EXPECTED LIKELIHOOD SPHERICITY TEST DISTRIBUTION FOR COMPLEX ANGULAR CENTRAL GAUSSIAN DATA

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ABSTRACT

Expected Likelihood based quality assessment of DOA estimates relies on the underlying signal and noise distributions having likelihood ratio probability density functions for the (unknown) true parameters that are independent of the actual true DOAs. This has been shown, both analytically and practically for a wide range of real and complex Gaussian solutions. Recent studies [1, 2] focusing on compound Gaussian mixtures have applied Expected Likelihood based on Monte-Carlo assessment of the "scenario-free" nature of the LR p.d.f.s. In this paper, through specified moments and the use of Mellin's transform, we derive the analytic p.d.f. for the Expected Likelihood sphericity test in the presence of data with the complex angular central Gaussian distribution associated with this compound Gaussian case.

***Index Terms*—** Maximum Likelihood Estimation, Angular Central Distribution

1. INTRODUCTION

Maximum likelihood (ML) direction of arrival (DOA) estimation of multiple sources in noise is a difficult problem that very rarely can provide a globally optimal solution in a computationally reasonable implementation. Instead , a series of different ML-proxy techniques are used, and in order to provide a "quality" assessment of those techniques, particularly in marginal SNR or sample-starved conditions which reside in the threshold regime where various ML-proxy algorithms have differing performance, we have suggested an "expected likelihood" (EL) approach [3, 4]. The EL approach is based on a quite straight-forward idea of comparison of the likelihood ratio (LR) generated by the DOA estimates $\Theta = [\hat{\theta}_1, \dots, \hat{\theta}_p]$ with the distribution of LRs generated by the true DOA's $\Theta_0 = [\theta_1, \dots, \theta_p]$. Naturally the true DOA's are unknown, but for a number of estimation circumstances, including the conditional and unconditional Gaussian models, the distribution of likelihood ratios $LR(\Theta_0)$ associated with the true DOAs does not depend on the actual DOAs, but instead is fully specified by the antenna dimension M and the number of i.i.d. Gaussian training samples T (and in the conditional model, the number of sources p), all of which can reasonably be known *a priori*. Thus the likelihood ratio produced by DOAs estimates can be compared to the range of LR values associated with the true DOAs and their quality assessed. It was demonstrated in [5–7] that for Gaussian mixtures, MUSIC-produced sets of DOA estimates that contain an outlier estimate generate an LR value which falls below the range of LR values statistically associated with the true DOAs. In fact, this "expected likelihood" technique has demonstrated high statistical efficiency in detecting ML-proxy (e.g. MUSIC) performance breakdown in threshold conditions where the SNR or number of snapshots T is insufficient to reliably generate accurate MUSIC DOA estimates, but MLE itself still produces CRB-consistent DOA estimates. Naturally, this method cannot help in regimes where MLE itself produces estimates with errors which deviate significantly from the CRB (the MLE "threshold" region), but is valuable in aiding the use of more computationally efficient ML-proxy algorithms in many circumstances. Since detection-estimation and DOA estimation in particular, is often performed in noise/clutter environments that are strongly non-Gaussian [8, 9], it is desirable to have a similar expected likelihood technique for quality assessment of the derived DOA estimates. Specifically, in [1], the following data model was considered:

$$\boldsymbol{x}_t = \boldsymbol{A}(\Theta)\boldsymbol{s}_t + \sqrt{\tau_t}\eta_t + \sigma_w w_t \tag{1}$$

where $\Theta = [\theta_1, \dots, \theta_p]$ is the vector of the DOAs for p sources and $\boldsymbol{A}(\Theta) = [\boldsymbol{a}(\theta_1), \dots, \boldsymbol{a}(\theta_p)]^T$ is the array manifold matrix. In what follows, we assume a conditional model for which the emitted waveforms $\boldsymbol{s}_t \in \mathcal{C}^p$ are treated as deterministic unknowns. $\boldsymbol{\eta}_t$ and $\boldsymbol{\omega}_t$ are independent and identically distributed Gaussian vectors: $\boldsymbol{\eta}_t \sim \mathcal{CN}(0, I)$ and $\boldsymbol{\omega}_t \sim \mathcal{CN}(0, I)$, σ_ω^2 stands for the thermal noise power and τ_t is a positive random variable.

The problem is that for most distributions of τ_t, used for "spikey" sea clutter modelling, for example, the closed form p.d.f. for the clutter and internal Gaussian noise mixture $(\sqrt{\tau_t}\boldsymbol{\eta}_t + \sigma_\omega\boldsymbol{\omega}_t)$ does not exist. Therefore, all derivations associated with the accurate maximum likelihood and CRB methodology are not applicable here, including the methodology which led to the expected likelihood technique in the Gaussian case. For this reason, in [1], we suggested an associated expected likelihood ratio test. Indeed, given a set of DOA estimates $\hat{\Theta}_p$ and a set of i.i.d. training samples $\boldsymbol{X}_T$ per (1), it is reasonable to assume that DOA estimates $\hat{\Theta}_p$ which deliver the minimum

$$\hat{\Theta} = \min_{\Theta} \mathrm{Tr}[\boldsymbol{P}_\perp(\Theta)\boldsymbol{X}_T\boldsymbol{X}_T^H] \tag{2}$$

should be close to the optimal solution and (trivially)

$$\min_{\Theta} \mathrm{Tr}[\boldsymbol{P}_\perp(\Theta)\boldsymbol{X}_T\boldsymbol{X}_T^H] \leq \mathrm{Tr}[\boldsymbol{P}_\perp(\Theta_0)\boldsymbol{X}_T\boldsymbol{X}_T^H] \tag{3}$$

where the projector matrix $\boldsymbol{P}_\perp$ is defined as

$$\begin{aligned}\boldsymbol{P}_\perp(\Theta) = \boldsymbol{I} - \boldsymbol{A}(\Theta)[\boldsymbol{A}^H(\Theta)\boldsymbol{A}(\Theta)]^{-1}\boldsymbol{A}^H(\Theta) = \\ = \boldsymbol{U}_{M-p}(\Theta)\boldsymbol{U}_{M-p}^H(\Theta)\end{aligned} \tag{4}$$

Yet, $\mathrm{Tr}[\boldsymbol{P}_\perp(\Theta_0)\boldsymbol{X}_T\boldsymbol{X}_T^H]$ still depends on τ_t and therefore its distribution is not available *a priori*. For this reason, in [1], we suggested the introduction (in the presence of p sources) of normalized samples

$$\boldsymbol{z}_t = \frac{\boldsymbol{U}_{M-p}(\Theta)\boldsymbol{x}_t}{||\boldsymbol{U}_{M-p}(\Theta)\boldsymbol{x}_t||} \tag{5}$$

where

$$\boldsymbol{U}^H_{M-p}(\Theta)\boldsymbol{U}_{M-p}(\Theta) = \boldsymbol{I}_{M-p} \tag{6}$$

We wish to assess proximity of $\boldsymbol{z}_t (t = 1, \ldots, T)$ to the set of random vectors

$$\boldsymbol{z}_{t0} = \frac{\boldsymbol{U}_{M-p}(\Theta_0)\boldsymbol{x}_t}{||\boldsymbol{U}_{M-p}(\Theta_0)\boldsymbol{x}_t||}, \tag{7}$$

where for the true DOAs Θ_0 (in contrast to the estimated DOAs Θ), the introduced vector z_{t0} is distributed with a complex angular central Gaussian distribution with an $(M-p)$-variate scatter matrix that does not depend on those DOAs or the τ_t distribution. We suggested in [1] that the quality of the DOA estimate $\hat{\Theta}_p$ be assessed based on the proximity of the scatter matrix of the normalized vectors $\boldsymbol{z}_t$ to the identity matrix, as per $(M-p)$-variate vectors $\boldsymbol{z}_{t0}$. In particular, we suggested the use for this purpose of the traditional sphericity test (ST):

$$ST(\Theta) = \frac{\det \hat{\boldsymbol{R}}(\Theta)}{\left[\frac{1}{M-p}\mathrm{Tr}\hat{\boldsymbol{R}}(\Theta)\right]^{M-p}} \tag{8}$$

where $\hat{\boldsymbol{R}}(\Theta) = \frac{1}{T}\boldsymbol{Z}_T\boldsymbol{Z}_T^H$. Yet, since $\boldsymbol{z}_t(\theta)$ is unit norm, $ST \propto \det \hat{\boldsymbol{R}}(\Theta)$, and so the EL technique proposed is to compare $ST(\hat{\Theta})$ against the support of the distribution $ST(\Theta_0)$, which does not depend on Θ_0 or τ_t and is fully specified by $(M-p)$ and T. In [1], rather than use an analytically derived p.d.f. of $ST(\Theta_0$, we conducted Monte-Carlo simulations. Therefore, in this paper, we derive the accurate analytic distribution for $ST(\Theta_0)$, similar to [10] for the real-valued case and [5] for the complex valued case.

2. SPHERICITY TEST FOR ANGULAR CENTRAL GAUSSIAN DISTRIBUTED DATA

Since

$$\boldsymbol{U}_{M-p}(\Theta_0)\boldsymbol{x}_t \sim \mathcal{CN}(0, (\tau_t + \sigma^2_\omega)I) \tag{9}$$

$$\boldsymbol{z}_{t0} \sim \mathcal{CAG}(0, \boldsymbol{I}_{M-p}) \tag{10}$$

where $\mathcal{CAG}$ denotes the complex angular central Gaussian distribution [11]. In what follows, we derive the p.d.f. distribution for the general case $ST = \det \hat{\boldsymbol{R}}$ where $\boldsymbol{R}$ is the sample covariance matrix formed from the normalized snapshots, i.e. $\hat{\boldsymbol{R}} = \frac{1}{T}\boldsymbol{Z}_T\boldsymbol{Z}_T^H$, and where $\boldsymbol{z}_t$ is complex angular central Gaussian distributed $\boldsymbol{z}_t \sim \mathcal{CAG}(0, \boldsymbol{I}_M)$. The complex angular central Gaussian distribution is specified in [11] as:

$$f(\boldsymbol{z}_t) = S_M^{-1}|\Sigma^{-1}|(\boldsymbol{z}_t^H\Sigma^{-1}\boldsymbol{z}_t)^{-M}. \tag{11}$$

For the considered case when

$$\boldsymbol{z}_t = \frac{\boldsymbol{x}_t}{||\boldsymbol{x}_t||}, \quad \boldsymbol{x}_t \sim \mathcal{CN}(0, c\boldsymbol{I}_M), \tag{12}$$

we have

$$f(\boldsymbol{z}_t) \sim S_M^{-1}; \quad S_M = \frac{2\pi^M}{\Gamma(M)} \tag{13}$$

Since integration over the surface $\sum_{i=1}^M |\boldsymbol{z}_t|_i^2 = 1$ is to be used in sequel, let us check the correctness of (13) by integrating

$$\mathcal{J}_1 = \int\limits_{|z_1|^2+\ldots+|z_M|^2=1} \ldots \int dz_1 dz_2 \ldots dz_M \tag{14}$$

The complex-valued surface integral in (14) can be calculated as the $2M$-variate real-valued surface integral $y_j, j = 1, \ldots, 2M$ where the complex valued $\boldsymbol{z}_t$ is comprised of $2M$ variates as the alternating real and imaginary components $z_j = y_j + iy_{j+1}$.

$$\mathcal{J}_1 = \int\limits_{\sum_{j=1}^{2M} y_j^2=1} \ldots \int dy_1 dy_2 \ldots dy_{2M} \tag{15}$$

For integration over the surface S described by equation $x_N = f(x_1, \ldots, x_{N-1})$, we use the formula [12]

$$\begin{aligned}\int \ldots \int\limits_S g(x_1, \ldots, x_N) dS = \\ = \int \ldots \int\limits_D g([x_1, \ldots, x_{N-1}, f(x_1, \ldots, x_{N-1})] \times \\ \times \sqrt{\left(\frac{\partial t}{\partial x_1}\right)^2 + \ldots + \left(\frac{\partial f}{\partial x_{N-1}}\right)^2 + 1} \; dx_1 dx_2 \ldots dx_{N-1}\end{aligned} \tag{16}$$

where D is the projection of the surface S on the $(N-1)$ plane. In our case (15), we have

$$y_{2M} = \pm\sqrt{1 - y_1^2 - \ldots - y_{2M-1}^2} \tag{17}$$

$$\frac{\partial f}{\partial y_j}_{j \neq 2M} = \frac{y_j}{\sqrt{1 - y_1^2 - \ldots - y_{2M-1}^2}} \tag{18}$$

and

$$\sqrt{\left(\frac{\partial f}{\partial y_1}\right)^2 + \ldots + \left(\frac{\partial f}{\partial y_{2M-1}}\right)^2 + 1} = \frac{1}{\sqrt{1 - y_1^2 - \ldots - y_{2M-1}^2}} \tag{19}$$

Therefore, with respect to the two solutions in (17), we get

$$\mathcal{J}_1 = \int\limits_{y_1^2+\ldots+y_{2M-1}^2\leq 1} \ldots \int \frac{dy_1 \ldots dy_{2M-1}}{\sqrt{1 - y_1^2 - \ldots - y_{2M-1}^2}} \tag{20}$$

According to equation 4.633 from [13], we get

$$\mathcal{J}_1 = \frac{2\pi^M}{\Gamma(M)} \; ; \; M > 1 \tag{21}$$

Let us make the transformation

$$x_j = \frac{y_j}{\sqrt{T}} \; ; \; y_j = \sqrt{Tx_j} \; ; \tag{22}$$

so that $z_j = \sqrt{T}(x_j + ix_{j+1})$ and

$$\mathrm{Tr}\tilde{\boldsymbol{X}}_T\tilde{\boldsymbol{X}}_T^H = 1 \; ; \; \tilde{x}_j = x_j + ix_{j+1} \tag{23}$$

For the transformed data, we have

$$\begin{aligned}\mathcal{J}_2 = 2 \int\limits_{x_1^2+\ldots+x_{2M-1}^2\leq \frac{1}{T}} \ldots \int \frac{dx_1 \ldots dx_{2M-1}}{\sqrt{\frac{1}{T} - x_1^2 - \ldots - x_{2M-1}^2}} = \\ = -\frac{2T^{-\left(\frac{2M-1}{2}\right)}\pi^M}{\Gamma(M)} \equiv S_{M.T}\end{aligned} \tag{24}$$

i.e.,

$$f(x_1, ...x_{2M}) = (S_{M,T})^{-1} \tag{25}$$

For T i.i.d. vectors $x_j, j = 1, \ldots, T$, we get

$$f(\boldsymbol{X}_T) = S_{M,T}^{-T} \equiv \left[\frac{2^T T^{-T(M-\frac{1}{2})} \pi^{TM}}{\Gamma^T(M)} \right]^{-1} \tag{26}$$

Let us now consider the transformation $\boldsymbol{X}_T = \boldsymbol{TL}$ where $\boldsymbol{T}$ is an $(M \times M)$ lower triangular matrix and $\boldsymbol{L}$ is $(T \times M)$ using the standard methodology [14]. The rank of $\boldsymbol{T} = M$, $t_{jj} > 0$ and $\boldsymbol{LL}^H = \boldsymbol{I}_M$. According to [15], the Jacobian of the transformation $\boldsymbol{X}_T = \boldsymbol{TL}$ is given by:

$$\mathcal{J}(\boldsymbol{X}_T \to \boldsymbol{T}, \boldsymbol{L}) = \mathcal{J}_1 \times \mathcal{J}_2 \times \mathcal{J}_3 \tag{27}$$

where

$$\mathcal{J}_1 = |\boldsymbol{T}|^{2T} \tag{28}$$

$$\mathcal{J}_3 = \prod_{j=1}^{M} \left[(t_{jj})^{-2j+1} \right] \tag{29}$$

and $\mathcal{J}_2$ is a function of $\boldsymbol{L}$ only, independent on $\boldsymbol{T}$. Let us denote $\mathcal{J}_2 = g(\boldsymbol{L})$. Then the joint density of $\boldsymbol{T}$ and $\boldsymbol{L}$ is

$$f(\boldsymbol{T}, \boldsymbol{L}) = g(\boldsymbol{L}) \prod_{j=1}^{M} t_{jj}^{2(T-j)+1} f(\boldsymbol{TT}^H) \tag{30}$$

where $f(\boldsymbol{TT}^H) = S_{M,T}^{-T}$. Therefore

$$f(\boldsymbol{T}, \boldsymbol{L}) = S_{M,T}^{-T} g(\boldsymbol{L}) \prod_{j=1}^{M} t_{jj}^{2(T-j)+1} \tag{31}$$

We find by integrating out $\boldsymbol{L}$ that the density of $\boldsymbol{T}$ is

$$f(\boldsymbol{T}) = S_{M,T}^{-T} \int_{\boldsymbol{LL}^H=1} \!\!\! ... \int g(\boldsymbol{L}) d\boldsymbol{L} \prod_{j=1}^{M} t_{jj}^{2(T-j)+1} \equiv$$

$$\equiv C_1 S_{M,T}^{-T} \prod_{j=1}^{M} t_{jj}^{2(T-j)+1} \; ; \; C_1 = \int_{\boldsymbol{LL}^H=1} \!\!\! ... \int g(\boldsymbol{L}) d\boldsymbol{L} \tag{32}$$

With respect to (22), (30), and (31), we have

$$f(\boldsymbol{T}) = C \prod_{j=1}^{M} t_{jj}^{2(T-j)+1} \; ; \; \sum_{j=1}^{M} t_{jj}^2 = 1 \tag{33}$$

Let us find the distribution of the diagonal elements of the matrix $\boldsymbol{T}$: $(t_{11}, t_{22}, \ldots, t_{MM})$. First let us make a transformation, preserving the diagonal elements of $\boldsymbol{T}$ i.e. $t_{ij_{i<j}} = t_{ii} r_{ij}$. The Jacobian of this transformation is given by is equal to [15]

$$\mathcal{J}(t_{11}, \ldots, t_{MM}, t_{12}, \ldots, t_{M-1,M} \to$$
$$\to t_{11}, \ldots, t_{MM}, r_{12}, \ldots, r_{M-1,M}) \tag{34}$$

$$\mathcal{J}(\boldsymbol{T} \to \boldsymbol{R}) = 2^{M-1} \prod_{i=1}^{M-1} t_{ii}^{2(M-i)} \tag{35}$$

Thus, the probability of distribution $f(t_{11}, ..., t_{MM})$ is defined as:

$$f(t_{11}, ..., t_{MM}) = \text{const.} \int_{i<j} \!\!\! \ldots \int dr_{ij} \times$$

$$\times \prod_{i=1}^{M-1} t_{ii}^{2(T+M-2j)+1} t_{MM}^{2(T-M)+1} 2^{M-1} \tag{36}$$

$$f(t_{11}, ..., t_{MM}) = 2^{M-1} C_t \prod_{i=1}^{M!1} t_{ii}^{2(T+M-2j)+1} t_{MM}^{2(T+M-2j)+1} \tag{37}$$

The constant C_t in (37) may now be found by integrating

$$\int_{t_{11}^2 + \cdots t_{MM}^2 = 1} \!\!\!\!\!\! \ldots \int f(t_{11}, ..., t_{MM}) dt_{11} \cdots d_{MM} = 1 \tag{38}$$

Using $t_{MM} = \sqrt{1 - t_{11}^2 - \cdots - t_{M-1,M-1}^2}$ for this integration (see (16)) along with (37), we then get

$$\mathcal{J}_3 = 2^{M-1} \int_{\sum_{i=1}^{M-1} t_{ii}^2 < 1} \!\!\!\!\!\! \ldots \int \frac{\prod_{i=1}^{M!1} t_{ii}^{2(T+M+1-2j)} \quad dt_{11} \cdots dt_{M-1,M-1}}{(1 - t_{11}^2 - \cdots - t_{M-1,M=1})^{-(T-M)}} \tag{39}$$

According to equation $(4.635.4^s)$ from [13], we get

$$\mathcal{J}_3 = \frac{2^{M-1}}{2^{M-1}} \frac{\Gamma(1+T-M) \prod_{j=1}^{M-1} \Gamma(T+M+1-2j)}{\Gamma(1+T-M) + \sum_{j=1}^{M-1} (T+M+1-2j)} =$$

$$= \frac{\prod_{j=1}^{M} \Gamma(T+M+1-2j)}{\Gamma(TM)} \tag{40}$$

Therefore, we finally get

$$f(t_{11}, ..., t_{MM}) = \frac{\Gamma(TM)}{\prod_{j=1}^{M} \Gamma(T+M+1-2j)} \prod_{i=1}^{M} t_{ii}^{2(T+M+1-2j)+1}$$

$$t_{11}^2 + \cdots t_{MM}^2 = 1 \tag{41}$$

Now, note that the sphericity test

$$ST(\hat{R}) \sim \det[\boldsymbol{X}_T \boldsymbol{X}_T^H] = \det[\boldsymbol{TT}^H] = \prod_{i=1}^{M} t_{ii}^2 \tag{42}$$

In order to find the p.d.f. for $\prod_{i=1}^{M} t_{ii}^2 \sim ST_0$, find an expression for the h-th moment and apply the inverse Mellin transform, as in [10].

$$\mathcal{E}[\det^h(\boldsymbol{X}_T \boldsymbol{X}_T^H)] =$$

$$= C_{T,M} \int_{\sum_{i=1}^{M} t_{ii}^2 = 1} \!\!\!\!\!\! \ldots \int t_{ii}^{2(h+T+M-2j)+1} \; dt_{11} \cdots dt_{MM} \tag{43}$$

$$C_{T,M} = \frac{\Gamma(TM)}{\prod_{j=1}^{M} \Gamma(T+M+1-2j)} \tag{44}$$

By applying the same methodology as in (16), we get

$$\mathcal{E}[\det^h \boldsymbol{V}] = \frac{\Gamma(TM) \prod_{j=1}^{M} \Gamma(h+T+M+1-2j)}{\prod_{j=1}^{M} \Gamma(T+M+1-2j) \Gamma[(h+T)M]} \tag{45}$$

where $\boldsymbol{V} = \boldsymbol{X}_T \boldsymbol{X}_T^H$. Note that the expression (45) is quite important in its own right for calculations of expected likelihood sphericity test moments.

The inverse Mellin transform for (45) is therefore defined as

$$f(\boldsymbol{x}) = \frac{C_{T,M}}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} \boldsymbol{x}^{-s+1} \frac{\prod_{i=0}^{M} \Gamma(T+M+1-2j+(s+1))}{\Gamma[(T+s)M]} ds \tag{46}$$

Let us adopt the Gauss-Legendre multiplication formula [13]:

$$\Gamma[M(T+s)] = (2\pi)^{\frac{1}{2}(1-M)} M^{(M(T+2)-\frac{1}{2})} \times \\ \times \prod_{j=1}^{M} \Gamma(T + \frac{j-1}{M} + s) \quad (47)$$

with respect to (47), we get

$$f(\boldsymbol{x}) = C_{T,M} (2\pi)^{\frac{1}{2}(1-M)} M^{MT-\frac{1}{2}} \times \\ \times \frac{1}{2\pi i} \int \boldsymbol{x}^{-s+1} M^{Ms} \frac{\prod_{j=1}^{M} \Gamma(T+M+1-2j+s+1)}{\prod_{j=1}^{M} \Gamma(T + \frac{j-1}{M} + s)} ds \quad (48)$$

By substituting the scalar $(T-M+1+s)$ with $-s$, this becomes

$$f(\boldsymbol{x}) = C_{T,M} (2\pi)^{\frac{1}{2}(1-M)} M^{(M-1)T-\frac{1}{2}} \left(\frac{x}{M^M}\right)^{T-M} \\ \times \frac{1}{2\pi i} \int \left(\frac{\boldsymbol{x}}{M^M}\right)^{s} \frac{\prod_{j=1}^{M} \Gamma(2M-2j+1-s)}{\prod_{j=1}^{M} \Gamma(\frac{M^2+j-1-M}{M} + s)} ds \quad (49)$$

Introducing the Meijer's G-function (G.301, [13]), we get

$$f(\boldsymbol{x}) = C_{T,M} \; (2\pi)^{\frac{1}{2}(1-M)} M^{(M-1)T-1} \times \\ \times G_{M,M}^{M,0} \left(\frac{x}{M^M} \Bigg|_{1,3,\ldots,2M-1}^{\frac{M^2-1}{M}, \frac{M^2-2}{M}, \ldots, \frac{M^2-M}{M}} \right) \quad (50)$$

where $G_{c,d}^{a,b}(\cdot)$ represents the Meijer's G-function.

Let us demonstrate for $T \to \infty$, we have $MV_M \to I_M$ and $\mathcal{E}[\det V] \to (\frac{1}{M})^M$. According to (45), we have

$$\lim_{T\to\infty} \mathcal{E}[\det V] \Rightarrow \frac{T^M}{(TM)^M} = \frac{1}{M^M} \quad (51)$$

Alternatively, for a minimally sampled case where $T = M$, we have

$$\mathcal{E}[\det V] = -\frac{\prod_{j=0}^{T-1}(1+2j)}{\prod_{j=0}^{T-1}(T^2+j)} \leq \left(\frac{2}{T^2}\right)^T \quad (52)$$

which as expected is a significantly smaller value.

Comparing the statistical properties of the above sphericity test for the complex angular central Gaussian distribution with the properties of the conventional sphericity test for complex Gaussian distributed data, introduced in [5], the interested reader can see the close analogy between the separate distributions, with the key property being the dependence of the LR probability density function only on parameters (M, T, and p) which can be determined *a priori*.

The derived analytical formulas are also in good agreement with separately generated direct Monte-Carlo simulations for this particular circumstance, published recently in [1].

3. CONCLUSION

In this paper, we derived the accurate expressions for the arbitrary moment of the "expected likelihood" sphericity test and its p.d.f. for the complex angular central Gaussian distribution, used in expected likelihood assessment of the quality of DOA estimates in signal mixtures with compound Gaussian (impulsive) interference and Gaussian noise.

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