Signalling-Optimal Beamformer For Multiuser MISO Broadcast Visible Light Communications

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***Abstract*—In this paper, we consider an optimal space constellation design for a multiuser multi-input single-output (MISO) visible light communication (VLC) broadcast system when channel state information is available at the transmitter. By utilizing the cooperation of multiuser interference, an optimal multidimensional additively uniquely decomposable constellation group (AUDCG) is designed to maximize the received worst-case minimum Euclidean distance of all users subject to a normalized average optical power within the orthants of a real-valued space. This optimal AUDCG is proved to be commonly used pulse amplitude (PAM) with an optimal beamformer, which is the optimal solution to a linear max-min programming problem. In addition, our optimal AUDCG admits fast demodulation of the sum signal from a noisy received signal as well as fast decoding individual signal from the estimated sum signal. Computer simulations indicate that our proposed design has significantly better error performance than currently available zero-forcing (ZF), minimum mean square error (MMSE) and time-division orthogonal access (TDMA) transmission schemes for multiuser MISO VLC broadcast systems.**

***Index Terms*—Multi-input single-output, visible light communications, broadcast channels, optimal multidimensional constellation and additively uniquely decomposable constellation group.**

I. Introduction

Visible light communication (VLC) has been viewed as a promising candidate for bandwidth-hungry multiuser applications [1]–[5] by using the already ubiquitously installed light emitting dioxides (LEDs). These multiple LEDs and a popular built-in complementary metal-oxide semiconductor (CMOS) cameras [6]–[13] in commonly used smart handsets naturally form a multiuser multi-input single-output (MISO) VLC broadcast system. In this paper, we focus on the energy-efficient multidimensional constellation design for this system, where channel state information at the transmitter (CSIT) is available. The key to the constellation design for multiuser broadcast system is how to look at the multiuser interference. On one hand, multiuser interference is traditionally treated as a detrimental factor and thus, to be suppressed or eliminated. For multiuser radio frequency (RF) broadcast systems, the commonly used linear designs, such as zero-forcing (ZF) [14], [15], minimum mean square error (MMSE) [16] beamforming methods and time-division multiple access (TDMA), have been proposed to suppress or eliminate it. These methods for MISO broadcast RF systems can not be applied to our VLC systems straightforwardly due to the nonnegativity requirement of intensity modulation [1], [3], [17]. To satisfy this requirement, modified ZF [18], MMSE [19], [20] and TDMA [3] have been proposed by adding direct current to the bipolar transmission. However, the energy-efficiency can not be guaranteed for the reason that the power constraint for VLC is on the mean of the signal amplitude. On the other hand, because of the nonnegativity property of the signal amplitude and channel coefficients for VLC, multiuser interference is impossible to be zero-forced. For this reason, the concept of an additively uniquely decomposable constellation group (AUDCG) was proposed in [21] to turn additive multiuser interference into uniquely identifiable signal components. One-dimensional AUDCG [21] was designed for single-input ideal additive white Gaussian noise (AWGN) VLC broadcast channel and shown to be more energy-efficient than TDMA [3]. In this paper, we consider the optimal multidimensional AUDCG for multiuser MISO VLC broadcast systems. Unfortunately, the optimal design of multidimensional constellations is a classic and long-standing problem in modern wireless communications [22]–[27]. Since the resulting discrete optimization problem for digital communications is extremely challenging to be formulated into a tractable optimization problem [28], the systematic design of the optimal constellation for both RF and VLC, to the best knowledge of authors, still remains unsolved thus far. Let alone say what the optimal multidimensional constellation for multiuser VLC systems is.

Our main contribution is that we will design an optimal multidimensional constellation optimizing the received worst-case minimum Euclidean distance among all users under an optical power constraint within the nonnegative orthants of a multidimensional real-valued space by utilizing the nonnegativity of VLC signal and channel coefficients. This optimal design is proved to be the commonly used pulse amplitude modulation (PAM) constellation multiplied by a nonnegative beamforming vector, which can be attained by solving a linear max-min programming problem [29], [30]. In fact, we have solved a long-standing open problem of the optimal multidimensional constellation design for multiuser MISO VLC broadcast systems. In addition, this optimal design has the properties that the receiver has low-complexity maximum likelihood (ML) demodulation and decoding algorithms. Simulations show that the designed constellation substantially outperforms the currently available ZF [18], MMSE [19], [20] and TDMA [3].

II. System Model

In this section, we briefly introduce the channel model to be considered and then, state the main problem to be solved.

Let us consider M-user MISO broadcast VLC systems, in which N transmitter apertures (such as LEDs) are equipped and the M users have single receiver photodiode (PD). For a given channel use, the message vector intended for User m is denoted by $\mathbf{s}^{(m)} \in \mathcal{S}^{(m)} \subseteq \mathbb{R}_+^N$, where $\mathbb{R}_+^N$ stands for the set of all the nonnegative $N\times 1$ vectors and the cardinality of $\mathcal{S}^{(m)}$ is given by $|\mathcal{S}^{(m)}| = 2^{K_m}$. Then, a total message vector carrying all the M users' message, $\mathbf{s} \in \mathcal{S} \subseteq \mathbb{R}_+^N$ is transmitted simultaneously by the N transmitter apertures to the M users through respective flat-fading channels $\mathbf{h}_u = [h_{u1}, \ \cdots, \ h_{uN}]^T$ for $u = 1, 2, \ \cdots, M$. Then, the signal received by the u-th user is represented by

$$r_u = \mathbf{h}_u^T \mathbf{s} + \xi_u, \tag{1}$$

where $\mathbf{h}_u$ for any u is a nonnegative real-valued vector. For indoor VLC systems, when the transmitter and receiver are at fixed locations, the channel coefficients h_{un} are considered to be deterministic and can be computed by [1], [31], [32]

$$h_{un} = \begin{cases} \frac{(\tau+1)A}{2\pi d_{un}^2} \cos^\tau(\phi_{un}) \cos(\psi_{un}), \\ \qquad 0 \le \psi_{un} \le \Psi; \\ 0, \psi_{un} > \Psi, \end{cases} \tag{2}$$

where ψ_{un} is the angle of incidence from the n-th transmitter to the u-th user, Ψ is the field-of-view angle of the receiver PD, A denotes the PD detection area, and $\tau = -\log_2 \cos \Phi_{\frac{1}{2}}$ with $\Phi_{\frac{1}{2}}$ being defined by the half power angle of LEDs. Therefore, it is realistic to assume that for indoor fixed broadcast links, CSIT is known by the transmitter. From [32]–[34], the channel noise ξ_u for the commonly used PD detection is well modelled as AWGN with zero mean and variance σ_ξ^2.

At the receiver side of User u, the received signal is given by r_u. The receiver detection consists of two successive steps.

1) **ML Demodulation:** Given the received signal r_u, the ML estimate of $\mathbf{s}$ from User u is determined by

$$\hat{\mathbf{s}}_u = \arg\min_{\mathbf{s}\in\mathcal{S}} |r_u - \mathbf{h}_u^T \mathbf{s}| \tag{3}$$

2) **Decoding:** The function of the decoder of User u is to determine the M signal components $\hat{\mathbf{s}}_u^{(m)}$ for $1 \le m \le M$ from the optimal estimate $\hat{\mathbf{s}}_u$.

The unique identification of $\hat{\mathbf{s}}_u^{(m)}$ from $\hat{\mathbf{s}}_u$ is assured only if there exists a one-to-one mapping between $(\mathbf{s}_u^{(1)}, \ \cdots, \ \mathbf{s}_u^{(M)})$ and $\hat{\mathbf{s}}_u$ for any $\hat{\mathbf{s}}_u \in \mathcal{S}$. If there exists such relationship with respect to the sum operations, then, $\mathcal{S}^{(1)}, \ \cdots, \ \mathcal{S}^{(M)}$ are called an AUDCG, whose formal definition is given below [21], [35]:

Definition 1: : A group of constellations $\mathcal{X}^{(m)}$ for $m = 1, 2, \ \cdots, M$ is said to constitute an AUDCG, if there exist $x^{(m)}, \tilde{x}^{(m)} \in \mathcal{X}^{(m)}$ such that $\sum_{m=1}^M x^{(m)} = \sum_{m=1}^M \tilde{x}^{(m)}$, then, we have $x^{(m)} = \tilde{x}^{(m)}$ for any $m = 1, 2, \ \cdots, M$. ■

By Definition 1, the M constellations $\mathcal{X}^{(m)}$ in an AUDCG naturally result in a specific sum constellation, $\mathcal{X} = \{\sum_{m=1}^M x^{(m)} : x^{(m)} \in \mathcal{X}^{(m)}\}$, which, for presentation convenience, is denoted by $\mathcal{X} = \mathcal{X}^{(1)} \uplus \mathcal{X}^{(2)} \uplus \ \cdots \ \uplus \mathcal{X}^{(M)}$. Given $\mathcal{X}^{(m)}$ with finite size, $\mathcal{X} = \mathcal{X}^{(1)} \uplus \mathcal{X}^{(2)} \uplus \ \cdots \ \uplus \mathcal{X}^{(M)}$ if and only if $|\mathcal{X}| = \prod_{m=1}^M |\mathcal{X}^{(m)}|$ [21], [35]. In this paper, our main task is to

1) design a group of N-dimensional constellations $\mathcal{S}^{(m)} \subseteq \mathbb{R}_+^N$ carrying the message set of User m and an N-dimensional constellation $\mathcal{S} \subseteq \mathbb{R}_+^N$ of 2^K nonnegative alphabets carrying the overall message of the N users,
2) and find an energy-efficient AUDCG $\mathcal{S} = \mathcal{S}^{(1)} \uplus \mathcal{S}^{(2)} \uplus \ \cdots \ \uplus \mathcal{S}^{(M)}$,

such that the worst-case error performance is optimized under an average transmitted optical power budget.

III. Optimal Space Constellation Design and Low-Complexity Receivers

In this subsection, we will characterize the structure of the optimal space constellation and develop a fast demodulation and decoding algorithm for this optimal design.

A. Optimal Space Constellation Design

When CSIT is available at the transmitter, the objective of the transmitted signal design for the ML demodulator is to maximize the received worst-case minimum Euclidean distance

$$\min_{1\le m\le M} \min_{\mathbf{s},\tilde{\mathbf{s}}\in\mathcal{S},\mathbf{s}\ne\tilde{\mathbf{s}}} \left|\mathbf{h}_u^T \mathbf{s} - \mathbf{h}_u^T \tilde{\mathbf{s}}\right| \tag{4}$$

under a power budget. Therefore, the design problem is formulated below:

Problem 1: Let the channel vectors of M users be defined by nonzero vectors $\mathbf{h}_u \in \mathbb{R}_+^N$ with $1 \le u \le M$ and K, N and M be arbitrarily given positive integers. Then, given M, $N \ge 2$ and $\mathbf{h}_u$, devise a group of constellations $\mathcal{S}^{(m)} \subseteq \mathbb{R}_+^N$ and an N-dimensional constellation $\mathcal{S} \subseteq \mathbb{R}_+^N$ of size 2^K such that 1) $\mathcal{S} = \mathcal{S}^{(1)} \uplus \mathcal{S}^{(2)} \uplus \ \cdots \ \uplus \mathcal{S}^{(M)}$, and 2) the minimum of the worst-case received minimum Euclidean distance of M users, $\min_{1\le u\le M} \min_{\mathbf{s},\tilde{\mathbf{s}}\in\mathcal{S},\mathbf{s}\ne\tilde{\mathbf{s}}} \left|\mathbf{h}_u^T \mathbf{s} - \mathbf{h}_u^T \tilde{\mathbf{s}}\right|$, is maximized subject to an average transmitted optical power constraint, $\frac{1}{2^K} \sum_{\mathbf{s}\in\mathcal{S}} \mathbf{s}^T \mathbf{1} = 1$. ■

Very interestingly, the solution to Problem 1 depends on an optimal solution to the following linear max-min programming problem:

Problem 2: Given any positive integer N, M and nonnegative $N \times 1$ vectors $\mathbf{h}_u \in \mathbb{R}_+^N$, find an $N \times 1$ nonnegative vector $\mathbf{w}$ such that $\min_{1\le u\le M} \mathbf{h}_u^T \mathbf{w}$ is maximized subject to $\mathbf{w}^T \mathbf{1} = 1$. ■

We are now in a position to formally state the main result in this paper.

Theorem 1: Let $\mathbf{w}_{\text{opt}}$ be the optimal solution to Problem 2. Then, the optimal solution to Problem 1 is determined as follows: $\tilde{\mathcal{S}} = \{\frac{2k\mathbf{w}_{\text{opt}}}{2^K-1}\}_{k=0}^{2^K-1}$, $\tilde{\mathcal{S}}^{(1)} = \{\frac{2k\mathbf{w}_{\text{opt}}}{2^K-1}\}_{k=0}^{k=2^{K_1}-1}$, $\tilde{\mathcal{S}}^{(m)} = \{2^{\sum_{i=1}^{m-1} K_i} \times \frac{2k\mathbf{w}_{\text{opt}}}{2^K-1}\}_{k=0}^{k=2^{K_m}-1}$, $2 \le m \le M$, and $\mathcal{S} = \mathcal{S}^{(1)} \uplus \mathcal{S}^{(2)} \uplus \ \cdots \ \uplus \mathcal{S}^{(M)}$ ■

To appreciate our optimal design in Theorem 1, whose proof is provided in Appendix A, we would like to make the following comments:

1) **Optimality Issues**. Our optimal space constellation structures are attained by solving the max-min-min and max-min-min-min, respectively, for the cases with and without CSIT. It should be noticed that the optimality of the space constellation is for our ML-demodulator-and-decoding receiver without any additional assumption on the signal set within the nonnegative orthants of an N-dimensional real space. In addition, this optimality is for any nonnegative MISO channels. When CSIT is available at the transmitter side, the optimal space constellation design is to solve a linear max-min programming problem, i.e., Problem 2. It is known that linear max-min programming is a classic problem, for which numerous efficient algorithms have been developed [29], [30].

2) **Design Techniques**. Generally speaking, attaining a closed-form solution to the max-min design problem with discrete and continuous mixed variables is very challenging [22]–[27], since the corresponding problem is hard to be transformed into a tractable problem [28]. It is noticed that our multiuser space constellation design problem is of the *max-min-min* or *max-min-min-min* form, which is more difficult and, to the best knowledge of authors, still remains unsolved up to now. Fortunately, by fully taking advantage of the unipolarity of the signal set and channel coefficients, we have successfully attained the optimal space constellation structures, therefore, significantly reducing the unknown variables from $N2^K$ to N, and equivalently, transformed the original nonlinear design problem into a linear design beamforming problem, which can be efficiently solved via numerical approach [29], [30]. These design techniques may provide useful insight into the optimal multidimensional constellation designs for VLC.

B. Low-Complexity Receivers

Here, we provide a fast ML demodulation and decoding algorithm for the optimal designs given by Theorem 1. Notice that for the optimal designs with $\mathbf{w}_{\text{opt}}$, the resulting MISO channel model for User u becomes

$$y_u = \bar{h}_u k + \xi_u \tag{5}$$

where $\bar{h}_u = \frac{2\mathbf{h}_u^T \mathbf{w}_{\text{opt}}}{2^K-1}$, $u \in \{1, \cdots, M\}$ and $k \in \{0, 1, \cdots, 2^K-1\}$. Since $\bar{h}_u$ is a positive scalar for any nonzero channel vector $\mathbf{h}_u \in \mathbb{R}_+^N$, such an equivalent channel model in (5) actually is a single-input-single-output ideal AWGN channel. Hence, the optimal estimate of the transmitted signal can be efficiently obtained below:

Algorithm 1: (**Fast ML Demodulation**): Given the received signal y_u defined by (5) and the non-zero channel vector $\mathbf{h}_u \in \mathbb{R}_+^N$, the output of the ML demodulator for User u is given by

$$\hat{\mathbf{s}}_u = \begin{cases} \mathbf{0}, & \frac{y_u}{\bar{h}_u} < 0, \\ \left\lfloor \frac{y_u}{\bar{h}_u} + \frac{1}{2} \right\rfloor \times \frac{2\mathbf{w}_{\text{opt}}}{2^K-1}, & 0 \le \frac{y_u}{\bar{h}_u} \le 2^K-1, \\ 2\mathbf{w}_{\text{opt}}, & \frac{y_u}{\bar{h}_u} > 2^K-1. \end{cases} \tag{6}$$

where $u = 1, \cdots, M$. ■

When the optimal ML estimate $\hat{\mathbf{s}}_u$ of $\mathbf{s}_u$ from User u has been obtained by Algorithm 1, now by Theorem 1, there exists a unique group of $\hat{\mathbf{s}}_u^m \in \tilde{\mathcal{S}}^{(m)}$ such that $\hat{\mathbf{s}}_u = \sum_{m=1}^M \hat{\mathbf{s}}_u^{(m)}$. For notational simplicity, we let $\hat{s}_u = \frac{2\mathbf{w}_{\text{opt}}^T \hat{\mathbf{s}}_u}{2^K-1} / \left\| \frac{2\mathbf{w}_{\text{opt}}}{2^K-1} \right\|_2^2$. By fully taking advantage of the properties of the optimal space structure $\tilde{\mathcal{S}}^{(m)}$, we know $\hat{\mathbf{s}}_u^{(1)} = \left(\bar{s}_u \text{mod } 2^{K_1}\right) \times \frac{2\mathbf{w}_{\text{opt}}^T}{2^K-1}$ and $\hat{\mathbf{s}}_u^{(2)} = \frac{2\mathbf{w}_{\text{opt}}^T}{2^K-1} \times 2^{K_1} \left(\frac{\bar{s}_u - \bar{s}_u \text{mod } 2^{K_1}}{2^{K_1}} \text{mod } 2^{K_2} \right)$. The above discussions can be summarized as the following fast decoding algorithm:

Algorithm 2: (**Fast decoding**) Let $\hat{\mathbf{s}}_u$ be the optimal ML estimate of $\mathbf{s}_u$ obtained by Algorithm 1 from User u, where $1 \le u \le M$. Then, the estimate of the m-th message component $\mathbf{s}_u^{(m)}$ from User u, say, $\hat{\mathbf{s}}_u^{(m)}$, is given by $\hat{\mathbf{s}}_u^{(1)} = \left(\bar{s}_u \text{mod } 2^{K_1}\right) \frac{2\mathbf{w}_{\text{opt}}^T}{2^K-1}$ and $\hat{\mathbf{s}}_u^{(m)} = \frac{2\mathbf{w}_{\text{opt}}^T}{2^K-1} \times 2^{\sum_{\ell=1}^{m-1} K_\ell} \left(\frac{\bar{s}_u - \bar{s}_u \text{mod } 2^{\sum_{\ell=1}^{m-1} K_\ell}}{2^{\sum_{\ell=1}^{m-1} K_\ell}} \text{mod } 2^{K_m} \right)$ for $2 \le m \le M$. ■

Now, we can see that the complexity of the proposed demodulation-decoding algorithm for the optimal design given by Theorem 1 is $O(M^2)$.

IV. Simulation Results

In this section, we carry out simulations to examine the performance of our optimal design in a $5\text{m} \times 5\text{m} \times 3\text{m}$ room with four LEDs located at $(1.25\text{m}, 1.25\text{m})$, $(3.75\text{m}, 1.25\text{m})$, $(1.25\text{m}, 3.75\text{m})$, $(3.75\text{m}, 3.75\text{m})$, respectively. The receiver PD is located at $(x\text{m}, y\text{m}, 0\text{m})$. To evaluate the average error performance within the illumination coverage area, we assume that both x and y are uniformly chosen from the interval $(0, 5)$. We assume that $\Phi_{\frac{1}{2}} = \Psi = 60°$, $A = 1\text{cm}^2$ and $N = 4$. The SNR is defined by $\frac{1}{\sigma_{xi}^2}$ with normalized average optical power. Let $\mathbf{H} = [\mathbf{h}_1, \cdots, \mathbf{h}_N]^T$. Then, all the schemes we would like to compare are described as follows:

1) **MMSE** [19], [20]: $\mathbf{s}_{\text{mmse}} = \frac{1}{\mathbf{d}_{\text{mmse}}^T \mathbf{1}} \left(\mathbf{W}_{\text{mmse}} \mathbf{x} + \mathbf{d}_{\text{mmse}}\right)$, where $x_m \in \{\pm(k-1)\}_{k=1}^{k=2^{K_m-1}}$, $\mathbf{W}_{\text{mmse}}$ is an $M \times N$ beamforming matrix and $\mathbf{d}$ is a direct-current vector to assure that all the entries of $\mathbf{s}$ are nonnegative. $\mathbf{W}_{\text{mmse}} = \mathbf{H}\left(\frac{1}{\rho}\mathbf{I}_{M \times M} + \mathbf{H}^T\mathbf{H}\right)^{-1}$ and $\mathbf{d}_{\text{mmse}}$ is an $N \times 1$ direct-current vector determined by minimizing $\mathbf{d}^T\mathbf{1}$ under the condition that all the entries of $\mathbf{s}_{\text{zf}}$ are nonnegative.
2) **ZF** [18]: $\mathbf{s}_{\text{zf}} = \frac{1}{\mathbf{d}_{\text{zf}}^T \mathbf{1}} \left(\mathbf{W}_{\text{zf}} \mathbf{x} + \mathbf{d}_{\text{zf}}\right)$, where $x_m \in \{\pm(k-1)\}_{k=1}^{k=2^{K_m-1}}$, $\mathbf{W}_{\text{zf}}$ is an $N \times M$ beamforming matrix and to assure that $\mathbf{W}_{\text{zf}} = \mathbf{H}\left(\mathbf{H}^T\mathbf{H}\right)^{-1}$ and $\mathbf{d}_{\text{zf}}$ is an $N \times 1$ direct-current vector minimizing $\mathbf{d}^T\mathbf{1}$ and satisfying the nonnegativity of $\mathbf{s}_{\text{zf}}$.
3) **TDMA** [3]: Within the successive M time slots, the transmitted signal matrix is given by $\mathbf{S}_{N \times M} = \frac{2M}{N \sum_{m=1}^M (2^{MK_m}-1)} [s_1 \mathbf{1}_{N \times 1}, \cdots, s_M \mathbf{1}_{N \times 1}]$, where $s_m \in \{k\}_{k=0}^{k=2^{K_m}-1}$ with $1 \le m \le M$.
4) **Constellation-Optimal Beamformer (COB)**: Our proposed COB is given by Theorem 1.

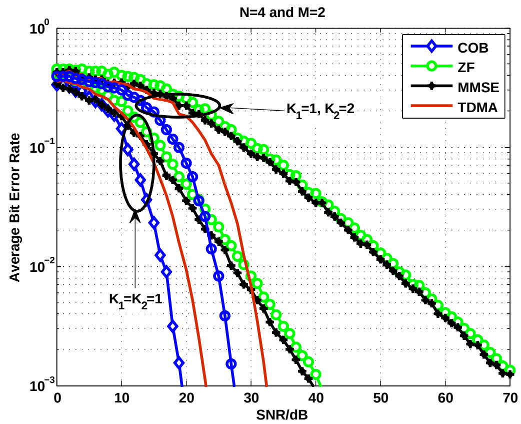

Fig. 1. Average BER of all users for proposed COB, ZF, MMSE and TDMA with $M = 2$ and $N = 4$.

All the above four schemes have the normalized transmitted optical power and the same bit rate K_m per transmission for User m, $m = 1, 2$. From Fig. 1, we can see that at the target average bit error rate (BER) of 10^{-3}, our proposed COB significantly outperforms ZF, MMSE and TDMA. When $K_1 = K_2 = 1$, the attained gains by our COB over TDMA, MMSE and ZF are 3.5 dB, 20 dB and 21 dB, respectively. In addition, at the same BER, the attained gains by our COB over TDMA and MMSE are above 40 dB and the performance advantage of COB over ZF is about 5 dB. Also, the performance gain of our proposed COB over ZF and MMSE will become larger with SNR increasing.

V. Conclusion

In this paper, we have considered the design of the optimal space constellation that maximizes the worst-case minimum Euclidean distance for a general MISO VLC broadcast system with CSIT. By fully making use of the nonnegativity of both the signal and channel coefficients, we have attained a closed-form optimal space constellation within the nonnegative orthants of a multidimensional space without any additional assumption on the signal set. We have shown that the optimal space constellation for any multiuser MISO VLC broadcast system is PAM with an optimal beamformer. Simulations have indicated that our optimal signalling method substantially outperforms the currently available ZF, MMSE and TDMA methods for multiuser MISO VLC broadcast systems.

Appendix

A. Proof of Theorem 1

Let $\mathbf{w}_{\text{opt}}$ be the optimal solution to Problem 2. Then, we define $D_{\text{opt}} = \min_{1\le u\le M} \mathbf{h}_u^T\mathbf{w}_{\text{opt}}$. Now, we can claim that for any constellation $\mathcal{S} \subseteq \mathbb{R}_+^N$ of size 2^K satisfying $\frac{1}{2^K}\sum_{\mathbf{s}\in\mathcal{S}} \mathbf{s}^T\mathbf{1} = 1$, it holds that

$$\min_{1\le u\le M} \min_{\mathbf{s},\tilde{\mathbf{s}}\in\mathcal{S},\mathbf{s}\ne\tilde{\mathbf{s}}} \left|\mathbf{h}_u^T\mathbf{s} - \mathbf{h}_u^T\tilde{\mathbf{s}}\right| \le 2D_{\text{opt}}/(2^K-1) \quad (7)$$

We prove this claim by contradiction. Suppose that there exists a size-2^K constellation $\hat{\mathcal{S}} \subseteq \mathbb{R}_+^N$, with normalized average optical power, satisfying that $\min_{1\le u\le M}\min_{\mathbf{s},\tilde{\mathbf{s}}\in\hat{\mathcal{S}},\mathbf{s}\ne\tilde{\mathbf{s}}} \left|\mathbf{h}_u^T\mathbf{s} - \mathbf{h}_u^T\tilde{\mathbf{s}}\right| > 2D_{\text{opt}}/(2^K-1)$. Then, we can attain that for any given u, it holds that

$$\left|\mathbf{h}_u^T\mathbf{s} - \mathbf{h}_u^T\tilde{\mathbf{s}}\right| > 2D_{\text{opt}}/(2^K-1) \quad (8)$$

Then, all the 2^K nonnegative scalars $\mathbf{h}_u^T\mathbf{s}_k$ for $k = 0, 1, \cdots, 2^K-1$ can be sorted in an strictly descending order, $\mathbf{h}_u^T\mathbf{s}_{2^K-1} > \cdots > \mathbf{h}_u^T\mathbf{s}_1 > \mathbf{h}_u^T\mathbf{s}_0$. Combining this result with (8) produces

$$\sum_{\mathbf{s}\in\hat{\mathcal{S}}} \mathbf{h}_u^T\mathbf{s} > \frac{2D_{\text{opt}}}{2^K-1}\sum_{k=0}^{2^K-1} k = 2^K D_{\text{opt}} \quad (9)$$

In addition, we have $\frac{1}{2^K}\sum_{\mathbf{s}\in\hat{\mathcal{S}}}\mathbf{h}_u^T\mathbf{s} = \frac{1}{2^K}\sum_{n=1}^N\sum_{\mathbf{s}=(s_1,\ \cdots,\ s_n)^T,\mathbf{s}\in\hat{\mathcal{S}}} h_{un}s_n = \sum_{n=1}^N\left(h_{un}\times\frac{1}{2^K}\sum_{\mathbf{s}=[s_1,\ \cdots,\ s_n]^T,\mathbf{s}\in\hat{\mathcal{S}}} s_n\right)$. From our power constraint that $\frac{1}{2^K}\sum_{\mathbf{s}\in\hat{\mathcal{S}}}\mathbf{s}^T\mathbf{1} = 1$, we know $\sum_{n=1}^N\frac{1}{2^K}\sum_{\mathbf{s}=(s_1,\ \cdots,\ s_n)^T,\mathbf{s}\in\hat{\mathcal{S}}} s_n = 1$. For presentation convenience, we denote $\frac{1}{2^K}\sum_{\mathbf{s}=[s_1,\ \cdots,\ s_n]^T,\mathbf{s}\in\hat{\mathcal{S}}} s_n$ by w_n, implying $\sum_{n=1}^N w_n = 1$ with $w_n \ge 0$. Then, the equality in (9) can be equivalently transformed into what follows: $2^K\mathbf{h}_u^T\mathbf{w} > 2^K D_{\text{opt}}$, where $\mathbf{w} = (w_1,\ \cdots,\ w_N)^T \in \mathbb{R}_+^N$ and $\sum_{n=1}^N w_n = 1$. In other words, we attain $\mathbf{h}_u^T\mathbf{w} > D_{\text{opt}}$. Recalling our definitions of $\mathbf{w}_{\text{opt}}$ and D_{opt}, we know that if $\mathbf{w}\in\mathbb{R}_+^N$ and $\mathbf{w}^T\mathbf{1} = 1$, then, there exists no vector $\mathbf{w}$ satisfying $\mathbf{h}_u^T\mathbf{w} > D_{\text{opt}}$, which is a contradiction. Therefore, we can conclude that the inequality (7) indeed holds.

Furthermore, when $\mathcal{S} = \tilde{\mathcal{S}} = \{\frac{2k\mathbf{w}_{\text{opt}}}{2^K-1}\}_{k=0}^{k=2^K-1}$, we have

$$\begin{aligned}
&\min_{1\le um\le M}\ \min_{\mathbf{h}_u^T\mathbf{1}=\mu_m>0}\ \min_{\mathbf{s},\tilde{\mathbf{s}}\in\tilde{\mathcal{S}},\mathbf{s}\ne\tilde{\mathbf{s}}} \left|\mathbf{h}_u^T\mathbf{s} - \mathbf{h}_u^T\tilde{\mathbf{s}}\right| \\
&= \frac{2}{2^K-1}\min_{1\le u\le M}\ \min_{0\le s,\tilde{s}\le 2^K-1,s\ne\tilde{s}} \mathbf{h}_u^T\mathbf{w}_{\text{opt}}\times|s-\tilde{s}| \\
&= \frac{2}{2^K-1}\min_{0\le s,\tilde{s}\le 2^K-1,s\ne\tilde{s}}|s-\tilde{s}|\times\min_{1\le u\le M}\mathbf{h}_u^T\mathbf{w}_{\text{opt}} \\
&= \frac{2}{2^K-1}\min_{1\le u\le M}\mathbf{h}_u^T\mathbf{w}_{\text{opt}} \quad (10)
\end{aligned}$$

Therefore, $\mathcal{S} = \tilde{\mathcal{S}}$ indeed maximizes the worst-case minimum Euclidean distance.

In the following, we prove that

$$\left\{\sum_{m=1}^M \tilde{\mathbf{s}}^{(m)} : \tilde{\mathbf{s}}^{(m)} \in \tilde{\mathcal{S}}^{(m)}, 1\le m\le M\right\} \subseteq \tilde{\mathcal{S}} \quad (11)$$

where $\tilde{\mathcal{S}}^{(m)}$ is defined in Theorem 1. For any $\mathbf{s} \in \{\sum_{m=1}^M \tilde{\mathbf{s}}^{(m)} : \tilde{\mathbf{s}}^{(m)} \in \tilde{\mathcal{S}}^{(m)}, 1\le m\le M\}$, it holds that $\mathbf{s} = \frac{2\mathbf{1}_{N\times 1}}{N(2^K-1)}\sum_{m=1}^M k_m$, where $k_1 \in \{k\}_{k=0}^{2^{K_1}-1}$ and $k_m \in \{2^{\sum_{i=1}^{m-1}K_i}\times k\}_{k=0}^{k=2^{K_m}-1}$ for $2\le m\le M$. It is observed that $\sum_{m=1}^M k_m \le 2^{K_1}-1+\sum_{m=2}^M\left(2^{K_m}-1\right)\times 2^{\sum_{i=1}^{m-1}K_i} = 2^{K_1}-1+2^{K_2+K_1}-2^{K_1}+\ \cdots\ +2^{\sum_{m=1}^M K_m}-2^{\sum_{m=1}^{M-1}K_m} = 2^{\sum_{m=1}^M K_m}-1 = 2^K-1$, inferring that $\sum_{m=1}^M k_m \in \{k\}_{k=0}^{2^K-1}$ and thus, the relationship (11) indeed holds. Combining this result with the fact that the cardinality of $\tilde{\mathcal{S}}$ is equal to 2^K yields that $|\mathcal{S}| = \prod_{m=1}^M 2^{K_m} = \prod_{m=1}^M |\mathcal{S}^{(m)}|$ and thus, $\mathcal{S} = \mathcal{S}^{(1)} \uplus \mathcal{S}^{(2)} \uplus\ \cdots\ \uplus \mathcal{S}^{(M)}$. Therefore, the proof of Theorem 1 is complete. □

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