms of Mean Square Error (MSE). In [24] ML estimator of a deterministic unknown parameter, based on sign measurements of observations are derived where both multiplicative and additive noises were involved. The authors also showed that the multiplicative noise exacerbates the performance of the ML estimator in most cases. However, provided that the additive noise variance is small in comparison with the energy of unknown parameter, suitable values of multiplicative noise may improve the MSE performance. Similar results have been reported in [16].

In this paper we consider the distributed estimation of an unknown Gaussian random variable. We assume sensors' observations are affected by both multiplicative and additive Gaussian observation noises. We derive the corresponding Bayesian CRLB [8] for two cases *i)* Fusion Center (FC) has the access to full precision observations, *ii)* the FC has access to quantized version of observations only. We also provide some closed-form approximations for both CRLB expressions which allow us to obtain a better understanding of the behavior of the CRLB expressions with respect to (w.r.t.) variations in different system parameters. Afterwards through analytic and simulation results we identify the conditions under which multiplicative observation noise can enhance the estimation accuracy. We call this phenomena as *enhancement mode* [25] [26] of multiplicative noise. Technically speaking we contend that there exists an *enhancement mode* for multiplicative noise provided that quantizers are *fine enough* and/or additive observation noise is *strong enough.*
Interestingly the results in this paper for Bayesian CRLB are different from those reported in [24] [16] [23], for classical CRLB (where the unknown parameter is deterministic). For instance according to our results there is no *enhancement mode* for binary quantizers, whereas [24] and [16] reported some scenarios where multiplicative observation noise may improve the performance. The authors in [24] and [16] also showed that the enhancement mode occurs when the variance of additive observation noise is small in comparison with the energy of unknown parameter. In contrast, for Bayesian CRLB, the enhancement mode is more likely to happen for larger additive noise variances. Also note that because of the deterministic nature of unknown parameter in [24] [16] [23] the improvement due to model randomness depends on specific values of unknown, which makes it elusive to exploit the *enhancement mode*.

II. System Model

We consider a network of K spatially distributed sensors and a FC, where the network is tasked with estimating a realization of an unknown zero mean Gaussian signal source θ, with known variance, i.e., $\theta \sim \mathcal{N}(0,\sigma_\theta^2)$. Each sensor makes a noisy observation of θ, where both additive and multiplicative observation noises are involved. In particular, we model the observation x_k at sensor k as:

$$x_k = h_k\theta + n_k, \qquad \text{for} \quad k = 1, ..., K, \tag{1}$$

where n_k's are additive noises that are uncorrelated with each other and θ. We assume $n_k \sim \mathcal{N}(0,\sigma_{n_k}^2)$. The multiplicative noises h_k's (*unknown* observation gains), are modeled as Gaussian random variables with known means and variances[2], i.e $h_k \sim \mathcal{N}(1,\sigma_{h_k}^2)$ that are uncorrelated with each

[1]This type of uncertainty may be referred as *model uncertainties*, *sensing matrix perturbations* or *multiplicative noise environments* in different literatures.

[2]Suppose $\mathbb{E}\{h_k\} = \mu_k$. Via scaling each x_k with $1/\mu_k$, the signal model in (1) transforms to $x'_k = h'_k\theta + n'_k$, where $x'_k = x_k/\mu_k$, $h'_k = h_k/\mu_k$, $n'_k = n_k/\mu_k$, such that $\mathbb{E}\{h'_k\} = 1$, var$(h'_k) = \sigma_{h_k}^2/\mu_k^2$ and $n'_k \sim \mathcal{N}(0,\sigma_{n_k}^2/\mu_k^2)$. Thus, without loss of generality, we assume $\mathbb{E}\{h_k\} = 1$.

other, θ, and additive noises n_k's. Each sensor transmits its quantized observation over an error-free communication channel to the FC, where collective received data are fused to estimate θ. Error-free communication channel model has been adopted before in [2], [3], [5], [8], [16] in the context of distributed estimation. Sensor k employs a quantizer with M_k quantization levels, quantization boundaries $\zeta_{k,i},\ i \in \{1, ...M_k+1\}$ and quantization levels $m_{k,i} = (\zeta_{k,i}+\zeta_{k,i+1})/2,\ i \in \{1,...M_k\}$, in order to map x_k into a quantization level $m_k \in \{m_{k,1},...,m_{k,M_k}\}$. The quantization level m_k can be mapped into a binary sequence of length $r_k = \log_2 M_k$ (bits). We refer to r_k as quantization rate of sensor k.

III. Bayesian Cramer Rao Lower Bound

In this section we derive the Bayesian CRLB for any unbiased estimator of random variable θ based on quantized and full precision observations. We also provide some insightful approximations for the both CRLB expressions, that enable us to find a better understanding of the CRLB behavior w.r.t. the variations of noise variances and to study the enhancement mode of the multiplicative observation noise.

A. Bayesian CRLB derivation

Let $\boldsymbol{m} = [m_1, ..., m_K]^T$ denote the vector of quantized observations of all sensors. One can verify that the log-likelihood function of quantized observations satisfies the regularity condition, i.e $\mathbb{E}\{\frac{\partial \ln p(\mathbf{m},\theta)}{\partial\theta}\} = 0$. Let F^q denote the Fisher information based on all quantized observations $\boldsymbol{m}$. It is well known that the MSE of any unbiased estimator of θ based on $\boldsymbol{m}$ is at least as large as the inverse of F^q (i.e., the CRLB based on $\boldsymbol{m}$). We can write F^q as:

$$\begin{aligned} F^q = & -\mathbb{E}\{\frac{\partial^2 \ln p(\mathbf{m},\theta)}{\partial^2\theta}\} \\ = & -\mathbb{E}\{\frac{\partial^2 \ln p(\mathbf{m}|\theta)}{\partial^2\theta}\} - \mathbb{E}_\theta\{\frac{\partial^2 \ln p(\theta)}{\partial^2\theta}\} \end{aligned} \quad (2)$$

where $\mathbb{E}_\theta$ is expectation operator w.r.t. distribution of θ. First, we consider the second term in (2). For $\theta \sim \mathcal{N}(0, \sigma_\theta^2)$ it is easy to verify that $\mathbb{E}_\theta\{\frac{\partial^2 \ln p(\theta)}{\partial^2\theta}\} = -\frac{1}{\sigma_\theta^2}$. We continue with characterizing the first term in (2). Note that h_k's and n_k's are all uncorrelated Gaussian and hence independent. Therefore, m_k's conditioned on θ are independent and $\ln p(\mathbf{m}|\theta) = \sum_{k=1}^K \ln p(m_k|\theta)$. This allows us to write the first and second derivatives of the log-likelihood function as the following:

$$\frac{\partial \ln p(\mathbf{m}|\theta)}{\partial\theta} = \sum_{k=1}^K \frac{1}{p(m_k|\theta)} \frac{\partial p(m_k|\theta)}{\partial\theta} \quad (3)$$

$$\frac{\partial^2 \ln p(\mathbf{m}|\theta)}{\partial^2\theta} = \overbrace{\sum_{k=1}^K \frac{1}{p(m_k|\theta)} \frac{\partial^2 p(m_k|\theta)}{\partial^2\theta}}^{=F_a} - \underbrace{\sum_{k=1}^K \frac{1}{p^2(m_k|\theta)} (\frac{\partial p(m_k|\theta)}{\partial\theta})^2}_{=F_b} \quad (4)$$

We observe that the first term in (2) is equal to $-\mathbb{E}\{F_a\} + \mathbb{E}\{F_b\}$. Next we find $\mathbb{E}\{F_a\}$ and $\mathbb{E}\{F_b\}$. We have $\mathbb{E}\{F_a\} = 0$ since:

$$\mathbb{E}\{F_a\} = \sum_{k=1}^K \mathbb{E}\{\frac{1}{p(m_k|\theta)} \frac{\partial^2 p(m_k|\theta)}{\partial^2\theta}\} = \sum_{k=1}^K \int p(\theta) \sum_{i=1}^{M_k} \frac{p(m_k = m_{k,i}|\theta)}{p(m_k=m_{k,i}|\theta)} \frac{\partial^2 p(m_k = m_{k,i}|\theta)}{\partial^2\theta} d\theta = \sum_{k=1}^K \int p(\theta) \left(\frac{\partial^2 (\overbrace{\sum_{i=1}^{M_k} p(m_k = m_{k,i}|\theta)}^{=1})}{\partial^2\theta} \right) d\theta \ = \ 0$$

For $\mathbb{E}\{F_b\}$ we have:

$$\mathbb{E}\{F_b\} = \sum_{k=1}^K \mathbb{E}\{\frac{1}{p^2(m_k|\theta)} (\frac{\partial p(m_k|\theta)}{\partial\theta})^2\} = \sum_{k=1}^K \int p(\theta) \sum_{i=1}^{M_k} \frac{1}{p(m_k=m_{k,i}|\theta)} (\frac{\partial p(m_k = m_{k,i}|\theta)}{\partial\theta})^2 d\theta \quad (5)$$

To obtain $\mathbb{E}\{F_b\}$ in (5) we need to characterize $p(m_k = m_{k,i}|\theta)$ and its derivative with respect to θ. One can show that:

$$\begin{aligned} & S_{k,i}(\theta) \triangleq p(m_k = m_{k,i}|\theta) = \\ & \Pr\{\zeta_{k,i} \le h_k\theta + n_k \le \zeta_{k,i+1}|\theta\} = \\ & \Phi(\frac{\zeta_{k,i+1}-\theta}{\sqrt{\theta^2\sigma_{h_k}^2+\sigma_{n_k}^2}}) - \Phi(\frac{\zeta_{k,i}-\theta}{\sqrt{\theta^2\sigma_{h_k}^2+\sigma_{n_k}^2}}) \end{aligned} \quad (6)$$

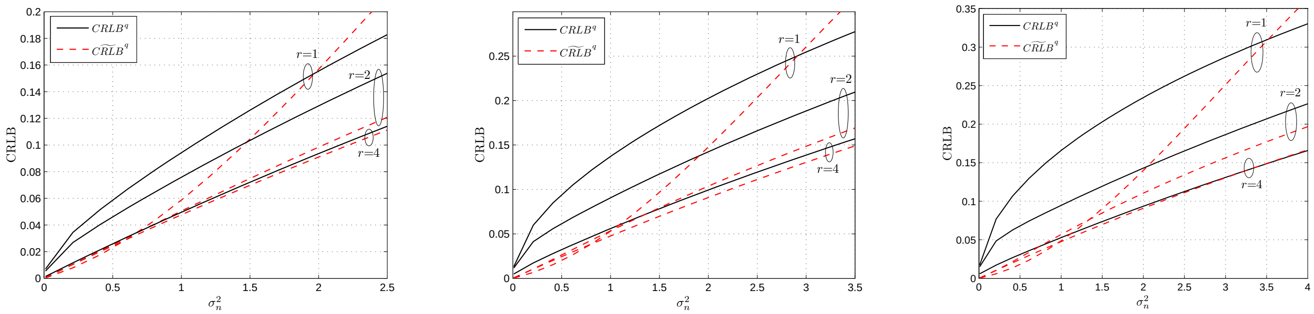
$$H_{k,i}(\theta) \triangleq \frac{\partial S_{k,i}(\theta)}{\partial\theta} \quad (7)$$

where $\Phi(.)$ is a standard normal CDF. Let $H_{k,i}(\theta) \triangleq \frac{\partial S_{k,i}(\theta)}{\partial\theta}$. Deriving $H_{k,i}(\theta)$ is straightforward and would be subtraction of two scaled standard normal PDFs. Substituting the expressions in (6) and $H_{k,i}(\theta)$ in (4) and (2), F^q in (2) becomes:

$$F^q = \sum_{k=1}^K \sum_{i=1}^{M_k} \mathbb{E}_\theta\{\frac{H_{k,i}^2(\theta)}{S_{k,i}(\theta)}\} + \frac{1}{\sigma_\theta^2} \quad (8)$$

Although the integral corresponding to the expectation in (8) does not render to a closed form, since $S_{k,i}(\theta) > 0,\ \forall\theta$ it can easily be calculated with numerical methods. Without loss of generality, in sequel of paper, we assume all sensors have the same observation noise variances, i.e $\sigma_{h_k}^2 = \sigma_h^2$, $\sigma_{n_k}^2 = \sigma_n^2,\ \ \forall k$, and same quantization rates, i.e $r_k = r = \log_2(M), \forall k$, and the same quantization boundaries $\zeta_1, ..., \zeta_{M+1}$. Thus index k for $S_{k,i}(\theta)$ and $H_{k,i}(\theta)$, can be dropped and F^q in (8) reduces to:

$$F^q = K \sum_{i=1}^M \mathbb{E}_\theta\{\frac{H_i^2(\theta)}{S_i(\theta)}\} + \frac{1}{\sigma_\theta^2} \quad (9)$$



(a) $\sigma_h^2 = 0$ (b) $\sigma_h^2 = 0.5$ (c) $\sigma_h^2 = 1$

Fig. 1: $CRLB^q$ and $\widetilde{CRLB}^q$ vs σ_n^2 for different values of r and σ_h^2.

B. Approximation of Bayesian CRLB and noise enhancement

We start with F^q in (9). Although the integral corresponding to expectation in (9) cannot be reduced to a closed form, for the case of $\sigma_h^2 \ll \sigma_n^2$ and large r, we can employ the second order Taylor approximations for $S_i(\theta)$ and $H_i(\theta)$, and after taking some tedious integral calculus steps, we reach the following approximate expression for F^q:

$$\tilde{F^q} = K \sum_{i=1}^{M} \frac{(\zeta_{i+1}-\zeta_i)(\sigma_\theta^4+\sigma_n^2\sigma_\theta^2+\sigma_n^2\zeta_i^2)}{\sigma_n^2(\sigma_\theta^2+\sigma_n^2)^{5/2}}\phi\left(\frac{\zeta_i}{\sqrt{\sigma_\theta^2+\sigma_n^2}}\right) + \frac{1}{\sigma_\theta^2} \quad (10)$$

where $\phi(.)$ is a standard normal PDF.

Let $\boldsymbol{x} = [x_1, ..., x_K]^T$ denote the vector of full precision observations of all sensors. For the clairvoyant case where the FC has access to (unquantized) full precision observations x_k's, one can verify that the log-likelihood function satisfies the regularity condition, i.e $\mathbb{E}\{\frac{\partial \ln p(\mathbf{x},\theta)}{\partial\theta}\} = 0$, and the Fisher information can be presented as following (we omitted the details due to space limitations):

$$F^{cv} = K\mathbb{E}_\theta\left\{\frac{\sigma_n^2+\theta^2\sigma_h^2+2\theta^2\sigma_h^4}{(\sigma_n^2+\theta^2\sigma_h^2)^2}\right\} + \frac{1}{\sigma_\theta^2} \quad (11)$$

It is easy to verify that $\frac{\partial F^{cv}}{\partial \sigma_n^2} < 0$, i.e. F^{cv} is decreasing in σ_n^2, which is intuitive. On the other hand F^{cv} is not a monotonic function of σ_h^2 which is unintuitive. In order to investigate the behavior of F^{cv} w.r.t. σ_h^2, let's have a closer look at (11). The integral corresponding to the expectation in (11) can not be expressed in a closed form, however it can be accurately approximated using Taylor expansion of ratio function. Expressing the Fisher information in (11) as $F^{cv} = K\mathbb{E}_\theta\{f(z)\} + \frac{1}{\sigma_\theta^2}$, where $f(z) = \frac{(1+2\sigma_h^2)z-2\sigma_n^2\sigma_h^2}{z^2}$ and expanding the Taylor series of $f(z)$ around $z_0 = \mu \triangleq \mathbb{E}\{z\} = \sigma_n^2 + \sigma_\theta^2\sigma_h^2$, gives us $f(z) \approx f(\mu) + \frac{f'(\mu)(z-\mu)}{2} + \frac{f''(\mu)(z-\mu)^2}{6}$, where the remaining terms are discarded. Substituting the expressions for first $f'(z)$ and second $f''(z)$ order derivatives and taking the expectations lead us to $\mathbb{E}_\theta\{f(z)\} \approx \frac{(1+2\sigma_h^2)\mu-2\sigma_n^2\sigma_h^2}{\mu^2} + (\frac{(1+2\sigma_h^2)\mu-6\sigma_n^2\sigma_h^2}{\mu^4})\text{Var}(z)$ (note that the expansion is valid for $z \neq 0$ which is the case here). Adopting the first term of Taylor expansion as an approximate of F^{cv}, we have:

$$F^{cv} \approx \tilde{F}^{cv} = K\left(\frac{\sigma_n^2+\sigma_h^2\sigma_\theta^2+2\sigma_h^4\sigma_\theta^2}{(\sigma_n^2+\sigma_h^2\sigma_\theta^2)^2}\right) + \frac{1}{\sigma_\theta^2} \quad (12)$$

Simulation results verify that second order Taylor expansion is a *good approximation*. Taking the derivative of (12) w.r.t. σ_h^2 (assuming $4\sigma_n^2 \geq \sigma_\theta^2$) we find that $\tilde{F}^{cv}$ is decreasing in σ_h^2 for $\sigma_h^2 < \tilde{\delta}^{cv} \triangleq \frac{1}{4-(\sigma_\theta^2/\sigma_n^2)}$, and increasing in σ_h^2 when $\sigma_h^2 > \tilde{\delta}^{cv}$ (worst case performance in terms of estimation error happens for $\sigma_h^2 = \tilde{\delta}^{cv}$). In other word provided that $\sigma_h^2 > \tilde{\delta}^{cv} > 0$, multiplicative observation noise can enhance the estimation accuracy[3] . Simulation results reveal that F^{cv} has a similar behavior to $\tilde{F}^{cv}$ w.r.t. σ_h^2.

Analyzing the behavior of F^q in (9) w.r.t. σ_h^2, however, is challenging and mathematically intractable. Simulation results show that F^q in (9) has different behavior w.r.t. σ_h^2 for small r, such that excessive multiplicative observation noise can not enhance the estimation accuracy. However for large r the behaviors are the same.

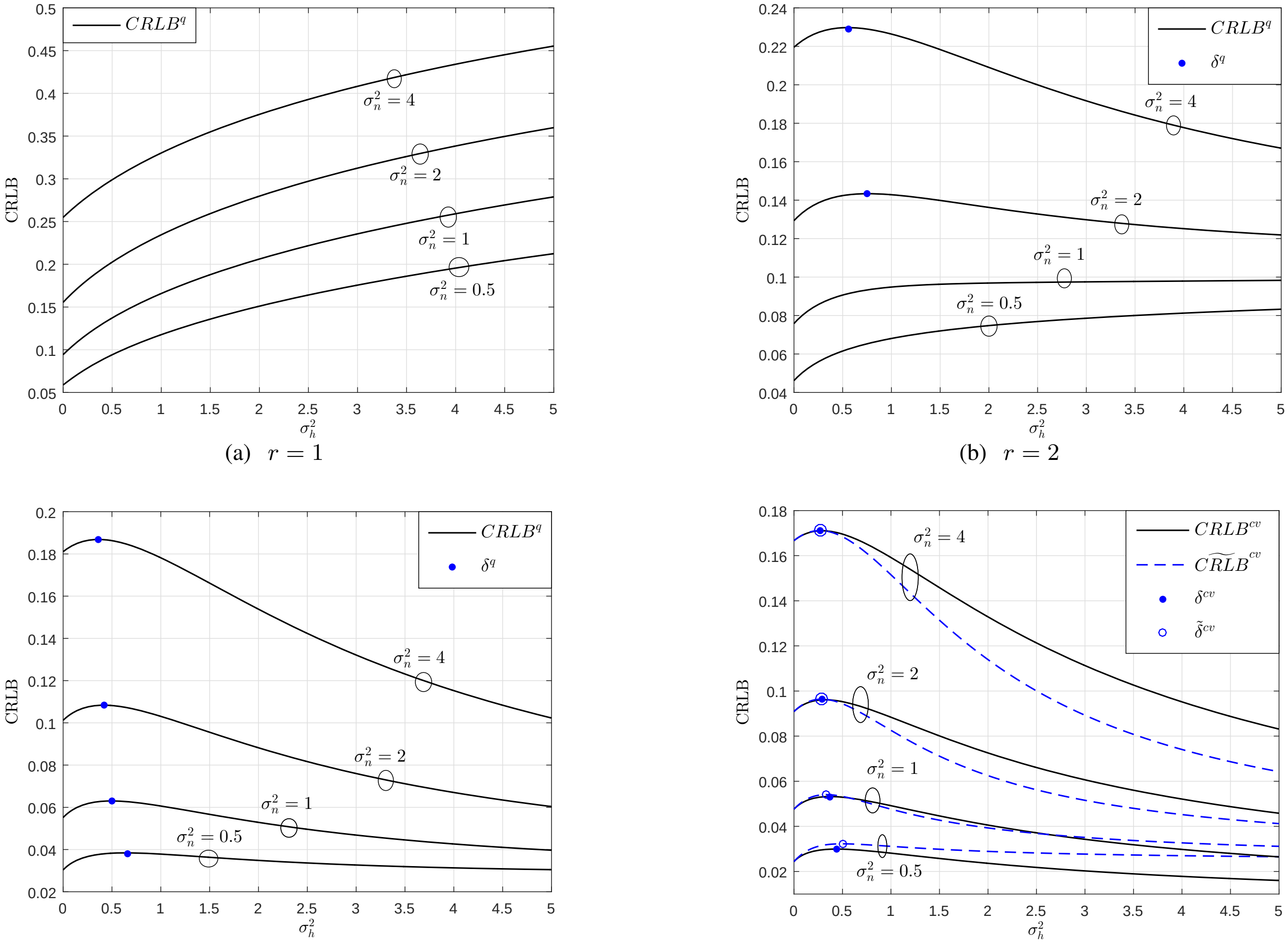
IV. Numerical and Simulation Results

In this section, with numerical examples and simulations, we corroborate our analytic results. The simulation results compare the proposed approximations for CRLB expressions in different scenarios. Numerical examples also verify that there exist scenarios that multiplicative observation noise can play an enhancive role improving the estimation accuracy. Without loss of generality we let $K = 20$ and $\sigma_\theta^2 = 1$. Due to the Gaussian nature of unknown θ and noises, we assume x_k lies in a bounded interval, i.e., $x_k \in [-\tau, \tau]$ for a reasonably large value of τ. We employ uniform quantizers in sensors with step size of $\Delta \triangleq \zeta_{i+1} - \zeta_i = \frac{2\tau}{M-1}$.

Let $CRLB^q$ and $\widetilde{CRLB}^q$ denote the inverses of F^q in (9) and $\tilde{F}^q$ in (10), respectively. Fig. 1 compares $CRLB^q$ $\widetilde{CRLB}^q$. As can be seen the approximation is very accurate for $r \geq 4$ and $\sigma_h^2 \ll \sigma_n^2$. Note that as r increases the approximation improves so we omitted r's larger than 4. For cases of $\sigma_h^2 = 0.5$ and $\sigma_h^2 = 1$ and $r \geq 4$, the approximation still remains quite accurate.

Let $CRLB^{cv}$ and $\widetilde{CRLB}^{cv}$ denote the inverse of F^{cv} in (11) and $\tilde{F}^{cv}$ in (12), respectively. Also let $\delta^q = \underset{\sigma_h^2}{\text{argmax}}(CRLB^q)$ and $\delta^{cv} = \underset{\sigma_h^2}{\text{argmax}}(CRLB^{cv})$. Fig. 2. *a,b,c*

[3]One may notice a resemblance of the phenomena with Stochastic Resonance (SR). However the SR happens in nonlinear systems with additive noise, where the performance is maximized for a particular noise intensity.



(a) $r = 1$ (b) $r = 2$

(c) $r = 3$ (d) clairvoyant

Fig. 2: $CRLB^q$, $CRLB^{cv}$ and $\widetilde{CRLB}^{cv}$ vs σ_h^2 for different values of r and σ_n^2.

depict $CRLB^q$, and Fig.2. *d* depicts $CRLB^{cv}$ and $\widetilde{CRLB}^{cv}$ vs σ_h^2. As can be seen in Fig.2. *d*, provided that $\sigma_h^2 > \delta^{cv}$, $CRLB^{cv}$ decreases as σ_h^2 increases. In another word the multiplicative noise can play an enhancive role improving the estimation accuracy. Note that $\widetilde{CRLB}^{cv}$ may be perceived as a rough approximate of $CRLB^{cv}$, however $\tilde{\delta}^{cv}$ is a very accurate approximate of δ^{cv}. As predicted by our analysis, *enhancement mode* starts with smaller values of σ_h^2 as σ_n^2 increases. For instance $\tilde{\delta}^{cv} = 0.5$, $\delta^{cv} = 0.420$ for $\sigma_n^2 = 0.5$ and $\tilde{\delta}^{cv} = 0.267$, $\delta^{cv} = .262$ for $\sigma_n^2 = 4$. This is also the case for $CRLB^q$ (Fig. 2. *b,c*), that *enhancement mode* (if there is any) starts with smaller values of σ_h^2 when σ_n^2 gets larger. However probing into Fig. 2. *a,b* reveals that there may not be any *enhancement mode* for $CRLB^q$ if r or σ_n^2 gets small. For instance for cases of $r = 1$ (Fig. 2. *a*) and $r = 2$, $\sigma_n^2 = 0.5, 1$ (Fig. 2. *b*) there is no *enhancement mode*.

V. Conclusions and Future Research

In this paper we derived the Bayesian CRLB for distributed estimation of a Gaussian random variable where both multiplicative and additive Gaussian noises are involved in observations, for both full precision and quantized observations. In sequel we provided some closed-form approximations for the CRLBs and studied the behavior of these approximations as quantization rates and variances of multiplicative and additive observation noises vary. In contrast to the additive noise which always degrades the estimation accuracy, our results reveal that in some scenarios the multiplicative noise can play an enhancive role in terms of estimation accuracy. We call this phenomena *enhancement mode* of multiplicative noise. Simulation results illustrate that there always exists an *enhancement mode* for the CRLB based on full precision observations. On the other hand for the CRLB based on quantized observations there may not exist an *enhancement mode* for small quantization rates, and/or small additive observation noise variance. In future work we plan to approach this problem from a system design perspective and using distributed optimization [27] techniques to optimize potential Bayesian estimators that can effectively utilize the enhancement mode of multiplicative observation noise, in order to improve the estimation accuracy.

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