PROBABILISTIC ANALYSIS OF TONE RESERVATION METHOD FOR THE PAPR REDUCTION OF OFDM SYSTEMS

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ABSTRACT

High peak values of transmission signals in wireless communication systems lead to wasteful energy consumption and degradation of several transmission performances. We continue the theoretical contributions made by B. and Farell [1, 2] towards the understanding of peak value reduction, using the strategy known as tone reservation for orthogonal transmission schemes. There it was shown that for OFDM systems, the combinatorial object called arithmetic progression plays an important role in setting limitations for the applicability of the tone reservation method. In this work, we consider ourselves with the performance of the tone reservation in the probabilistic asymptotic setting. We show in particular that for a sufficiently large number N of carriers, choosing each element of that set independently with arbitrary small probability, yields in turn a set of carriers, for which the PAPR reduction problem is not solvable with certain explicitly given threshold constants with probability 1 as N goes to infinity.

Index Terms— Orthogonal Transmission Scheme, OFDM, PAPR, Tone Reservation, Arithmetic Progressions.

1. INTRODUCTION

The rapid development of technologies and the astronomic growth in data usage over the past two decades are inter alia the driving force for the development of flexible, efficient, and reliable transmission technologies for mobile communications networks. The present and future transmission technologies for mobile communications can certainly not be imagined without the development of the so called *Orthogonal Frequency Division Multiplexing (OFDM)*. OFDM constitutes a transmission scheme with which several data can be transmitted instantaneously orthogonally in one single shot within a given time frame by means of the wave functions/carriers having the form of complex sines. It is certainly one of the most promising techniques for achieving high-rate data transmissions due to its high spectral efficiency and inherent robustness toward a multipath channel [3]. Moreover, it has become an important part and a foundation of various current, and future standards, such as DSL, IEEE 802.11, DVB-T, LTE and LTE-advanced/4G [4], and 5G [5, 6].

The high-power amplifier (HPA) in the radio frequency front end of a base station constitute the part of the transmitter which consumes most of the energy put in that system. The amount of energy consumed by the HPA is in particular directly related to the so-called peak-to-average power ratio (PAPR) of the possible input signals. Moreover, in the case that some of the possible transmit signals possess high PAPR value, and the input back off of the used HPA, i.e. the linear range of HPA, is not large enough, such that the magnitude of the peaks are not contained within it, then non-linear distortion of the transmit signals occurs [7]. In particular, this results in the alteration of the transmit waveforms, in the destruction of the desired structure–the orthonormality of the wave functions, and correspondingly in the occurrence of negative effects for several transmission performances (see e.g. [8, 9, 10, 11, 12]). A naive solution to that problem is to use another amplifier with a higher threshold value. However, this is practically not the best solution, since an amplifier with a high input back off is expensive, not only to purchase, but also to maintain. Such HPA would in general have inefficiently high power consumption. For instance, in a typical OFDM device, the transmit energy consumption accounts for only 8% of the total energy consumption at the transmitter, whereas 41% of the energy is wasted by HPA, and the energy consumption of all other circuit devices is about 51% [13, 14]. By the similar argumentation (substitute HPA simply by amplifier!), high PAPR value impacts not only the energy efficiency of the base stations but also of the mobile terminals, which might lead for instance to the reduction the battery lifetime of mobile terminals. Furthermore, the importance of the PAPR reduction in the mobile communications can be seen by the fact that a major company set this topic into one of its research strategies [15].

It is well-known, both theoretically and practically, that the waveforms of orthogonal transmission schemes possess high PAPR value. This undesired behaviour might be caused by the fact that such waveforms are generated by a superposition of large numbers of wave functions. However, the so-called *tone reservation method* [16, 17, 18] is without doubt one of the canonical ways to reduce the PAPR value of a waveform. There, the (indexes of) available wave functions are separated into two (fixed) subsets. The so-called *information set* $\mathcal{I}$ is reserved for those, which carry the information data. Another set consists of those, whose role is to reduce/compensate the peak value of all possible waveforms, s.t. it is below a certain threshold constant $C_{\text{Ex}} > 0$, called *extension constant*. In particular, this should done, if possible, by choosing the coefficients via convex optimization. Further, we refer to the applicability of tone reservation method (for a given set of available wave functions) for the information set $\mathcal{I}$ with the threshold constant $C_{\text{Ex}} > 0$ simply as the *solvability of the PAPR problem for $\mathcal{I}$ with extension constant C_{Ex}*. Certainly, there are other methods to reduce the peak value [19, 20]. However, tone reservation is canonical and robust, in the sense that the only information required on the receiver's side is the indexes of the information-bearing coefficients. The auxiliary coefficients may simply be ignored by the receiver. Therefore, there is no need for additional overhead in the transmission symbols. Besides, the compensation set can also be used for channel estimation purposes [21]. For more comprehensive and further discussions on those issues, we recommend the overview article [22], which gives for instance a discussion on alternative metrics beyond the PAPR, which are relevant to the behaviour of the energy consumption of an transmission system, and new mathematical concepts aiming to overcome the high PAPR behavior of OFDM. We mention that to date there exists no technique, which chooses, for a given extension constant and for a given information set, a possibly small compensation set.

In [1, 2], it was shown that the existence of the so-called arithmetic progressions in the information set gives a limitation to the solvability of PAPR reduction problem for OFDM. To be more specific, the existence of a long arithmetic progression implies the non-solvability of the PAPR reduction problem with small extension constant. The problem of finding an arithmetic progression in a given set of numbers has been a long lasting problem in mathematics. A great achievement was made by Szemerédi [23], who shows that if the considered set is large enough, the existence of an arithmetic progression of a given length in subsets of the considered set of numbers of a given density is ensured. However, Szemerédi Theorem is unsatisfactory for the asymptotic case, since it merely ensures the existence of *arithmetic progressions of arbitrary length* for subsets of $\mathbb{N}$ with *positive upper density*. A tightening of this statement is due to Green and Tao [24]. They showed the existence of a subset of natural numbers possessing the density 0 in $\mathbb{N}$, containing arithmetic progressions of arbitrary length.

Up to date, no deterministic constructions of information set $\mathcal{I} \subset [N]$, whose size is small in contrasts to the set $[N]$ of available tones (even asymptotically) and which perform badly, in the sense that the PAPR reduction problem is not solvable for small extension constants, are known. In several fields, such as information theory and compressed sensing, probabilistic methods have been applied to construct relevant objects, such as codes [25] and measurement matrices [26]. In that spirit, we continue the contributions made in [1, 2] by giving an application of Conlon-Gower's recent result [27, 28] on arithmetic progressions for the PAPR reduction problem in the probabilistic asymptotic setting. For a sufficiently large number N of carriers, it will be shown, that by choosing each element of that set of carriers independently with arbitrary small probability, there exists a set of sub-carriers $\mathcal{I}$, for which the PAPR reduction problem is not solvable for all subsets of $\mathcal{I}$ with cardinality at least $\delta\,|\mathcal{I}|$, where $\delta > 0$ is given, with certain threshold constants, and with probability 1 as N goes to infinity. Furthermore, the corresponding threshold constants is also given explicitly.

2. BASIC NOTIONS

Let $\mathcal{K}$ be a discrete set. $\ell^2(\mathcal{K})$ denotes the space of square-summable sequences indexed by $\mathcal{K}$, equipped with the canonical norm. If it is clear from the context, we write $\ell^2(\mathcal{K})$ simply by ℓ^2. For $p \in [1,\infty)$, the space of p-integrable functions on the interval $[0,1]$ (resp. $[0,2\pi]$) is denoted by $L^p([0,1])$ (resp. $L^p([0,2\pi])$). The space of essentially bounded functions on $[0,1]$ (resp. $[0,2\pi]$) is denoted by $L^\infty([0,1])$ (resp. $L^\infty([0,2\pi])$). For $p \in [1,\infty]$, in case that there is no danger of confusions, we denote $L^p([0,1])$ and $L^p([0,2\pi])$ simply by L^p. L^p is equipped with the usual (normalized) norm.

Given a duration $T_s > 0$ (w.l.o.g. $T_s = 1$) of a transmit signal, and given a transmit data $\{a_k\}_{k\in\mathcal{K}}$, where $\mathcal{K} \subset \mathbb{N}$, which in our case constitute simply a sequence in $\mathbb{C}$. The transmit signal of an orthogonal transmission scheme has the form $s(t) = \sum_{k\in\mathcal{K}} a_k\phi_k(t)$, where the collection $\{\phi_n\}_{k\in\mathcal{K}}$ of *wave functions* or *carriers* constitutes an *orthonormal system* (ONS) in the space of square integrable functions on $[0,1]$. The subspace of integrable functions representable as linear combinations of $\{\phi_n\}_{k\in\mathcal{K}}$ is denoted by $\mathfrak{F}^1(\mathcal{K})$. We consider in this work mainly OFDM, where $\{\phi_n\}_{k\in\mathcal{K}}$ is simply a subcollection of the sine functions $\{e^{i2\pi(n-1)(\cdot)}\}_{n\in\mathbb{N}}$, where $e_n(\cdot) := e^{i2\pi(k-1)(\cdot)}$.

For $N \in \mathbb{N}$, we denote the *(Nth) Dirichlet kernel* by $D_N(t) = \sum_{k=-N}^{N} e^{ikt}$, $t \in \mathbb{R}$. The Dirichlet kernel is 2π-periodic, and an even function. By elementary computations, it can also be written as:

$$D_N(t) = \frac{\sin\left(\left[\frac{2N+1}{2}\right]t\right)}{\sin\left(\frac{t}{2}\right)}. \qquad (1)$$

From both representation of D_N, it is clear that $|D_N(t)| \leqslant 2N + 1, \quad \forall t \in [0,1]$.

The following mathematical objects constitute central object in our study:

Definition 1 (Arithmetic Progression, (δ, m)-Szemerédi Set): *An arithmetic progression of length $m \in \mathbb{N}$ is defined as a subset of $\mathbb{N}$ of the form $\{a, a+d, a+2d, \ldots, a+(m-1)d\}$, with $a \in \mathbb{Z}$ and $d \in \mathbb{N}$. Given $\delta > 0$ and $m \in \mathbb{N}$. A set $\mathcal{I} \subset \mathbb{N}$ is called (δ, m)-Szemerédi set if every subset of $\mathcal{I}$ of cardinality at least $\delta\,|\mathcal{I}|$ contains an arithmetic progression of length m.*

3. MAIN RESULTS

3.1. PAPR Reduction Problem and Its Equivalent Formulation

In general, the PAPR of a function is defined as follows: Given an orthonormal system $\{\phi_n\}_{n\in\mathbb{N}}$, and index set $\mathcal{K} \subset \mathbb{N}$. For a waveform f of an orthogonal transmission scheme the PAPR of f (which depends only on the transmission data $\mathbf{a} \in \ell^2(\mathcal{K})$, and the used orthonormal system) can simply be given by

$$\mathrm{PAPR}(\{\phi_k\}_{k\in\mathcal{K}}, \mathbf{a}) = \operatorname*{ess\,sup}_{t\in[0,1]} \frac{\left|\sum_{k\in\mathcal{K}} a_k\phi_k(t)\right|}{\|\mathbf{a}\|_{\ell^2(\mathcal{K})}}.$$

Notice that our version of PAPR differs slightly with that given in some literature by a square. One can show [29, 2], that:

$$\sqrt{N} \leqslant \sup_{\|\mathbf{a}\|_{\ell^2} \leqslant 1} \mathrm{PAPR}(\{\phi_k\}_{k\in[N]}, \mathbf{a}).$$

Moreover, a sequence $\mathbf{a}$, with $\|\mathbf{a}\|_{\ell^2} = 1$, for which above inequality holds, can easily be constructed. As already mentioned in the introduction, the high dynamical behaviour has in particular a negative impact to the reliability, the cost, and energy efficiency of an orthogonal transmission scheme [1, 2, 30].

A canonical strategy to reduce the peak value of a waveform generated by orthonormal functions is the so called tone reservation [16, 17, 18], which is formalized as follows:

Definition 2 (PAPR Reduction Problem): *Let $\{\phi_n\}_{n\in\mathbb{N}}$ be an orthonormal system in $L^2([0,1])$, and $\mathcal{I} \subset \mathbb{N}$. We say the PAPR reduction problem is solvable for the pair $(\{\phi_n\}_{n\in\mathbb{N}}, \mathcal{I})$ with constant $C_{Ex} > 0$, if for every $\mathbf{a} \in l^2(\mathcal{I})$, there exists $\mathbf{b} \in l^2(\mathcal{I}^c)$, satisfying $\|\mathbf{b}\|_{\ell^2(\mathcal{I}^c)} \leqslant C_{Ex}\,\|\mathbf{a}\|_{\ell^2(\mathcal{I})}$, for which the following holds:*

$$\operatorname*{ess\,sup}_{t\in[0,1]} \left|\sum_{k\in\mathcal{I}} a_k\phi_k(t) + \sum_{k\in\mathcal{I}^c} b_k\phi_k(t)\right| \leqslant C_{Ex}\,\|\mathbf{a}\|_{\ell^2(\mathcal{I})} \qquad (2)$$

We further refer $\mathcal{I}$ as *information set*, $\mathcal{I}^c$ as *compensation set*, $\{\phi_n\}_{n\in\mathcal{I}}$ as *information tones*, and respectively $\{\phi_n\}_{n\in\mathcal{I}^c}$ as *compensation tones*. A necessary condition for the solvability of the PAPR reduction problem is surely, that $\{\phi_n\}_{n\in\mathcal{I}}$ is uniformly bounded, in the sense that $\phi_n \in L^\infty([0,1])$, and $\|\phi_n\|_{L^\infty([0,1])} \leqslant C$ for all $n \in \mathcal{I}$, with $C > 0$ a certain constant. To avoid any further undesirable behaviour, we assume that all of the considered orthonormal systems are uniformly bounded. The condition $\|\mathbf{b}\|_{\ell^2(\mathcal{I}^c)} \leqslant C_{\text{Ex}} \|\mathbf{a}\|_{\ell^2(\mathcal{I})}$ serves in some sense as a restriction of the possible solutions of the PAPR reduction problem [1]. Notice also, that we allow infinitely many carriers for the compensation of the PAPR value. This is of practical interests, since the solvability of the PAPR reduction problem in the setting, where the corresponding compensation set is infinite, is a necessary condition for the solvability of the PAPR reduction problem in the setting, where the available compensation tones are of finite number. In particular, limitations for the tone reservation method in the setting, where the available compensation tones are of infinite number, are also limitations for the tone reservation method in the setting, where the available compensation tones are of finite number. This is suited for our aim to investigate information sets for which the PAPR reduction is not solvable with a given extension constant.

It was shown in [1], that the PAPR reduction problem is connected to the embedding problem of $\mathfrak{F}^1(\mathcal{I})$ seen as a subspace of $L^1([0,1])$ into $L^2([0,1])$:

Theorem 1 (Boche and Farell [1]): *Let $\{\phi_n\}_{n\in\mathbb{N}}$ be a Complete ONS (CONS) in $L^2([0,1])$. Given a subset $\mathcal{I} \subset \mathbb{N}$ and a constant $C_{Ex} > 0$. The PAPR reduction problem is solvable for $(\{\phi_n\}_{n\in\mathbb{N}}, \mathcal{I})$ with extension constant C_{Ex} if and only if the following holds:*

$$\|f\|_{L^2([0,1])} \leqslant C_{Ex} \|f\|_{L^1([0,1])}, \quad \forall f \in \mathfrak{F}^1(\mathcal{I}). \tag{3}$$

Given a fixed $C_{\text{Ex}} > 0$. Above Theorem asserts that to show that the PAPR reduction problem is not solvable for $(\{\phi_n\}_{n\in\mathbb{N}}, \mathcal{I})$, where $\mathcal{I}$ is a given information set, with extension constant C_{Ex}, it is enough to find a function $f \in \mathfrak{F}^1(\mathcal{I})$, for which the embedding equation (3) does not hold. Such a function can be constructed by means of an arithmetic progression within the information set:

Lemma 2: *Let be $\mathcal{I} \subset \mathbb{N}$. Assume that there exists an arithmetic progression of length $m \geqslant 2$ in $\mathcal{I}$. Then, if the PAPR reduction problem is solvable for $(\{e_n\}_{n\in\mathbb{N}}, \mathcal{I})$ with a given $C_{Ex} > 0$, it follows*

$$C_{Ex} > \frac{\sqrt{m}}{\left\|\sum_{k=1}^m e_k\right\|_{L^1([0,1])}}$$

.

Proof: Consider the signal $f(t) = \sum_{k=1}^m \frac{1}{\sqrt{m}} e^{i2\pi(a+dk)(t)}$. It is obvious, that $f \in \mathfrak{F}^1(\mathcal{I})$. Furthermore, we have:

$$\|f\|_{L^1([0,1])} = \left\|\sum_{k=1}^m \tfrac{1}{\sqrt{m}} e^{i2\pi(a+dk)(\cdot)}\right\|_{L^1} = \left\|\sum_{k=1}^m \tfrac{1}{\sqrt{m}} e^{i2\pi dk(\cdot)}\right\|_{L^1}$$

By substituting the variable of the integral, and by noticing that $\sum_{k=1}^m \frac{1}{\sqrt{m}} e^{i2\pi k(\cdot)}$ is 1-periodic and that $e^{i2\pi t}$ is of modulus 1, we have:

$$\|f\|_{L^1} = \left\|\sum_{k=1}^m \tfrac{1}{\sqrt{m}} e^{i2\pi dk(\cdot)}\right\|_{L^1} = \tfrac{1}{\sqrt{m}} \left\|\sum_{k=1}^m e_k\right\|_{L^1}.$$

It is not hard to see that $\|f\|_{L^2([0,1])} = 1$. Finally, by the assumption that PAPR reduction problem is solvable for $(\{e_n\}_{n\in\mathbb{N}}, \mathcal{I})$ with constant C_{Ex}, previous observation and Theorem 1, and the fact $f \in \mathfrak{F}^1(\mathcal{I})$, we have $1 = \|f\|_{L^2([0,1])} \leqslant C_{\text{Ex}} \|f\|_{L^1([0,1])} \leqslant C_{\text{Ex}} \left\|\sum_{k=1}^m e_k\right\|_{L^1([0,1])} / \sqrt{m}$, as desired. ∎

Above Lemma asserts that in case the existence of an arithmetic progression of a given length in an information set can be ensured, the limitation of the PAPR reduction problem can be found out by giving an upper bound of the L^1-norm of $\sum_{k=1}^m e_k$.

3.2. Bound for the Dirichlet Kernel and its Application to the PAPR Reduction Problem

As we soon see, the desired upper estimate of the L^1-norm of $\sum_{k=1}^m e_k$ given later relies in turn on the upper estimate of the L^1-norm of the Dirichlet kernel. It is known in the literature [31, 32], that the L^1-norm of the Dirichlet kernel D_N is asymptotically like $(4/(\pi^2))\log(N)$. However, a self-contained proof is hard to find. By this reason, we give in the following a quantitative proof that D_N is upper bounded by $(4/(\pi^2))\log(N)$ up to an universal constant:

Theorem 3: *For every $N \in \mathbb{N}$, it holds:*

$$\|D_N\|_{L_1([0,2\pi])} = \tfrac{1}{2\pi}\int_0^{2\pi} |D_N(t)|\,\mathrm{d}t < \tfrac{4}{\pi^2}\log(N) + C,$$

where $C > 0$ is a universal constant given by:

$$C = 3 + \tfrac{2}{24-\pi^2} + \tfrac{4}{\pi^2} \tag{4}$$

Proof: By the 2π-periodicity of D_N and by the fact that D_N is an even function, it follows that $\int_0^{2\pi} |D_N(t)|\,\mathrm{d}t = \int_{-\pi}^{\pi} |D_N(t)|\,\mathrm{d}t = 2\int_0^{\pi} |D_N(t)|\,\mathrm{d}t$. Thus to give a bound for $\|D_N\|_{L^1([0,2\pi])}$, it is sufficient to give a bound for $\int_0^{\pi} |D_N(t)|\,\mathrm{d}t$. Using the representation (1), it holds:

$$\begin{aligned}\int_0^{\pi} |D_N(t)|\,\mathrm{d}t = &\left(\sum_{k=1}^N \int_{(k-1)\frac{2\pi}{2N+1}}^{k\frac{2\pi}{2N+1}} \frac{\left|\sin\left(\left[\frac{2N+1}{2}\right]t\right)\right|}{\left|\sin\left(\frac{t}{2}\right)\right|}\mathrm{d}t\right)\\ &+ \int_{\frac{2N\pi}{2N+1}}^{\pi} \frac{\left|\sin\left(\left[\frac{2N+1}{2}\right]t\right)\right|}{\left|\sin\left(\frac{t}{2}\right)\right|}\mathrm{d}t.\end{aligned} \tag{5}$$

Notice that the first integral can be bounded by 2π and the last integral can be bounded by π. Now we aim to bound the remaining $N-1$ integrals on the right hand side of (5). Let be $k \in [N-1]$. For $t \in [2\pi(k-1)/(2N+1), 2\pi k/(2N+1)]$, it follows by the fact that sin is monotonically increasing on $[0,\pi/2]$, and since it is 2π-periodic:

$$\tfrac{1}{\sin\left(\frac{t}{2}\right)} \leqslant \tfrac{1}{\sin\left(\frac{(k-1)\pi}{2N+1}\right)}. \tag{6}$$

Now for $\tilde{t} > 0$, it holds $\tilde{t} - \frac{\tilde{t}^3}{3!} \leqslant \sin(\tilde{t})$. Thus we have for for $\tilde{t} \leqslant \pi/2$, $\frac{1}{\sin(\tilde{t})} \leqslant \frac{1}{\tilde{t}} + \frac{4\tilde{t}}{24-\pi^2}$. By applying this inequality to (6), we obtain $\frac{1}{\sin\left(\frac{t}{2}\right)} \leqslant \frac{2N+1}{(k-1)\pi} + C_1\frac{(k-1)\pi}{2N+1}$, where $C_1 := 4/(24-\pi^2)$ is a universal positive constant. Therefore, we have:

$$\int_{(k-1)\frac{2\pi}{2N+1}}^{k\frac{2\pi}{2N+1}} \frac{\left|\sin\left(\left[\frac{2N+1}{2}\right]t\right)\right|}{\left|\sin\left(\frac{t}{2}\right)\right|}\mathrm{d}t = \tfrac{4}{(k-1)\pi} + C_1\tfrac{4(k-1)\pi}{(2N+1)^2}, \tag{7}$$

since $\int_{(k-1)\frac{2\pi}{2N+1}}^{k\frac{2\pi}{2N+1}} \left|\sin\left(\left[\frac{2N+1}{2}\right]t\right)\right|\mathrm{d}t = 4/(2N+1)$.

By summing (7) over all $k \in \{2,\dots,N\}$, (5), and by the given bound of the first and the last integral of (5), $\|D_N\|_{L_1([0,2\pi])}$ can be bounded as follows:

$$\pi\|D_N\|_{L_1([0,2\pi])} \leqslant 3\pi + \sum_{k=2}^N \left(\tfrac{4}{(k-1)\pi} + C_1\tfrac{4(k-1)\pi}{(2N+1)^2}\right). \tag{8}$$

Furthermore, it is elementary to give:

$$\sum_{k=2}^N \tfrac{4}{(k-1)\pi} < \tfrac{4}{\pi}(\log(N)+1) \text{ and } \sum_{k=2}^N C_1\tfrac{4(k-1)\pi}{(2N+1)^2} < \tfrac{C_1\pi}{2}. \tag{9}$$

The first relation follows from the usual bound of the partial sum of harmonic series, and the second relation follows by the equality $\sum_{k=1}^{N}(k-1) = N(N-1)/2$. Combining (8) and (9), the desired result yields. ■

By more easier computations, one can obtain $\|D_N\|_{L_1([0,2\pi])} > (4/(\pi^2))\log(N)$. Thus, the upper bound given in above Theorem is in some sense tight. Having established an upper bound of the L^1-norm of the Dirichlet kernel, it is immediate to give an estimate of the L^1-norm of $\sum_{k=1}^{m} e_k$, since it can be seen as a "truncated" Dirichlet kernel:

Corollary 4: *Let be $m \geqslant 2$. It holds $\left\|\sum_{k=1}^{m} e_k\right\|_{L^1([0,1])} < \frac{4}{\pi^2}\log\left(\frac{m}{2}\right) + C_*$, where $C_* > 0$ is a universal constant given by:*

$$C_* = 4 + \tfrac{2}{24-\pi^2} + \tfrac{4}{\pi^2} \tag{10}$$

Proof: It holds:

$$\sum_{k=1}^{m} e_k(t) = \sum_{k=0}^{m-1} e^{i2\pi kt} = e^{i2\pi\left\lceil\frac{m-1}{2}\right\rceil t}\sum_{k=-\left\lceil\frac{m-1}{2}\right\rceil}^{k=\left\lceil\frac{m-1}{2}\right\rceil} e^{i2\pi kt} - \mathrm{Res}_m(t),$$

where $\mathrm{Res}_m(t) = 0$, if m is an odd number, and $\mathrm{Res}_m(t) = e^{i2\pi mt}$, if m is an even number. By triangle inequality, and by the fact that $e^{i\left\lceil\frac{m-1}{2}\right\rceil t}$ has modulus 1, and by (3) and the substitution of the variable of the integral, we have $\left\|\sum_{k=0}^{m-1} e^{ik(\cdot)}\right\|_{L^1([0,1])} < \frac{4}{\pi^2}\log\left(\left\lceil\frac{m-1}{2}\right\rceil\right) + C + \|\mathrm{Res}_m\|_{L^1([0,1])}$, where $C > 0$ is a universal constant given by (4). Clearly, $\|\mathrm{Res}_m\|_{L^1([0,1])}$ can be upper bounded by 1. Thus, since log is monotonically increasing, we have $\left\|\sum_{k=0}^{m-1} e^{i2\pi k(\cdot)}\right\|_{L^1([0,1])} < \frac{4}{\pi^2}\log\left(\frac{m}{2}\right) + C + 1$, as desired. ■

Finally, the estimate of the L^1-norm of $\sum_{k=1}^{m} e_k$ given in corollary 4 allows us to specify Lemma 2 as follows:

Lemma 5: *Let be $\mathcal{I} \subset \mathbb{N}$. Assume that there exists an arithmetic progression of length $m \geqslant 2$ in $\mathcal{I}$. Then, if the PAPR reduction problem is solvable for $(\{e_n\}_{n\in\mathbb{N}}, \mathcal{I})$ with a given $C_{Ex} > 0$, it follows:*

$$C_{Ex} > \frac{\sqrt{m}}{\frac{4}{\pi^2}\log\left(\frac{m}{2}\right) + C_*}, \tag{11}$$

for a universal constant $C_* > 0$ given by (10).

3.3. Asymptotic Probabilistic Result for the Solvability of PAPR Reduction Problem

In case that the existence of an arithmetic progression of length $m \in \mathbb{N}$ can be ensured, Lemma 5 gives the limitation of the tone reservation method, in the sense that the admissible extension constant can not be smaller than $\sqrt{m}/(\frac{4}{\pi^2}\log\left(\frac{m}{2}\right) + C_*)$, where C_* is a universal constant, which can be given explicitly. One can apply Szemerédi [23] and Green-Tao's [24] result to check whether an arithmetic progression of length $m \in \mathbb{N}$ exists in a given information set, as already mentioned in the introduction.

As already announced in the last paragraph in the introduction, we aim to construct probabilistically an information set, which is (asymptotically) small relative to the set of available tones and for which the tone reservation method performs badly. In doing that, we use the following recent result on the existence of arithmetic progressions, more specifically (δ, m)-Szemerédi set, for the asymptotic probabilistic setting due to Conlon and Gowers:

Theorem 6 (Conlon, Gowers [27]): *Given $\delta > 0$, and a natural number $m \in \mathbb{N}$. There exists a constant $C > 0$, s.t.:*

$$\lim_{N\to\infty} \mathbb{P}([N]_p \text{ is } (\delta, m)\text{-Szemerédi}) = 1, \quad \text{if } p > CN^{\frac{-1}{(m-1)}}.$$

Furthermore, an overview related to above result is given in [28]. The corresponding application to the PAPR reduction problem can finally be given:

Theorem 7: *Let be $m \in \mathbb{N}$, and $\delta \in (0,1)$. Given a constant $C_{Ex} > 0$. Then, there is a constant C, s.t.:*

$$\lim_{N\to\infty} \mathbb{P}\left(A_{N,m,p}\right) = 1, \quad \text{if } p > \frac{C}{N^{\frac{1}{m-1}}},$$

where $A_{N,m,p}$ denotes the event: "The PAPR problem is not solvable for $(\{e_n\}_{n\in\mathbb{N}}, \mathcal{I})$ with

$$C_{Ex} \leqslant \frac{\sqrt{m}}{\frac{4}{\pi^2}\log\left(\frac{m}{2}\right) + C_*}, \tag{12}$$

where $C_* > 0$ is an absolute constant given by (10)*, for every subset $\mathcal{I} \subset [N]_p$ of size $|\mathcal{I}| \geqslant \delta N$ "*

Proof: Choose m sufficiently large, s.t. (11) does not hold. Thm. 6 asserts the existence of a constant $C > 0$, such that the set $[N]_p$ resulted by choosing elements of $[N]$ independently by probability $p > C/(N^{\frac{1}{m-1}})$, is a (δ, m)-Szemerédi with probability tends to 1 as N tends to infinity. By the definition of (δ, m)-Szemerédi set and Lemma 5, the result follows immediately. ■

In more convenient words, we have shown that by probabilistic construction (i.i.d.), where the selection probability is not too small (dependending to the size of the available tones), the resulted information set is asymptotically almost surely behaves badly with respect to tone reservation method.

4. DISCUSSION - OUTLOOK - RELATION TO PRIOR WORKS

In this paper, we have studied the limitations of the tone reservation method in the probabilistic asymptotic setting. We show in particular that for a sufficiently large number N of carriers, choosing each element of that set independently with arbitrary small probability, yields in turn a set of carriers, for which the PAPR reduction problem is not solvable asymptotically almost surely with certain given threshold constants, which can explicitly be given.

Furthermore, the statement given in this work is stronger than aforementioned, since the previous mentioned statement holds not only for the mentioned probabilistic set of carriers $\mathcal{I}$ but also for all subsets of $\mathcal{I}$ having cardinality at least $\delta|\mathcal{I}|$. The corresponding proof is based on the results given in [1, 2] on the connection between the existence of an arithmetic progression in an information set and the solvability of the PAPR reduction problem, and the recent result given in [27] on probabilistic "sparse" Széméredi set. However, we have explicitly specified in aforementioned case the extensions constant for which the PAPR reduction is not solvable. The corresponding approach is based on the upper bound of the L^1-norm of the Dirichlet kernel. Furthermore, until now, there is no approach to analyze the behaviour of the tone reservation method in the asymptotic probabilistic setting.

The results given in this work asserts that the tone reservation method behaves badly for OFDM systems with large/massive number of carriers, since even by choosing a small sets (relative to the whole available carriers) active carriers randomly, the tone reservation method is not applicable for certain threshold constants.

5. REFERENCES

[1] H. Boche and B. Farell, “PAPR and the density of information bearing signals in OFDM,” *EURASIP J. Adv. Sig. Proc.*, vol. 2011, no. 1, pp. 1–9, 2011.

[2] ——, “On the peak-to average power ratio reduction problem for orthogonal transmission schemes,” *Internet Mathematics*, vol. 9, no. 2-3, pp. 265–296, 2013.

[3] L. J. Cimini, “Performance studies for high-speed indoor wireless communications,” *Wireless Personal Communications*, vol. 2, no. 1, pp. 67–85, 1995.

[4] A. Ghosh, R. Ratasuk, B. Mondal, N. Mangalvedhe, and T. Thomas, “LTE-advanced: next-generation wireless broadband technology,” *IEEE Wireless Communications*, vol. 17, no. 3, pp. 10–22, 2010.

[5] G. Wunder, P. Jung, M. Kasparick, T. Wild, F. Schaich, Y. Chen, S. T. Brink, I. Gaspar, N. Michailow, A. Festag, L. Mendes, N. Cassiau, D. Ktenas, M. Dryjanski, S. Pietrzyk, B. Eged, P. Vago, and F. Wiedmann, “5GNOW: non-orthogonal, asynchronous waveforms for future mobile applications,” *IEEE Communications Magazine*, vol. 52, no. 2, pp. 97–105, 2014.

[6] B. Farhang-Boroujeny and H. Moradi, “Derivation of GFDM based on OFDM principles,” in *2015 IEEE International Conference on Communications (ICC)*, 2015, pp. 2680–2685.

[7] F. H. Raab, P. Asbeck, S. Cripps, P. B. Kenington, Z. B. Popovic, N. Pothecary, J. F. Sevic, and N. O. Sokal, “Power amplifiers and transmitters for RF and microwave,” *IEEE Transactions on Microwave Theory and Techniques*, vol. 50, no. 3, pp. 814–826, Mar. 2002.

[8] A. Chini, Y. Wu, M. El-Tanany, and S. Mahmoud, “Hardware nonlinearities in digital tv broadcasting using OFDM modulation,” *IEEE Transactions on Broadcasting*, vol. 44, no. 1, pp. 12–21, Mar. 1998.

[9] D. Dardari, V. Tralli, and A. Vaccari, “A theoretical characterization of nonlinear distortion effects in OFDM systems,” *IEEE Transactions on Communications*, vol. 48, no. 10, pp. 1755–1764, 2000.

[10] S. W. Chen, W. Panton, and R. Gilmore, “Effects of nonlinear distortion on CDMA communication systems,” *IEEE Transactions on Microwave Theory and Techniques*, vol. 44, no. 12, pp. 2743–2750, Dec. 1996.

[11] C. van den Bos, M. H. L. Ksuwenhoven, and W. A. Serdijn, “Effect of smooth nonlinear distortion on OFDM symbol error rate,” *IEEE Transactions on Communications*, vol. 49, no. 9, pp. 1510–1514, Sep. 2001.

[12] X. Li and L. J. Cimini, “Effects of clipping and filtering on the performance of OFDM,” in *Vehicular Technology Conference, 1997, IEEE 47th*, vol. 3, 1997, pp. 1634–1638.

[13] T. Jiang, W. Xiang, H. H. Chen, and Q. Ni, “Multicast broadcast services support in OFDMA-based WiMAX systems,” *IEEE Communications Magazine*, vol. 45, no. 8, pp. 78–86, 2007.

[14] B. Bougard, “Cross-layer energy management in broadband wireless transceivers,” Ph.D. dissertation, Katholieke Univ. Leuven, Louvain, Belgium, Mar. 2006.

[15] H. Boche, “Future research challenges for mobile networks,” in *Next Generation Mobile Networks: (R)evolution in Mobile Communications*, Technology Radar Edition III/2010. Deutsche Telekom AG Laboratories, 2010, feature Paper.

[16] J. Tellado, “Peak to average power reduction for multicarrier modulation,” Ph.D. dissertation, Stanford University, 1999.

[17] J. Tellado and J. M. Cioffi, “Peak to average power ratio reduction,” U.S.A Patent 09/062, 867, Apr. 20, 1998.

[18] ——, “Efficient algorithms for reducing PAR in multicarrier systems,” in *Information Theory, 1998. Proceedings. 1998 IEEE International Symposium on*, 1998.

[19] S. Han and J. Lee, “An overview of peak-to-average power ratio reduction techniques for multicarrier transmission,” *IEEE Transactions on Wireless Communications*, vol. 12, no. 2, pp. 56–65, Apr. 2005.

[20] S. Litsyn, *Peak Power Control in Multicarrier Communications*. Cambridge University Press, 2007.

[21] P. Walk, H. Becker, and P. Jung, “OFDM channel estimation via phase retrieval,” in *2015 49th Asilomar Conference on Signals, Systems and Computers*, 2015, pp. 1161–1168.

[22] G. Wunder, R. F. H. Fischer, H. Boche, S. Litsyn, and J. S. No, “The PAPR problem in OFDM transmission: New directions for a long-lasting problem,” *IEEE Signal Processing Magazine*, vol. 30, no. 6, pp. 130–144, Nov. 2013.

[23] E. Szemerédi, “On sets of integers containing no k elements in arithmetic progression,” *Acta Arith.*, vol. 27, pp. 199–245, 1975.

[24] B. Green and T. Tao, “The primes contain arbitrarily long arithmetic progressions.” *Annals of Mathematics*, vol. 167, pp. 481–547, 2008.

[25] I. Csiszár and J. Körner, *Information Theory*, 2nd ed. Cambridge University Press, Dec. 2015.

[26] S. Foucart and H. Rauhut, *A Mathematical Introduction to Compressive Sensing*. Birkhäuser Basel, 2013.

[27] D. Conlon and W. T. Gowers, “Combinatorial theorems in sparse random sets,” *Ann. of Math*, vol. 184, no. 2, pp. 367–454, 2016.

[28] D. Conlon, “Combinatorial theorems relative to a random set,” in *Proc. of the Int. Cong. of Math.*, vol. 3, 2014, pp. 303–328.

[29] H. Boche and V. Pohl, “Signal Representation and Approximation - Fundamental Limits.” *European Trans. Telecomm. (ETT)*, vol. 19, no. 5, pp. 445–456, 2007.

[30] H. Boche and E. Tampubolon, “Mathematics of signal design for communication systems,” in *Mathematics and Society*, W. König, Ed. European Mathematical Society Publishing House, 2016, pp. 185–220.

[31] A. Zygmund, *Trigonometric Series*, 3rd ed. Cambridge University Press, 1935, vol. 1.

[32] Y. Katznelson, *An Introduction to Harmonic Analysis*, 3rd ed. Cambridge University Press, 2004.