A NEW GENERALIZATION OF THE DISCRETE TEAGER-KAISER ENERGY OPERATOR - APPLICATION TO BIOMEDICAL SIGNALS

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ABSTRACT

The discrete Teager-Kaiser operator (TKO) was firstly introduced in [1]. Generalized versions of this operator (GTKO) were proposed later in [2]. Both the TKO and GTKO were able to detect instantaneous amplitude changes of signals and they significantly improved the signal to noise ratios (SNR). The TKO, as well as the GTKO, can be viewed as the determinant of an embedding square matrix of size 2×2 that is built using a window sliding over signals. In the present paper, we propose a new extension of these operators and we define the extended GTKO (EGTKO) as the determinant of an embedding matrix of size $d \times d$ with $d \geq 2$. We discuss different structures of such an embedding matrix.

The detection of instantaneous amplitude changes can be achieved by applying a threshold to the proposed EGTKO. To theoretically determine the optimal threshold that allows for the most accurate detection, we present a statistical characterization of the EGTKO based on the determinant theory of random matrices. The receiver operating characteristic (ROC) curves obtained at different SNR show that the accuracy of the EGTKO outperforms that of the TKO and GTKO. An application to a real biomedical signal is also presented and illustrates the superiority of the proposed EGTKO.

Index Terms— Teager-Kaiser operator, random matrix determinant, detection, biomedical signal, electromyography.

1. INTRODUCTION

The Teager-Kaiser energy operator (TKO) was defined in [1] for a continuous time t and a signal $x(t)$ as $\Psi(t) = \left(\frac{\partial x(t)}{\partial t}\right)^2 - x(t)\frac{\partial^2 x(t)}{\partial t^2}$, and then for a discrete time n as

$$\Psi[n] = x^2[n] - x[n-1]\, x[n+1]. \tag{1}$$

This nonlinear operator was able to localize instantaneous amplitude changes of signals and it significantly improved the signal to noise ratios (SNR) [3, 4, 5, 6]. The TKO usefulness has been proven in many research fields, including biomedical signal processing [3, 4], speech analysis [5] and communications [6].

In [2], a generalized version of the Teager-Kaiser operator (GTKO) was defined as

$$\Psi[n] = x[n]x[n+k] - x[n-h]x[n+k+h], \tag{2}$$

where k and h are discrete time lags. This operator (2) corresponds to an asymmetric discrete energy velocity measure for k=1 and h=1 whereas it yields a discrete energy acceleration measure for k=2 and h=1, accordingly to [2]. The main advantage of this GTKO (2) over the TKO (1) lies in its superior localization of changes of instantaneous amplitudes.

The GTKO (2) can be viewed as the determinant of a square matrix of size 2×2 built by embedding the observed signal in its time delayed coordinates [7]

$$\Psi[n] = \det \begin{pmatrix} x[n] & x[n+k+h] \\ x[n-h] & x[n+k] \end{pmatrix}. \tag{3}$$

The TKO (1) is obtained by setting k=0 and h=1 in (3) and it appears to be the determinant of a 2×2 Toeplitz matrix [7].

In the present paper, we propose an extend generalized Teager-Kaiser operator (EGTKO) that we define as the determinant of an embedding square matrix of a higher size $d \times d$, with $d \geq 2$. This EGTKO is expected to further improve the localization of changes of instantaneous amplitudes.

We discuss various possibilities of the embedding matrix structure: independent entries, Toeplitz and symmetric. We also focus on the statistical properties of the EGTKO based on the determinant theory of random matrices. It is worth noting the characterization lack of the TKO (1) and GTKO (2) probability density function (pdf).

The detection of instantaneous amplitude changes of signals can then be achieved by comparing the proposed EGTKO to an optimal threshold. To the best of our knowledge, we theoretically derive for the first time the expression of the optimal threshold that allows for the most accurate detection, whatever the d value.

To demonstrate the accuracy superiority of the proposed EGTKO over the TKO and GTKO, receiver operating characteristic (ROC) curves obtained at different SNR are presented.

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An application to a real biomedical signal is also provided.

The paper is organized as follows. Section 2 presents the proposed EGTKO whereas Section 3 details its statistical feautres. Section 3 is dedicated to the theoretical derivation of the optimal threshold for detection purposes. The feasibility and interest of the EGTKO compared to TKO and GTKO are illustrated in Section 5 using ROC curves and a real biomedical signal. Finally, Section 6 draws the conclusions and work in progress.

2. PROPOSED EGTKO

Let us consider a 1-D signal $x[n]$ where $n = 0, 1, \ldots, N-1$ and N is the sample number. This signal can be transformed into vectors $\mathbf{x_n}$ of length d by embedding it in its time delayed coordinates

$$\mathbf{x}_n = [x[n], x[n-m], x[n-2m], \ldots, x[n-(d-1)m]]^T, \quad (4)$$

where $(.)^T$ denotes the transpose operator and m is a time lag. An embedding matrix $\mathbf{X}_n$ of size $d \times d$ is built as

$$\mathbf{X}_n = \left[\mathbf{x}_n, \mathbf{x}_{n+s}, \ldots, \mathbf{x}_{n+(d-1)s}\right], \quad (5)$$

where $s \neq 0$ is a time lag.

The matrix entries are $(d-1)(s+m)+1$ samples taken using a sliding window $[n-(d-1)m, n+(d-1)s]$ over the considered signal. The explicit expression of $\mathbf{X}_n$ is

$$\mathbf{X}_n = \begin{pmatrix} x[n] & x[n+s] & \ldots & x[n+(d-1)s] \\ x[n-m] & x[n-m+s] & \ldots & x[n-m+(d-1)s] \\ \vdots & \vdots & \ddots & \vdots \\ x[n-(d-1)m] & x[n-(d-1)m+s] & \ldots & x[n-(d-1)(m-s)] \end{pmatrix}. \quad (6)$$

We then define the EGTKO, denoted by $\Psi_{d,m,s}[n]$ as

$$\Psi_{d,m,s}[n] = \det(\mathbf{X}_n). \quad (7)$$

The TKO (1) can be viewed as a special case of (7) by fixing d=2 and s=m=1 in (7). Additionally, the GTKO (2) turns out to be also a special case of (7) when selecting d=2 and k=$s-m$ and h=m.

This definition of the EGTKO (7) proposed is also motivated by the fact that the determinant is an algebraic operator that helps quantify the notion of near singularity. Moreover, the determinant is an alternating-multilinear form of the matrix entries that verifies the property of similarity invariance (basis independent) [8].

3. STATISTICAL PROPERTIES OF EGTKO

The statistical properties of the EGTKO (7) are dependent not only on the structure of the embedding square matrix $\mathbf{X}_n$ (Toeplitz, symmetric, independent entries,...) but also on the assumptions made on the distribution law of the matrix entries (Gaussian, exponential, uniform,...). Due to lack of space, only some scenarios are presented.

3.1. EGTKO with d=2

Basic statistics of TKO (1) and GTKO (2) were addressed in [1, 7, 9]. However, it must be pointed out that only mean and variance were obtained when the input signal is an additive zero-mean white Gaussian noise and/ or sum of sinusoids. We here aim at filling the lack of distribution law of EGTKO (7) with d=2 and, *a fortiori*, that of TKO and GTKO.

3.1.1. Independent random normal matrix entries d=2

Let us suppose that the matrix entries (6): $x[n], x[n+s], x[n-m]$ and $x[n+s-m]$ are mutually independent random variables ($s\neq 0$, $m\neq 0$, $s\neq m$ and $s\neq -m$), each normally distributed, with means μ_0, μ_s, μ_m and μ_{s-m} and a common variance σ_w^2. For the sake of notation simplicity, we denote by Ψ the random determinant Ψ=$\Psi_{d,m,s}[n]$.

The assessment of the pdf and cumulants of the determinant of such a matrix is an old mathematical subject [10, 11]. When all the means μ_i vanish the exact law of Ψ is calculated to be the Laplace distribution [10, 11]

$$p_\Psi(\psi) = \frac{1}{4\sigma_w^2} \exp\left(-\frac{|\psi|}{2\sigma_w^2}\right) \quad \text{for} \quad \psi \in \mathbb{R}. \quad (8)$$

When the means μ_i are not zero, the distribution of Ψ is skewed, and is not expressible in a simple closed form. However, a straightforward inversion of the characteristic function of Ψ with σ_w^2=1 is possible and yields the exact pdf of Ψ [11]. This characteristic function and its associated pdf are

$$\begin{array}{l} \varphi_\Psi(z) = \mathbb{E}\left[e^{jz\Psi}\right] = \frac{1}{1+z^2} \exp\left(\frac{-\Lambda z^2 + 2j\Delta z}{2(1+z^2)}\right), \\ p_\Psi(\psi) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-j\psi z}\varphi_\Psi(z)dz, \end{array} \quad (9)$$

respectively, where j^2=-1, $\mathbb{E}[.]$ is the expectation and

$$\begin{array}{rcll} \Lambda & = & \mu_0^2 + \mu_s^2 + \mu_m^2 + \mu_{s-m}^2, & 0 \leq \Lambda < +\infty, \\ \Delta & = & \mu_0\mu_{s-m} - \mu_s\mu_m, & -\frac{\Lambda}{2} \leq \Delta < \frac{\Lambda}{2}. \end{array} \quad (10)$$

The cumulants of Ψ can be obtained by expanding the logarithm of $\varphi_\Psi(z)$ [11].

3.1.2. Random Gaussian Toeplitz matrix entries $d = 2$

Let us assume s=m, the matrix (5) is then a 2×2 Toeplitz matrix. To our knowledge, we derive for the first time the expression of the determinant pdf of such a matrix. To that end, we first determine the expression of the characteristic function under the assumption of mutually independent entries $x[n], x[n+s]$ and $x[n-m]$ ($s\neq 0$, $m\neq 0$ and $s\neq -m$), each normally distributed with a common variance σ_w^2=1 and means μ_0, μ_s and μ_m, respectively:

$$\varphi_\Psi(z) = \mathbb{E}\left[e^{jz\Psi}\right] = \frac{e^{\frac{jz\mu_0^2}{1-2jz}}}{(1-2jz)^{\frac{1}{2}}} \frac{e^{-\frac{\left(\mu_s^2+\mu_m^2\right)z^2+2j\mu_s\mu_m z}{2\left(1+z^2\right)}}}{\sqrt{(1+z^2)}} \quad (11)$$

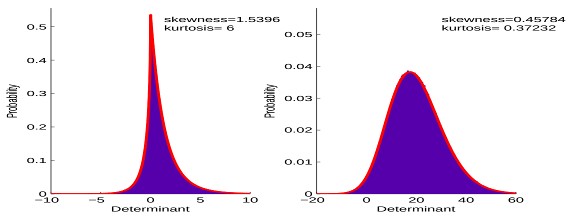

Fig. 1. EGTKO pdf for normally distributed random entries of a 2×2 Toeplitz matrix $\mathbf{X_n}$. The common variance σ_w^2=1 and means $\{\mu_0, \mu_s, \mu_m\}$ are equal to $\{0, 0, 0\}$ (left column) and $\{1, 2, 3\}$ (right column). Theoretical pdfs (11) (red curve) are superimposed to the normalized histograms.

The EGTKO pdf is then obtained by the Fourier transform of (11). Figure 1 displays such a pdf. We also derive the cumulants of Ψ by expanding the logarithm of $\varphi_\Psi(z)$ (11)

$$\begin{aligned} c_{2h} &= (2h)!\left(\frac{\mu_s^2+\mu_m^2}{2} + \frac{1}{2h} + 2^{2h-2}\left(\frac{1}{h} + 2\mu_0^2\right)\right) \\ c_{2h+1} &= (2h+1)!\left(2^{2h}\left(\mu_0^2 + \frac{1}{2h+1}\right) - u_s\mu_m\right). \end{aligned} \tag{12}$$

In particular the mean, variance, skewness and kurtosis of Ψ are given by $1+\mu_0^2-\mu_s\mu_m$, $4\mu_0^2+\mu_s^2+\mu_m^2+3$, $\frac{6(4\mu_0^2+\frac{4}{3}-\mu_s\mu_m)}{(4\mu_0^2+\mu_s^2+\mu_m^2+3)^{\frac{3}{2}}}$, and $\frac{12(16\mu_0^2+\mu_s^2+\mu_m^2+\frac{9}{2})}{(4\mu_0^2+\mu_s^2+\mu_m^2+3)^2}$, respectively.

We note that the pdf of $\Psi_{2,m,s}[n]$ shown in Fig.1 can be easily generalized for a common variance σ_w^2 not necessarily equal to one, provided that means $\{\mu_0, \mu_s, \mu_m\}$ are equal to $\{0, 0, 0\}$. This generalization can be viewed as a scaled probability $\frac{1}{\sigma_w^2} p_\Psi(\frac{\psi}{\sigma_w^2})$. We also note that for this latter case, we retrieve a classical result: the EGTKO mean is exactly the variance of the Gaussian process $\mathbb{E}[\Psi_{2,m,s}[n]] = \sigma_w^2$.

3.1.3. *Random Gaussian symmetric matrix entries d=2*

Let us assume s=$-m$, the matrix (5) is then a 2×2 symmetric matrix, called a Hankel matrix [12]. To our knowledge, we derive the determinant pdf of such a matrix for the first time. So, we first derive the expression of the characteristic function by substituting $-z$ to z and μ_{s-m} to μ_m in (11), under the assumption of mutually independent entries $x[n], x[n+s]$ and $x[n-m+s]$ ($s{\neq}0$, $m{\neq}0$ and $s{\neq}m$), each being normally distributed with a common variance σ_w^2=1 and means μ_0, μ_s and μ_{s-m}, respectively. The pdf is then obtained by Fourier transform of this characteristic function.

3.2. EGTKO with $d \geq 3$

There are very few analytic expressions of the exact pdf of random determinants with $d \geq 3$ and one can instead find asymptotic expressions [10]. The case of independent identically distributed (i.i.d) normal matrix elements with zero-mean and unit common variance is considered in [10]. Complex expressions of the asymptotic behaviours of the determinant pdfs are provided for all $d \geq 3$ whereas exact pdfs are only derived up to $d = 4$:

$$\begin{aligned} p_\Psi(\psi) &= \frac{1}{\sqrt{2\pi}}\int_0^\infty r\exp\left(-\frac{r^2}{2} - \frac{\psi}{r}\right)dr && \text{with } \Psi = \Psi_{3,m,s}[n] \\ p_\Psi(\psi) &= \frac{\psi}{2}K_2(2\sqrt{\psi}) && \text{with } \Psi = \Psi_{4,m,s}[n]. \end{aligned} \tag{13}$$

The related exact moments of the determinant (7) are also provided in [10]. Both pdfs (13) can be easily generalized for a common variance not necessarily unit.

In [13], it is shown the log-normality of the determinant of a standard real Gaussian random matrix whose entries are i.i.d. when d tends toward ∞. More precisely, the Lyapunov's central limit theorem holds for $\log(|\Psi|)$:

$$\frac{\log(|\Psi|) - \frac{1}{2}\log((d-1)!)}{\sqrt{\frac{\log(d)}{2}}} \overset{\text{weakly converges}}{\longrightarrow} \mathcal{N}(0,1). \tag{14}$$

4. OPTIMAL THRESHOLD FOR DETECTION

We determine the theoretical optimal threshold to apply on the EGTKO output for an accurate detection of instantaneous amplitude changes. The idea is to monitor the threshold selection by only setting the probabilities of false alarm $\mathbb{P}_{\text{fa}}$ and/or good detection $\mathbb{P}_{\text{gd}}$ of events of interest (occurring changes). To that end, when only noise is present we define for any $\xi \geq 0$ the false alarm probability as

$$\mathbb{P}_{\text{fa}_\xi} = \mathbb{P}[\,|\Psi_{d,m,s}[n]| \geq \xi; \mathbf{X}_n \text{ entries (5) are noise samples}]. \tag{15}$$

It is the probability that specific noise samples lead an EGTKO output above a threshold ξ and can be misinterpreted as signal samples. Similarly, when the signal of interest and noise are both present, we define for any $\xi \geq 0$ detection probability as

$$\mathbb{P}_{\text{gd}_\xi} = \mathbb{P}[\,|\Psi_{d,m,s}[n]| \geq \xi; \mathbf{X}_n \text{ entries (5) are noisy signal samples}], \tag{16}$$

It is the probability that signal samples corrupted by noise lead an EGTKO output above a threshold ξ. Therefore, the best threshold ξ is the one that minimizes the $\mathbb{P}_{\text{fa}_\xi}$ and maximizes $\mathbb{P}_{\text{gd}_\xi}$.

Let us denote by $F^{-1}_{\Psi_{d,m,s}[n]}$ the inverse of the cumulative distribution related to the (scaled) pdf of the EGTKO (7). Let us also assume both the noise and free noise signal samples to be zero-mean Gaussian with a standard deviation σ_w and σ_s, respectively. The theoretical expressions of the optimal threshold ξ (15) and (16) that we derive are given by

$$\xi = \sigma_w^d F^{-1}_{\Psi_{d,m,s}[n]}\left(1 - \frac{\mathbb{P}_{\text{fa}_\xi}}{2}\right), \tag{17}$$

$$\xi = (\sigma_w^2 + \sigma_s^2)^{\frac{d}{2}} F^{-1}_{\Psi_{d,m,s}[n]}\left(1 - \frac{\mathbb{P}_{\text{gd}_\xi}}{2}\right). \tag{18}$$

Under the assumption of Section 3.1.1 and using (8), (15) and (16), (17) and (18) write

$$\mathbb{P}_{\text{fa}_\xi} = \exp\left(-\frac{\xi}{2\sigma_w^2}\right) \text{ and } \mathbb{P}_{\text{gd}_\xi} = \exp\left(-\frac{\xi}{2(\sigma_w^2+\sigma_s^2)}\right). \tag{19}$$

This leads the following relationship, with $\text{SNR} = 10\log_{10}\left(\frac{\sigma_s^2}{\sigma_w^2}\right)$

$$\log\left(\mathbb{P}_{\text{fa}_\xi}\right) \;=\; \log\left(\mathbb{P}_{\text{gd}_\xi}\right)\left(1+10^{\frac{\text{SNR}}{10}}\right). \qquad (20)$$

Figure 2.a shows the receiver operating characteristic (ROC) curves related to (20) for different SNR values.

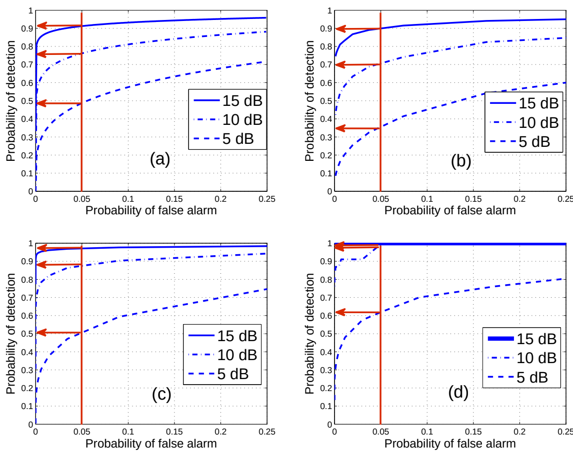

Fig. 2. ROC curves of EGTKO detector at different SNR values, 5, 10 and 15 dB, obtained under the assumptions assumed in: (a) Section 3.1.1 with $d{=}-m{=}2$ and $s{=}1$ *i.e.* GTKO with $k{=}3$ and $h{=}-2$, (b) Section 3.1.2 with $d{=}2$ and $s{=}m{=}1$ *i.e.* TKO, (c) Section 3.2 with $d{=}-m{=}3$ and $s{=}1$ and finally (d) with $d{=}-m{=}4$ and $s{=}1$.

Under the assumption of Section 3.1.2 or Section 3.1.3 or Section 3.2, analytic expressions of false alarm and good detection probabilities are difficult to explicit. However, since the pdf can be numerically evaluated from (11) or (13), ROC curves can be numerically deduced from (17) as shown in Fig.2b, c and d. Actually, Fig.2a corresponds to the GTKO (2) whereas Fig.2b corresponds to the TKO (1). As, one can notice from both figures, the detection probability is reduced using TKO compared to that of GTKO for a given false alarm probability. This result supports the use of the first proposal of the GTKO [2] instead of the conventional TKO [1].

Additionally, as one can notice from Fig.2, the EGTKO defined with $d{=}4$ provides the best ROC curves compared to the EGTKO defined with $d{=}3$ and $d{=}2$. Once again, this result lends support for the use of the proposed EGTKO (7) instead of the GTKO (2). Indeed, increasing the matrix size of $\mathbf{X}_n$ (5) seems to reduce the false alarm probability while the rate of good detection is increased. A better discrimination between only noise and the signal presence is expected.

Under the assumption of Section 3.2 and when $d \to \infty$ (in statistic practice for $d \geq 33$), it would be preferable to derive the optimal threshold for the logarithm of the EGTKO output using (14)

$$\begin{aligned}
\xi_{\log} = \; & -\tfrac{1}{2}\log\left((d-1)!\right) + \sqrt{2}\sqrt{\tfrac{\log(d)}{2}}\ \text{erfinv}\left(1-\mathbb{P}_{\text{fa}_\xi}\right) + \\
& \log\left(\sigma_w^d\right), \\
\xi_{\log} = \; & -\tfrac{1}{2}\log\left((d-1)!\right) + \sqrt{2}\sqrt{\tfrac{\log(d)}{2}}\ \text{erfinv}\left(1-\mathbb{P}_{\text{gd}_\xi}\right) + \\
& \log\left(\left(\sigma_w^2+\sigma_s^2\right)^{\frac{d}{2}}\right),
\end{aligned} \qquad (21)$$

where erfinv$(.)$ is the inverse error function [14].

5. APPLICATION TO BIOMEDICAL SIGNALS

We here apply the proposed EGTKO to a surface electromyography (sEMG) signal acquired at a sampling rate 1999 Hz from the right soleus of a healthy subject during gait. The EGTKO is built with an embedding matrix size $d{=}2, 3, 4$ and 33 and with $s{=}1$ and $m{=}-d$ for all cases. The desired detection probability is set equal to 90%. As one can notice from Fig.3, the best localization of on-off timing of the sEMG signal is obtained using $d{=}33$.

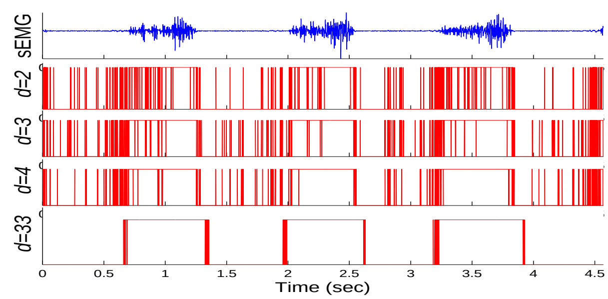

Fig. 3. Application to the localization of instantaneous amplitude changes of a sEMG signal: $d{=}2, 3, 4$ and 33 from top to bottom and with $s{=}1$ and $m{=}-d$ for all cases. The desired detection probability is set equal to 0.9.

6. CONCLUSIONS

we propose a new generalization of Teager-Kaiser operator, named EGTKO, that helps improve the SNR and contributes to a higher accuracy of localization of signal changes. The proposed EGTKO is defined as the determinant of an embedding matrix built from the signal itself. The localization accuracy of the proposed EGTKO is dependent on the selection of the matrix size $d \geq 2$, the matrix structure (independent entries, Toeplitz or symmetric) tuned by parameters m and s, and finally the threshold ξ theoretically selected in such a way as to maximize the detection probability while minimizing the false alarm. In future work, we aim at comparing the EGTKO to existing tools of biomedical signal segmentation.

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