JOINT ALPHA-FAIRNESS BASED DSM AND USER ENCODING ORDERING FOR ZERO-FORCING NONLINEAR PRECODING IN G.FAST DOWNSTREAM TRANSMISSION

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ABSTRACT

In the G.fast frequency range with strong levels of crosstalk, nonlinear precoding (NLP) is proposed as a near-optimal technique for crosstalk precompensation in downstream transmission. While existing methods for multi-tone NLP user encoding ordering (UEO) are rather heuristic in how they approach fairness and suffer from substantial suboptimality, we develop a novel algorithm for joint dynamic spectrum management (DSM) and UEO that enforces a generalized alpha-fairness policy. Since finding the optimal UEO is a combinatorial optimization problem with excessive computational complexity, the proposed algorithm uses a low-complexity iterative method which provides near-optimal approximate solutions. Simulations demonstrate that the novel algorithm achieves a trade-off between fairness and performance that outperforms current UEO methods.

Index Terms— DSM, NLP, User encoding ordering

1. INTRODUCTION

The latest generation digital subscriber lines (DSL) access technology approved by the International Telecommunication Union (ITU) is called G.fast [1]. G.fast offers "fiber-like" (over 1 Gb/s) transmission speeds for very short copper telephony lines (below 100 m), by using a broad spectrum up to 212 MHz. Notwithstanding its tremendous improvement, the broad spectrum also makes crosstalk cancellation very challenging in G.fast, especially at high frequencies where strong levels of inter-user crosstalk interference are encountered.

Foremost among the challenges is that the traditional linear zero forcing (ZF) precoder [2] for crosstalk precompensation in downstream transmission suffers from a reduced signal-to-noise ratio (SNR) due to large transmit power penalties. No longer near-optimal for G.fast, the linear ZF precoder starts to get outperformed by the linear minimum mean squared error (MMSE) precoder which tolerates some residual crosstalk at the benefit of an improved SNR [3].

An alternative for improving the SNR is nonlinear precoding (NLP) which can be combined with either the ZF or the MMSE criterion. NLP sequentially encodes the user transmit signals in order to "pre-subtract" the crosstalk from previously encoded users without transmit power penalties. The NLP performance is greatly affected by the per-tone user encoding ordering (UEO), since users encoded first typically achieve higher rate gains than users encoded last. In this paper, we pragmatically focus on ZF-NLP which allows for easier expressions compared to theoretically optimal MMSE-NLP, against only a very small performance loss [3].

However, the algorithms available for multi-tone NLP UEO are rather heuristic in how they approach fairness and suffer from substantial suboptimality. The most well-known solution is the V-BLAST method [4] which maximizes the minimum SNR at each tone separately. Recently, a new efficient method called dynamic ordering (DO) has been proposed in [5], which is shown in simulations to provide a higher minimum total data rate than V-BLAST in G.fast.

In this paper, we develop a novel algorithm for joint dynamic spectrum management (DSM) and UEO for ZF-NLP in G.fast. Overcoming the heuristic fairness approach, the algorithm enforces the generalized alpha (α)-fairness policy (as widely used in wireless scenarios [6]) and will be referred to as α-fair per-tone exhaustive search (α-fair PTES). Unfortunately, finding the optimal UEO entails a combinatorial problem with an exponential complexity in the number of tones and users, which makes the considered optimization problem numerically infeasible. Therefore, α-fair PTES resorts to a low-complexity iterative method which provides near-optimal approximate solutions. Simulations demonstrate that the novel algorithm achieves a trade-off between fairness and performance that outperforms current UEO methods.

2. PROBLEM STATEMENT

2.1. System Model

We consider NLP based downstream transmission in a G.fast DSL cable binder consisting of N interfering users and K

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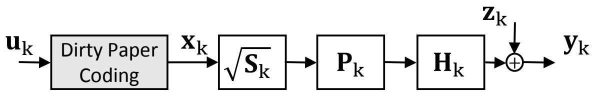

Fig. 1: NLP model for downstream transmission on tone k.

tones or frequency sub-carriers (see Fig. 1 for a block diagram). Assuming the standard synchronous discrete-multi-tone (DMT) modulation, the linear part of transmission can be modeled independently on each tone $k = [1, \cdots, K]$ as

$$\mathbf{y}_k = \mathbf{H}_k \mathbf{P}_k \sqrt{\mathbf{S}_k} \mathbf{x}_k + \mathbf{z}_k. \tag{1}$$

$\mathbf{x}_k \triangleq [x_k^1, \cdots, x_k^N]$ is the N-vector containing the dirty paper coded user data signals $\mathbf{u}_k \triangleq [u_k^1, \cdots, u_k^N]$ on tone k with identity covariance matrix. $\mathbf{y}_k$ is the N-vector of received signals on tone k. $\mathbf{z}_k$ is the N-vector of uncorrelated additive noise samples on tone k such that $\mathbb{E}\{\mathbf{z}_k \mathbf{z}_k^H\} = \sigma_k \mathbf{I}_N$. $\mathbf{P}_k$ denotes the $N \times N$ precoder matrix on tone k, which represents the linear processing applied to the user data signals. $\sqrt{\mathbf{S}_k} \triangleq \text{diag}\{\sqrt{\mathbf{s}_k}\}$ is a diagonal gain scaling matrix on tone k, with $\mathbf{s}_k \triangleq [s_k^1, \cdots, s_k^N]$ being the power allocation vector. Finally, $\mathbf{H}_k \triangleq [h_k^{n,m}]$ denotes the $N \times N$ channel matrix on tone k. The diagonal elements of $\mathbf{H}_k$ contain the direct channels whilst the off-diagonal elements contain the crosstalk channels. Although the direct channels of $\mathbf{H}_k$ typically are dominant below 30 MHz (i.e. $|h_k^{n,n}| \gg |h_k^{m,n}|, m \neq n$), recent measurements show that this is not valid anymore for higher frequencies of G.fast where the direct channels may even be smaller than the crosstalk channels [7]. This in particular makes it important to consider improved vectoring schemes based on NLP and UEO for G.fast.

NLP is based on the theoretical concept of dirty paper coding (DPC) which is a sequential interference pre-subtraction technique that achieves capacity in the downstream channel [8]. To implement DPC a UEO is required: the first encoded user experiences crosstalk from all other users, the last encoded user has all other users' crosstalk pre-subtracted. DPC serves as an upper bound for practical NLP implementations like the well-known Tomlinson-Harashima precoding (THP) [9]. Importantly, there is a small performance gap between the DPC concept and the THP implementation due to the necessary modulo operations resulting in some power penalties [10]. Furthermore, we assume perfect channel state information and ideal signal processing, although NLP schemes are recently shown to be sensitive to channel estimations errors and other non-idealities [11].

NLP can be extended with the ZF criterion in order to cancel the remaining crosstalk, by using the QR decomposition (QRD) of the conjugate transposed channel matrix [9]. To include UEO, the QRD expression is extended with a linear permutation matrix $\mathbf{E}_k$ obtaining

$$(\mathbf{E}_k \mathbf{H}_k)^H \stackrel{\text{qr}}{=} \mathbf{Q}_k \mathbf{R}_k, \tag{2}$$

where $\mathbf{Q}_k$ is a unitary matrix and $\mathbf{R}_k$ is an upper triangular matrix. Now, $\mathbf{H}_k = \mathbf{E}_k^T \mathbf{R}_k^H \mathbf{Q}_k^H$. Furthermore, $\mathbf{E}_k$ is computed according to the UEO vector $\boldsymbol{\pi}_k \triangleq [\pi_k^1, \cdots, \pi_k^N]$ as

$$[\mathbf{E}_k]_{n,m} = \begin{cases} 1 & \text{if } m = \pi_k^n \\ 0 & \text{else,} \end{cases}$$

where π_k^i denotes the i-th encoded user and $\mathbf{E}_k^T \mathbf{E}_k = \mathbf{I}_N$ for each tone k. Then in combination with DPC, the ZF criterion is achieved by setting the precoder matrix to $\mathbf{P}_k = \mathbf{Q}_k \text{diag}(\mathbf{R}_k^H)^{-1} \mathbf{E}_k$, and the estimated signal vector after decoding is

$$\hat{\mathbf{y}}_k = \sqrt{\mathbf{S}_k} \mathbf{u}_k + \mathbf{z}_k. \tag{3}$$

For this ZF-NLP system, the number of achievable bits that can be loaded on tone k for user n is modeled by

$$b_k^n = \log_2 \left(1 + s_k^n (\Gamma \sigma_k)^{-1}\right), \tag{4}$$

where Γ denotes the capacity gap which takes practical QAM implementations into account, and is a function of the desired BER, coding gain, and noise margin [12]. The total data rate of user n in bits per second is $R_n = f_s \sum_k b_k^n$ where f_s is the DMT symbol rate.

2.2. Problem Statement

Instead of the traditional weighted-sum-rate (WSR), we consider an objective function $f(\{R_n\})$ based on the α-fairness utility function of the data rates [6]. The joint α-fair DSM or power allocation and UEO problem is then described by

$$\underset{\{\boldsymbol{\pi}_k\},\{\mathbf{s}_k\}}{\text{maximize}} \; f(\{R_n\}) = \begin{cases} \sum_n w_n \dfrac{(R_n)^{1-\alpha}}{1-\alpha} & \alpha \neq 1, \alpha \geq 0 \\ \sum_n w_n \log(R_n) & \alpha = 1. \end{cases}$$

$$\begin{aligned} \text{s.t.} \quad & \sum_k \sum_m \left| [\mathbf{P}_k]_{n,m} \right|^2 s_k^m \leq P^{\text{line}} \quad \forall n \\ & \sum_m \left| [\mathbf{P}_k]_{n,m} \right|^2 s_k^m \leq P_k^{\text{mask}} \quad \forall n, k \\ & 0 \leq s_k^n \leq s_k^{\text{cap}} = \Gamma \sigma_k \left(2^{b^{\text{cap}}} - 1 \right), \forall n, k. \end{aligned} \tag{5}$$

In the objective function, w_n is the weight of user n, and $\alpha \in [0, \infty]$ controls the trade-off between performance (user mean-rate) on the one hand, and fairness (user min-rate) on the other hand. In particular, $\alpha = \infty$ corresponds to the max-min fairness (that can be considered to be the most fair allocation), $\alpha = 1$ corresponds to the proportional fair policy, and $\alpha = 0$ corresponds to WSR maximization (that is most performant).

The first two constraints are realistic per-line transmit power constraints. P^{line} denotes the aggregate transmit power (ATP) for every line, while $\{P_k^{\text{mask}}\}$ are the per-tone spectral masks which are kept low in G.fast in order to not generate too much interference into other technologies. The third constraint corresponds to the bit cap b^{cap} for the bit loading, translated into a power cap s_k^{cap} by re-writing (4).

3. α-FAIR PTES

Problem (5) is nonconvex since finding the optimal per-tone UEOs $\{\boldsymbol{\pi}_k\}$ entails a combinatorial problem whose complexity increases exponentially with the number of tones K and the number of users N. For any fixed set of $\{\boldsymbol{\pi}_k\}$ though, the remaining optimal power allocation problem is convex and can be efficiently solved. The combinatorial problem thus can be considered as the maximization over various concave functions (one for each set of $\{\boldsymbol{\pi}_k\}$) leading to a non-smooth objective function. Unfortunately, for any practical G.fast scenario, the naive attack on problem (5) with exhaustive search over the $(N!)^K$ possible UEOs to find the globally optimal solution is intractable. This motivates us to develop an efficient algorithm that computes a near-optimal approximate solution of (5).

To first decouple the per-line ATP constraints between the tones, we dual decompose problem (5) by applying Lagrangian relaxation. This consists in incorporating the ATP constraints into the objective function, which results into the following constrained Lagrange dual function

$$\begin{aligned}\underset{\{\boldsymbol{\pi}_k\},\{\mathbf{s}_k\in\mathcal{D}_k\}}{\text{maximize}} \quad & f(\{R_n\}) \\ & -\sum_n \theta^n \left(\sum_k \sum_m |[\mathbf{P}_k]_{n,m}|^2 s_k^m - P^{\text{line}}\right) \\ \text{s.t.} \quad & \sum_m \left|[\mathbf{P}_k]_{n,m}\right|^2 s_k^m \leq P_k^{\text{mask}} \quad \forall n,k, \quad (6)\end{aligned}$$

where $\mathcal{D}_k \triangleq \{s_k^n | 0 \leq s_k^n \leq s_k^{\text{cap}}, \forall n\}$ and $\{\theta^1, \cdots, \theta^N\}$ are the non-negative dual Lagrange variables. The idea is to solve the dual function (6) for each set of $\{\theta^n\}$. Then, the solution to the original problem (5) may be found by choosing $\{\theta^n\}$ (with e.g. standard subgradient updating) such that the ATP constraint for each line n is either tight or inactive, i.e., $\theta^n \left(\sum_k \sum_m |[\mathbf{P}_k]_{n,m}|^2 s_k^m - P^{\text{line}}\right) = 0$. For this dual decomposition step, we may claim optimality by referring to the "zero duality gap"-result for multi-carrier systems when the number of tones is sufficiently large [13, 14].

Next, we propose an iterative method to approximately solve (6) based on coordinate ascent of the objective function. The iterative method is organized in a per-tone fashion, where for each tone k the optimal $\{s_k^n, \forall n\}$ and $\{\pi_k^n, \forall n\}$ are calculated, while keeping $\{\mathbf{s}_{k'}\}$, $\{\boldsymbol{\pi}_{k'}\}$ of all other tones $k' \neq k$ fixed. In order to decouple the objective function $f(\{R_n\})$ between the tones as well, the method also linearizes $f(\{R_n\})$ in $\{R_n\}$ each iteration. The linearized objective function is expressed as

$$\begin{aligned} f^{\text{lin}}(\{R_n\}) &\triangleq f(\{\bar{R}_n\}) + \sum_n w_n \frac{\partial f}{\partial R_n}\bigg|_{\bar{R}_n} \left(R_n - \bar{R}_n\right) \\ &= \sum_n \frac{w_n}{\left(\bar{R}_n\right)^{\alpha}} R_n + c, \qquad (7)\end{aligned}$$

where $\{\bar{R}_n\}$ correspond to the set of data rates obtained after the previous iteration of tone $k-1$, and c makes the approximation tight. The linearized objective is an upper bound for $f(\{R_n\})$ when the UEO is fixed. Moreover, it may be interpreted as a WSR function where the weights $w_n/(\bar{R}_n)^{\alpha}$ balance the user rates. Then, the independent subproblem for each tone k is (when omitting the constants c and $P^{\text{line}}\sum\theta^n$)

$$\begin{aligned}\underset{\boldsymbol{\pi}_k,\mathbf{s}_k\in\mathcal{D}_k}{\text{maximize}} & \sum_n \left\{ \frac{w_n f_s b_k^n}{\left(\bar{R}_n\right)^{\alpha}} - \theta^n \sum_m |[\mathbf{P}_k]_{n,m}|^2 s_k^m \right\} \\ \text{s.t.} \quad & \sum_m \left|[\mathbf{P}_k]_{n,m}\right|^2 s_k^m \leq P_k^{\text{mask}}, \forall n. \quad (8)\end{aligned}$$

As optimizing over $\boldsymbol{\pi}_k$ burdens subproblem (8) with a non-smooth objective function, (8) remains nonconvex. Hence, solving (8) optimally still requires an exhaustive search over all $\boldsymbol{\pi}_k$, where for each $\boldsymbol{\pi}_k$ the optimal power allocation has to be calculated. This procedure is mathematically described in (9) and (10).

$$\underset{\boldsymbol{\pi}_k}{\text{maximize}} \quad h_k(\boldsymbol{\pi}_k) \qquad (9)$$

with

$$\begin{aligned} h_k(\boldsymbol{\pi}_k) = \underset{\mathbf{s}_k\in\mathcal{D}_k}{\text{maximize}} & \sum_n \left\{ \frac{w_n f_s b_k^n}{\left(\bar{R}_n\right)^{\alpha}} - \theta^n \sum_m |[\mathbf{P}_k]_{n,m}|^2 s_k^m \right\} \\ \text{s.t.} \quad & \sum_m \left|[\mathbf{P}_k]_{n,m}\right|^2 s_k^m \leq P_k^{\text{mask}}, \forall n \qquad (10)\end{aligned}$$

This iterative process of sequentially linearizing $f(\{R_n\})$ with (7) and solving (8) optimally with (9) and (10) for each tone k is guaranteed to converge, because each per-tone iteration strictly increases the objective function which is at the same time upper bounded. The convergence point can be shown to be a locally optimal stationary point of (6). For $\alpha = 0$, it is in fact the global optimum since the α-fair objective function simplifies to the WSR function [15].

For given $\boldsymbol{\pi}_k$, the per-tone optimal power allocation problem (10) can be solved in a straightforward manner. First, note that now the objective function is concave [6] and the constraints form a convex set in $\mathbf{s}_k$. As a result the KKT conditions are sufficient for optimality. Examining these leads to following water-filling type power update formula

$$s_k^n = \left[\frac{w_n f_s / \log(2)}{\left(\sum_m (\theta^m + \lambda_k^m) |[\mathbf{P}_k]_{m,n}|^2\right) \left(\bar{R}_n\right)^{\alpha}} - \Gamma\sigma_k \right]_0^{s_k^{\text{cap}}}, \qquad (11)$$

where $[x]_a^b \triangleq \max(a, \min(x,b))$. Furthermore, $\boldsymbol{\lambda}_k \triangleq [\lambda_k^1, \cdots, \lambda_k^N]$ are non-negative per-tone Lagrange multipliers, which should be chosen such that for each user n and tone k the spectral mask constraint is either tight or inactive, i.e., such that the KKT complementary condition is satisfied

$$\lambda_k^n \left(\sum_m |[\mathbf{P}_k]_{n,m}|^2 s_k^m - P_k^{\text{mask}} \right) = 0, \forall n. \qquad (12)$$

Algorithm 1: α-fair PTES

Initialize $\forall k$: $\boldsymbol{\pi}_k = [1, \cdots, N], \mathbf{s}_k = \mathbf{0}_N$
repeat
  Set/update $\theta^n, \forall n$
  repeat
    for $k = [1 \cdots K]$ **do**
      Calculate $\bar{R}_n, \forall n$
      foreach $[\pi_k^1, \cdots, \pi_k^N]$ **do**
        $\mathbf{P}_k = \mathbf{Q}_k \text{diag}(\mathbf{R}_k^H)^{-1}\mathbf{E}_k$ using (2)
        $\mathbf{s}_k \leftarrow$ Solve problem (10)
        if $f(\{R_n\})$ is increased **then**
          store $\{\mathbf{s}_k, \boldsymbol{\pi}_k, \mathbf{P}_k\}$
        end
      end
    end
  until convergence
until $\theta^n \left(\sum_k \sum_m |[\mathbf{P}_k]_{n,m}|^2 s_k^m - P^{\text{line}}\right) < \epsilon \quad \forall n$

The optimal $\boldsymbol{\lambda}_k$ may be computed with either the ellipsoid method or with the standard sub-gradient approach [13].

A complete algorithmic description is given in Alg. 1, which is referred to as α-fair PTES, since the K per-tone subproblems (8) are solved using an exhaustive search over $N!$ possible orderings. Thus the complexity for computing the UEO is linear in K and exponential in N. For more than five users, this algorithm continues to be prohibitively complex. However, we add here that also larger scenarios can be handled by adopting specific suboptimal methods with only polynomial complexity in N to replace the exhaustive search in (8). This will be the topic of a forthcoming report, further details are omitted here for brevity. As a preview, we include in the performance evaluation in section 4 two such α-fair UEO methods, called successive ordering search (SOS) and derivative based ordering (DBO), that reduce the complexity of α-fair PTES against only little performance loss.

4. PERFORMANCE EVALUATION

In this section the performance of α-fair PTES is evaluated for the G.fast 212 MHz profile. We assume G.9700 spectral masks [1], 8 dBm per-line ATP constraints, a 14 bitcap and a background noise of -140 dBm/Hz. The tone spacing is 51.75 kHz and the symbol rate is 48 KHz. The capacity gap Γ is set to 9.45 dB. The channel matrices have been obtained by measurements of a cable binder consisting out of $N = 5$ lines of 80 m. We consider equal user weights (i.e. $\{w_n = 1|\forall n\}$).

The comparison between the different UEO methods is given in Fig. 2, showing the obtained mean-rates (performance) and min-rates (fairness) versus the fairness parameter α. Included in the comparison are the heuristic V-BLAST [4] and DO [5] UEO methods. Given the obtained $\{\boldsymbol{\pi}_k\}$, the optimal power allocations are then calculated such that the WSRs are maximized. The best of four randomly generated $\{\boldsymbol{\pi}_k\}$ is also included for comparison (labeled as 'Random').

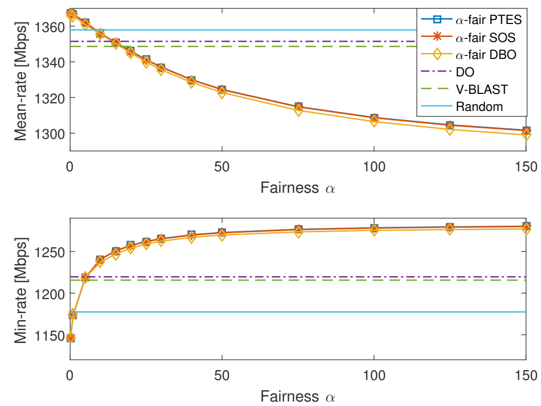

Fig. 2: $N = 5$ scenario where the proposed α-fair UEO methods achieve a trade-off between performance (mean-rate) and fairness (min-rate) that outperforms current UEO methods.

Interestingly, the proposed α-fair UEO methods achieve a trade-off between performance and fairness which outperforms current UEO methods. For instance, for large α (max-min fairness), the user min-rate obtained by α-fair PTES is more than 5% higher than for DO. In addition, for the same user min-rate, α-fair PTES achieves a small higher ($\approx 0.6\%$) user mean-rate than DO. We remark also that DO achieves a higher min- and mean-rate than V-BLAST, confirming the results in [5].

Clearly, the economies of scale are also active, meaning that first for a small performance loss a lot of fairness is gained, after which increasingly more performance is sacrificed for only little extra fairness. This steep initial fairness gain mainly stems from the UEOs being assigned in favor of the weakest users. Only for large α's, the power allocation also starts to really kick in as strong users back off their transmit power in favor of weak users.

5. CONCLUSION

An optimization framework for joint α-fairness based DSM and UEO for ZF-NLP in G.fast downstream transmission is presented in this paper. Since computing the optimal UEO scales exponentially with the number of tones K and users N, computing a globally optimal solution for any G.fast scenario is intractable. Motivated by this, α-fair PTES has been proposed, which solves the optimization problem in a near-optimal per-tone iterative fashion, leading to a linear complexity in K. The complexity of this algorithm can be further reduced with efficient suboptimal methods, which will be the topic of a forthcoming report. Simulations reveal that the α-fair PTES achieves a trade-off between performance and fairness that outperforms current UEO methods like DO.

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