A "POLYPHASE" STRUCTURE OF TWO-CHANNEL SPECTRAL GRAPH WAVELETS AND FILTER BANKS

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ABSTRACT

This paper addresses a polyphase¹ structure of spectral graph wavelets and filter banks. We consider two-channel critically sampled graph filter banks. In classical signal processing, polyphase structure of filter banks is very useful since downsampler (upsampler) can be placed before analysis filtering (after synthesis filtering). We theoretically derive that a similar structure is also possible for spectral graph filter banks. The structure can be used for any twochannel critically sampled spectral graph filter banks as long as an underlying graph is bipartite.

Index Terms— Graph signal processing, spectral graph wavelets, polyphase structure, graph filter banks

1. INTRODUCTION

Graph signal processing is becoming popular in the community of signal and information processing [1,2]. It has many promising applications by analyzing signals on complex networks, e.g., sensor, social, neuronal, and transportation networks. Recent progress in this field is to *reinterpret* classical signal processing tools into graph spectral domain with spectral graph theory. They include wavelets and filter banks [3–9], sampling theory [10–12], uncertainty principle [13, 14], and robust principal component analysis [15, 16] on graphs. Many applications have been found so far; for example, denoising and interpolation [17–19], sensor position selection [20,21], and multilinear discriminant analysis for brain–computer interface [22].

In this paper, we study an efficient structure of wavelets and filter banks on graphs. We consider two-channel critically sampled spectral graph filter banks. Since naive filtering with eigendecomposition is usually not recommended due to its computational cost, we often use polynomial filter kernels. There have been mainly two approaches to design spectral graph filter banks with polynomial kernels. One is spectral factorization [5,6], and the other is polynomial approximation of a spectral response [3,4,9]. Due to *spectral folding phenomenon*, which is a counterpart of aliasing effect in classical signal processing, critically sampled spectral graph filter banks require a careful design. There are two well-known wavelets in such a class. One is the orthogonal solution [4] and the other is biorthogonal [5]. Though they can be applied to signals on arbitrary graphs, the original graph should be divided into some bipartite subgraphs



Fig. 1. Simple bipartite graphs. Black and white nodes represent two disjoint sets L and H, respectively. (Top left) ring graph. (Top right) star graph. (Bottom) path graph.

since the transform has to be applied to the signals on a bipartite graph to guarantee the perfect reconstruction.

Like classical signal processing, filtered graph signals are downsampled to obtain transformed coefficients in graph spectral domain. In the synthesis side, they are upsampled followed by filtering to reconstruct the signal. It is so-called *direct form*. We know that, in the classical case, we can obtain the same results by using *polyphase structure* [23–25]. It can switch the places of downsampler and upsampler: Downsampling then filtering in the analysis bank, and filtering then upsampling in the synthesis bank. Its purpose is that we can work at a fast rate in mutirate signal processing. We can separate even and odd indexed-signals and perform filtering to them in parallel.

Such polyphase structure will also be useful in graph signal processing. Since we often have to treat signals on large graphs, downsampling before filtering will be highly beneficial. However, there are few approaches so far. Noble identities in graph signal processing have been partially shown in [26, 27], which will be useful for the polyphase structure of graph signal processing. Unfortunately, the condition is restricted: In the two-channel case, the number of transformed coefficients in the lowpass and highpass channels must be the same, i.e., each channel has N/2 coefficients where N is the length of the graph signal². It is generally satisfied in the classical case since the downsampling in classical signal processing takes every other sample. However, it is not the case for graph signal processing: The number of coefficients in each channel heavily depends on underlying graphs. For example, ring and path graphs will be similar to the classical case, whereas star graph only has one coefficients for the lowpass channel and the remaining (N-1) samples are for the highpass channel (and vice versa). Illustrative examples are shown in Fig. 1. It could be solved by using a similarity trans-

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¹Indeed "phase" has not been strictly defined in the context of graph signal processing. We just use this word (and also, "polyphase") as a counterpart of classical signal processing.

 $^{^2 {\}rm For}$ the general M -channel case, each channel has N/M coefficients [26,27].

form matrix [27]. However, it changes graph Fourier basis and thus, it has to be carefully designed. The similarity transformation also adds extra computation complexity.

In this paper, we consider an efficient polyphase structure of two-channel critically sampled spectral graph wavelets and filter banks. We systematically follow the classical approach, and relaxed the condition on the number of transformed coefficients in each channel. Arbitrary spectral graph filters can use the proposed polyphase structure as long as the graph G is connected. Additionally, the perfect reconstruction condition can be represented very easily.

The remaining of this paper is organized as follows. Preliminaries are shown in Section 2, which includes graph signal processing tools used in this paper and polyphase structure in z-domain for classical signal processing. Section 3 is the main contribution in this paper: Polyphase structure in graph spectral domain is introduced with proofs. Finally, this paper is concluded in Section 4.

2. PRELIMINARIES

2.1. Basics of Graph Signal Processing

A graph G is represented as $G = (\mathcal{V}, \mathcal{E})$, where \mathcal{V} and \mathcal{E} denote sets of nodes and edges, respectively. The number of nodes is $N = |\mathcal{V}|$, unless otherwise specified. The (m, n)-th element of the adjacency matrix **A** is w_{mn} if m and n are connected, and 0 otherwise, where w_{mn} denotes the weight of the edge between m and n. The degree matrix **D** is a diagonal matrix, and its mth diagonal element is $d_{mm} = \sum_n w_{mn}$. We consider an undirected graph in this paper. The normalized adjacency matrix is also defined as $\mathcal{A} := \mathbf{D}^{-1/2} \mathbf{A} \mathbf{D}^{-1/2}$. Various variation operators, such as graph Laplacian $\mathbf{L} := \mathbf{D} - \mathbf{A}$, are often considered instead of the adjacency matrix. We use the normalized adjacency matrix since it is the simplest form to realize our polyphase structure. Extensions to the other variation operators are our future work.

The key symbols used here are:

- 1. f: Graph signal ($f \in \mathbb{R}^N$).
- 2. $\boldsymbol{u}_{\lambda_i}$: *i*th eigenvector of $\boldsymbol{\mathcal{A}}$.
- 3. λ_i : *i*th eigenvalue of \mathcal{A} . $(\mathcal{A}u_{\lambda_i} = \lambda_i u_{\lambda_i})$, where $-1 \leq \lambda_0 \leq \lambda_1 \leq \ldots \leq \lambda_{N-2} \leq \lambda_{N-1} = 1$ for connected graphs. $\lambda_0 = -1$ only for bipartite graphs.
- 4. $\sigma(\mathbf{A})$: Spectrum of the graph, i.e., $\sigma(\mathbf{A}) := \{\lambda_i\}_{i=0,...,N-1}$.

The eigenvectors $\mathbf{U} = [\boldsymbol{u}_{\lambda_0}, \dots, \boldsymbol{u}_{\lambda_{N-1}}]$ satisfy $\mathbf{U}\mathbf{U}^\top = \mathbf{I}_N$, where \cdot^\top is the transpose of a matrix or a vector and \mathbf{I}_N is the identity matrix with the size $N \times N$. The graph Fourier transform is defined as $\bar{f}(\lambda_i) = \langle \boldsymbol{u}_{\lambda_i}, \boldsymbol{f} \rangle = \sum_{n=0}^{N-1} u_{\lambda_i}(n) f(n)$ [3]. Let $H(\lambda)$ be the spectral kernel of filter **H** defined on the real line $\lambda \in [-1, 1]$. The spectral domain filter can be written as

$$\mathbf{H} = H(\boldsymbol{\mathcal{A}}) = \sum_{\lambda_i \in \sigma(\boldsymbol{\mathcal{A}})} H(\lambda_i) \sum_{\lambda = \lambda_i} \boldsymbol{u}_{\lambda} \boldsymbol{u}_{\lambda}^{\top}.$$
 (1)

Spectral domain filtering of graph signals will simply be denoted as $f_{out} = \mathbf{H} f_{in}$.

2.2. Two-Channel Critically Sampled Spectral Graph Wavelets

A bipartite graph, whose nodes can be decomposed into two disjoint sets L and H such that every edge connects a node in L to one in H, is represented as $\mathcal{G} = (L, H, \mathcal{E})$. The downsampling operation is represented as $(\downarrow J_L)$ and $(\downarrow J_H)$, where $(\downarrow J_L)$ picks up the nodes



Fig. 2. Critically sampled spectral graph filter bank.

in L and $(\downarrow J_H)$ selects ones in H. Conversely, the upsampling operator makes a signal with length N by inserting zeros. When we have a downsampled signal $f_L \in \mathbb{R}^{|L|}$ which corresponds to the signal on the set L,

$$(\uparrow J_L) \boldsymbol{f}_L = \begin{bmatrix} \boldsymbol{f}_L \\ \boldsymbol{0}_{|H| \times 1} \end{bmatrix}.$$
 (2)

The critically sampled spectral graph wavelet transforms decompose N input signals into |L| lowpass coefficients and |H| highpass coefficients, where |L| + |H| = N. The direct form is illustrated in Fig. 2. Since any arbitrary graph can be decomposed into K bipartite subgraphs where $K = \lceil \log_2 C \rceil$ and C is the chromatic number of the graph, spectral graph wavelet transforms for bipartite graphs can be applied to any non-bipartite graph [4,28].

The perfect reconstruction condition of the critically sampled spectral graph wavelet transforms can be expressed as follows [4]:

$$F_0(\lambda)H_0(\lambda) + F_1(\lambda)H_1(\lambda) = 2, \qquad (3)$$

$$-F_0(\lambda)H_0(2-\lambda) + F_1(\lambda)H_1(2-\lambda) = 0.$$
 (4)

Although the normalized graph Laplacian has been originally considered in (3) and (4) [4,5], we have the same perfect reconstruction condition even if we use \mathcal{A} as a variation operator since their difference is only a range of eigenvalues: Those of \mathcal{A} is $\lambda \in [-1,1]$ instead of [0, 2] of the normalized graph Laplacian. Eigenvectors are obviously the same.

There are two well-known wavelets: GraphQMF [4], which is an orthogonal transform and non-compact support, and graphBior [5], which is a biorthogonal transform and satisfies the perfect reconstruction and compact support conditions.

2.3. Polyphase Structure in z-Domain

Here we briefly review the polyphase structure of two-channel filter banks in classical signal processing. When we have an FIR filter $G(z) = \sum_{m} g_m z^{-m}$, where $\boldsymbol{g} = [g_0, g_1, \ldots]^{\top}$ is the impulse response of the filter, its type 1 and type 2 polyphase structures are respectively represented as

$$G(z) = \begin{cases} G_0(z^2) + z^{-1}G_1(z^2) & \text{Type 1} \\ G_1(z^2) + z^{-1}G_0(z^2) & \text{Type 2} \end{cases},$$
 (5)

where $G_0(z)$ and $G_1(z)$ correspond to the even and odd powers of z in G(z), respectively.

The type 1 polyphase matrix is used for the analysis bank and it is represented as [25]

$$\mathbf{H}(z) = \begin{bmatrix} H_{00}(z) & H_{01}(z) \\ H_{10}(z) & H_{11}(z) \end{bmatrix}$$
(6)

where $H_i(z) = H_{i0}(z^2) + z^{-1}H_{i1}(z^2)$ and $i = \{0, 1\}$ is the filter index. The type 2 polyphase matrix is also used for the synthesis bank and defined as

$$\mathbf{F}(z) = \begin{bmatrix} F_{00}(z) & F_{10}(z) \\ F_{01}(z) & F_{11}(z) \end{bmatrix}.$$
(7)



Fig. 3. Two-channel critically sampled filter bank in classical signal processing.

By using the polyphase structure, the perfect reconstruction condition can easily be represented as

$$\mathbf{F}(z)\mathbf{H}(z) = z^{-\ell}\mathbf{I},\tag{8}$$

where ℓ is some constant. The direct and polyphase forms are illustrated in Fig. 3.

3. POLYPHASE STRUCTURE IN GRAPH SPECTRAL DOMAIN

In this section, we prove that the polyphase structure is also possible in graph signal processing. Without loss of generality, A is assumed to be

$$\boldsymbol{\mathcal{A}} = \begin{bmatrix} \mathbf{0} & \mathbf{B} \\ \mathbf{B}^{\top} & \mathbf{0} \end{bmatrix}, \tag{9}$$

where $\mathbf{B} \in \mathbb{R}^{|L| \times |H|}$ represents edges between L and H. Additionally, let us define $\widetilde{\mathbf{B}} = \mathbf{B}\mathbf{B}^{\top}$.

3.1. Analysis Polyphase

We assume that the lowpass and highpass filters in the analysis bank are the n_0 and n_1 th order polynomials, respectively. They are represented as follows:

$$\mathbf{H}_0(\boldsymbol{\mathcal{A}}) = \sum_{m=0}^{n_0} a_m \boldsymbol{\mathcal{A}}^m \tag{10}$$

$$\mathbf{H}_1(\boldsymbol{\mathcal{A}}) = \sum_{m=0}^{n_1} b_m \boldsymbol{\mathcal{A}}^m.$$
(11)

The filtering-then-downsampling operation is represented as

$$\begin{bmatrix} \tilde{f}_L \\ \tilde{f}_H \end{bmatrix} = \begin{bmatrix} (\downarrow J_L) \mathbf{H}_0(\mathcal{A}) \\ (\downarrow J_H) \mathbf{H}_1(\mathcal{A}) \end{bmatrix} \begin{bmatrix} f_L \\ f_H \end{bmatrix}.$$
 (12)

We have the following proposition for the analysis polyphase matrix:

Proposition 1. (Analysis polyphase structure) For a connected bipartite graph G, the analysis transform matrix of two-channel critically sampled spectral graph filter banks can always be represented as

$$\begin{bmatrix} (\downarrow J_L) \mathbf{H}_0 \\ (\downarrow J_H) \mathbf{H}_1 \end{bmatrix} = \mathbf{H}_p(\boldsymbol{\mathcal{A}}) \begin{bmatrix} \mathbf{I}_{|L|} & \mathbf{0} \\ \mathbf{0} & \mathbf{B} \end{bmatrix}, \quad (13)$$

where

$$\mathbf{H}_{p}(\boldsymbol{\mathcal{A}}) := \begin{bmatrix} \mathbf{H}_{00}(\tilde{\mathbf{B}}) & \mathbf{H}_{01}(\tilde{\mathbf{B}}) \\ \mathbf{H}_{11}(\tilde{\mathbf{B}}^{\top}) & \mathbf{H}_{10}(\tilde{\mathbf{B}}^{\top}) \end{bmatrix}$$
(14)

in which³

$$\mathbf{H}_{00}(\widetilde{\mathbf{B}}) = \sum_{m} a_{2m} \widetilde{\mathbf{B}}^{m}$$
$$\mathbf{H}_{01}(\widetilde{\mathbf{B}}) = \sum_{m} a_{2m+1} \widetilde{\mathbf{B}}^{m}$$
$$\mathbf{H}_{10}(\widetilde{\mathbf{B}}^{\top}) = b_{0} \mathbf{P} + \sum_{m} b_{2m+2} (\widetilde{\mathbf{B}}^{\top})^{m} \mathbf{B}^{\top}$$
$$\mathbf{H}_{11}(\widetilde{\mathbf{B}}^{\top}) = \sum_{m} b_{2m+1} (\widetilde{\mathbf{B}}^{\top})^{m} \mathbf{B}^{\top},$$
(15)

and $\mathbf{P} \in \mathbb{R}^{|H| \times |L|}$ satisfies $\mathbf{PB} = \mathbf{I}_{|H|}$.

Proof. First, we consider the lowpass filter $\mathbf{H}_0(\boldsymbol{\mathcal{A}})$. It is represented as

$$\mathbf{H}_{0}(\boldsymbol{\mathcal{A}}) = \sum_{m} a_{2m} \boldsymbol{\mathcal{A}}^{2m} + \sum_{m} a_{2m+1} \boldsymbol{\mathcal{A}}^{2m+1}.$$
 (16)

Since $\boldsymbol{\mathcal{A}}$ is bipartite and can be represented as (9), $\sum_{m} a_{2m} \boldsymbol{\mathcal{A}}^{2m}$ in (16) is rewritten as

$$\sum_{m} a_{2m} \mathcal{A}^{2m} = a_0 \begin{bmatrix} \mathbf{I}_{|L|} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_{|H|} \end{bmatrix} + a_2 \begin{bmatrix} \widetilde{\mathbf{B}} & \mathbf{0} \\ \mathbf{0} & \widetilde{\mathbf{B}}^{\top} \end{bmatrix} + \cdots$$
$$= \operatorname{diag} \left(\sum_{m} a_{2m} \widetilde{\mathbf{B}}^m, \sum_{m} a_{2m} (\widetilde{\mathbf{B}}^{\top})^m \right).$$
(17)

Similarly, $\sum_{m} a_{2m+1} \mathcal{A}^{2m+1}$ in (16) is rewritten as

$$\sum_{m} a_{2m+1} \mathcal{A}^{2m+1}$$

$$= \operatorname{diag}\left(\sum_{m} a_{2m+1} \widetilde{\mathbf{B}}^{m}, \sum_{m} a_{2m+1} (\widetilde{\mathbf{B}}^{\top})^{m}\right) \mathcal{A}.$$
(18)

Since the downsampler $(\downarrow J_L)$ keeps the first |L| elements,

$$(\downarrow J_L)\mathbf{H}_0(\mathcal{A}) = \begin{bmatrix} \mathbf{H}_{00}(\widetilde{\mathbf{B}}) & \mathbf{H}_{01}(\widetilde{\mathbf{B}}) \end{bmatrix} \begin{bmatrix} \mathbf{I}_{|L|} & \mathbf{0} \\ \mathbf{0} & \mathbf{B} \end{bmatrix}.$$
 (19)

With a similar technique, the highpass filter followed by the downsampling by $(\downarrow J_H)$ can be represented as

$$(\downarrow J_H)\mathbf{H}_1(\boldsymbol{\mathcal{A}}) = \begin{bmatrix} \mathbf{H}_{11}(\widetilde{\mathbf{B}}^{\top}) & \mathbf{H}_{10}(\widetilde{\mathbf{B}}^{\top}) \end{bmatrix} \begin{bmatrix} \mathbf{I}_{|L|} & \mathbf{0} \\ \mathbf{0} & \mathbf{B} \end{bmatrix}.$$
 (20)

If G is connected, there is at least one nonzero element at each row (column) in **B**. Therefore, we can always find **P** which satisfies $\mathbf{PB} = \mathbf{I}_{|H|}$. By combining (19) and (20), we completed the proof.

3.2. Synthesis Polyphase

Similar to the analysis bank, we assume that the lowpass and highpass filters in the synthesis bank are the k_0 and k_1 th order polynomials, respectively. They are represented as follows:

$$\mathbf{F}_0(\boldsymbol{\mathcal{A}}) = \sum_{m=0}^{k_0} c_m \boldsymbol{\mathcal{A}}^m$$
(21)

$$\mathbf{F}_1(\boldsymbol{\mathcal{A}}) = \sum_{m=0}^{k_1} d_m \boldsymbol{\mathcal{A}}^m.$$
 (22)

³Hereafter, the index m starts from 0 (m = 0, 1, ...).

Similarly to the above, the upsampling-then-filtering operation of the downsampled signals $\tilde{f}_L \in \mathbb{R}^{|L|}$ and $\tilde{f}_H \in \mathbb{R}^{|H|}$ is written as

$$\widehat{\boldsymbol{f}} = \left[\mathbf{F}_0(\boldsymbol{\mathcal{A}})(\uparrow J_L) \widetilde{\boldsymbol{f}}_L + \mathbf{F}_1(\boldsymbol{\mathcal{A}})(\uparrow J_H) \widetilde{\boldsymbol{f}}_H \right], \qquad (23)$$

and the synthesis bank has the following polyphase structure.

Proposition 2. (Synthesis polyphase structure) For a connected bipartite graph G, the synthesis transform matrix of two-channel critically sampled spectral graph filter banks can always be represented as

$$\begin{bmatrix} \mathbf{F}_{0}(\boldsymbol{\mathcal{A}})(\uparrow J_{L})\widetilde{\boldsymbol{f}}_{L} + \mathbf{F}_{1}(\boldsymbol{\mathcal{A}})(\uparrow J_{H})\widetilde{\boldsymbol{f}}_{H} \end{bmatrix}$$
$$= \begin{bmatrix} \mathbf{0} & \mathbf{I}_{|L|} \\ \mathbf{B}^{\top} & \mathbf{0} \end{bmatrix} \mathbf{F}_{p}(\boldsymbol{\mathcal{A}}) \begin{bmatrix} \widetilde{\boldsymbol{f}}_{L} \\ \widetilde{\boldsymbol{f}}_{H} \end{bmatrix}$$
(24)

where

$$\mathbf{F}_{p}(\boldsymbol{\mathcal{A}}) := \begin{bmatrix} \mathbf{F}_{01}(\widetilde{\mathbf{B}}) & \mathbf{F}_{10}(\widetilde{\mathbf{B}}^{\top}) \\ \mathbf{F}_{00}(\widetilde{\mathbf{B}}) & \mathbf{F}_{11}(\widetilde{\mathbf{B}}^{\top}) \end{bmatrix}$$
(25)

in which

$$\mathbf{F}_{00}(\widetilde{\mathbf{B}}) = \sum_{m} c_{2m}(\widetilde{\mathbf{B}})^{m}$$
$$\mathbf{F}_{01}(\widetilde{\mathbf{B}}) = \sum_{m} c_{2m+1}(\widetilde{\mathbf{B}})^{m}$$
$$\mathbf{F}_{10}(\widetilde{\mathbf{B}}^{\top}) = d_{0}\mathbf{Q} + \sum_{m} d_{2m+2}(\widetilde{\mathbf{B}}^{\top})^{m}\mathbf{B}^{\top}$$
$$\mathbf{F}_{11}(\widetilde{\mathbf{B}}^{\top}) = \sum_{m} d_{2m+1}(\widetilde{\mathbf{B}}^{\top})^{m}\mathbf{B}^{\top},$$
$$(26)$$

and $\mathbf{Q} \in \mathbb{R}^{|L| \times |H|}$ satisfies $\mathbf{B}^{\top} \mathbf{Q} = \mathbf{I}_{|H|}$.

Proof. Similar to Proposition 1, $\mathbf{F}_0(\mathbf{A})$ can be rewritten as

$$\mathbf{F}_{0}(\boldsymbol{\mathcal{A}}) = \sum_{m} c_{2m} \boldsymbol{\mathcal{A}}^{2m} + \sum_{m} c_{2m+1} \boldsymbol{\mathcal{A}}^{2m+1}, \qquad (27)$$

where

$$\sum_{m} c_{2m} \mathcal{A}^{2m} = \operatorname{diag}\left(\sum_{m} c_{2m} \widetilde{\mathbf{B}}^{m}, \sum_{m} c_{2m} (\widetilde{\mathbf{B}}^{\top})^{m}\right) \qquad (28)$$

$$\sum_{m} c_{2m+1} \mathcal{A}^{2m+1}$$

$$= \mathcal{A} \operatorname{diag} \left(\sum_{m} c_{2m+1} \widetilde{\mathbf{B}}^{m}, \sum_{m} c_{2m+1} (\widetilde{\mathbf{B}}^{\top})^{m} \right).$$
(29)

After the upsampling by $(\uparrow J_L)$, the last |H| elements are zero as shown in (2), and thus,

$$\mathbf{F}_{0}(\boldsymbol{\mathcal{A}})(\uparrow J_{L})\widetilde{\boldsymbol{f}}_{L} = \begin{bmatrix} \mathbf{0} & \mathbf{I}_{|L|} \\ \mathbf{B}^{\top} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{F}_{01}(\mathbf{\dot{B}}) \\ \mathbf{F}_{00}(\mathbf{\ddot{B}}) \end{bmatrix} \widetilde{\boldsymbol{f}}_{L}.$$
 (30)

 $\mathbf{F}_1(\mathcal{A})(\uparrow J_H)\mathbf{f}_H$ can also be calculated similarly. If G is connected, we can always find \mathbf{Q} which satisfies $\mathbf{B}^\top \mathbf{Q} = \mathbf{I}_{|H|}$. By combining the above expressions, we completed the proof.



Fig. 4. Equivalent polyphase structure of a two-channel spectral graph filter bank.

3.3. Perfect Reconstruction Condition

As a result, the transformation of the graph signal f by a twochannel critically sampled spectral graph filter bank can be written with the above polyphase matrices as:

$$\widehat{f} = \begin{bmatrix} \mathbf{0} & \mathbf{I}_{|L|} \\ \mathbf{B}^{\top} & \mathbf{0} \end{bmatrix} \mathbf{F}_{p}(\mathcal{A}) \mathbf{H}_{p}(\mathcal{A}) \begin{bmatrix} \mathbf{I}_{|L|} & \mathbf{0} \\ \mathbf{0} & \mathbf{B} \end{bmatrix} f.$$
(31)

It is clear that

$$\begin{bmatrix} \mathbf{0} & \mathbf{I}_{|L|} \\ \mathbf{B}^{\top} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{I}_{|L|} & \mathbf{0} \\ \mathbf{0} & \mathbf{B} \end{bmatrix} = \mathbf{A}.$$
 (32)

Therefore, the perfect reconstruction condition becomes very clear by using the polyphase structure:

$$\mathbf{F}_p(\boldsymbol{\mathcal{A}})\mathbf{H}_p(\boldsymbol{\mathcal{A}}) = \mathbf{I}.$$
(33)

Since the multiplication by \mathcal{A} can be considered as the shift of the graph signal [29], the output signal is a "delayed" graph signal. It can be considered as a counterpart of classical signal processing.

3.4. Efficient Implementation

Thanks to the polyphase structure, we can move the downsampler *before* the analysis filtering, and the upsampler *after* the synthesis filtering. The polyphase form is illustrated in Fig. 4 and it is very easy to parallelize. The polyphase matrices (14) and (25) mean that the *internal* filtering within each set (L or H) is firstly performed, and the results are combined to obtain the transformed coefficients.

4. CONCLUSION AND FUTURE DIRECTIONS

This paper presents a polyphase structure of two-channel critically sampled spectral graph filter banks. Similar to classical signal processing, it can move downsampling and upsampling operators before and after the analysis and synthesis filtering operations, respectively. The number of transformed coefficients in the lowpass and highpass channels are not necessarily to be N/2.

As a future extension, a generalization to the M-channel case and the cascaded two-channel structure will be an interesting step. The non-bipartite case should also be considered. Furthermore, design of spectral graph filter banks with the proposed polyphase structure is a possible approach. For example, a lifting structure, which is a generalization of the classical signal processing counterpart [30, 31], would be interesting.

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