DETECTION RATE OPTIMIZATION IN RADAR SYSTEMS WITH UNKNOWN DISTURBANCE POWER

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ABSTRACT

In this work, we study the fundamental tradeoff between integration time and scan rate in radar systems. The contrasting needs for a large probability of detection and a short scan time are carefully balanced by optimizing the detection rate, defined as the average number of detections from a target per unit of time. A closed-form solution for the optimum pulse train length is provided in the relevant case of Gaussian observations with unknown noise variance. Some examples are given to show the possible tradeoffs among the principal system parameters.

***Index Terms*—** Detection rate (DR), false alarm rate (FAR), constant FAR (CFAR), pulse radars, scan time.

1. INTRODUCTION

For fixed transmit power, the probability of detection of a radar system can be improved by increasing the length of the pulse train transmitted in each angular direction (i.e., the dwell time), so as to increase the time-on-target (TOT) and then the amount of integrated energy at the receiver. However, this has the drawback to increase the scan duration and then the reaction time of the radar [1]. To balance these contrasting needs, the cumulative probability of detection, defined as the probability of detecting the target at least once in a preassigned time interval where multiple scans take place, has been used in the past for system design, since it can quantify the capability of a search radar to detect a newly born target within a preassigned time interval [2–4] or, if the target is closing, before it reaches a given range [5–7].

Recently, this concept has been put forward in [8–10], where the authors optimize the detection rate (DR), defined as the average number of detections per unit of time from a target, under a constraint on the false alarm rate (FAR), defined as the average number of false alarms per unit of time from the inspected area. Indeed, in [10], it is shown that DR is the inverse of the average value of the detection time (defined as the random time elapsing before a new detection from a prospective target arrives), and that it bounds the quantile of the detection time and the cumulative probability of detection. Therefore, a large DR, on top of being *per se* desirable, implies a short time interval between consecutive detections from a persistent target or a fast detection of a newly born target, thus reducing the reaction time of the radar. Also, a larger DR facilitates subsequent track-before-detect [11–17] and/or tracking [18–21] algorithms, which follow the detector in the radar processing chain, since more frequent hits result in a smaller association gate in the track estimation process. On the other hand, FAR is commonly adopted by radar engineers to measure whether a radar is troubled by excessive false alarms; in particular, it has a direct impact on the computational requirement for real-time data processing and on the capacity of the human operator to take actions by monitoring the hits visualized on the radar scope [22, 23].

In this work, we optimize DR under a constrain on FAR in a radar system with range, azimuth, and Doppler resolution cells, the latter being in number equal to the pulse train length (a parameter for system optimization). Different from [10], we tackle here the case where the noise power is unknown. A closed-form solution for the optimum length of the pulse train is provided, showing that it should be set so as to have a *relatively small* probability of detection, which is approximately in the range $(1/9, 1/e)$. Examples showing the impact of the system parameters, such as the level of the probability of false alarm, the signal-to-noise (SNR) ratio, and the sample-size of secondary data used for the noise power estimation, are provided.

The reminder of the paper is organized as follows. In the next section, the radar model is presented. Sec. 3 is devoted to system optimization, while the performance analysis is provided in Sec. 4. Finally, concluding remarks are given in Sec. 5.

2. SYSTEM MODEL

Consider a pulse radar monitoring an azimuth sector of width $\Psi = M\Delta$, where Δ is the antenna beamwidth and M the number of azimuth bins. A pulse train of length N is emitted in each azimut direction. The pulse repetition time (PRT) is T and the bandwidth W, so that $K = \lfloor TW \rfloor$ range bins are defined. The transmit beam is rotated at steps of Δ, which brings the scan time to $T_{\mathrm{s}} = MNT$.

At the receive side, a standard discretization process is carried out. The signal is projected onto a delayed and frequency-shifted version of the pulse train, where the inspected pairs of delay and Doppler shift, say (τ, f_d), are dictated by the bandwidth and duration of the pulse train, i.e., $\tau \in \{1/W, 2/W, \dots, K/W\}$, and $f_d \in \{0, 1/(NT), \dots, (N-1)/(NT)\}$. Assuming that delay and Doppler shift of prospective targets lie on the inspected grid (i.e., neglecting straddling losses) and do not change during the TOT, the discretized signal from the range-azimuth-Doppler bin under inspection can be written as

$$r = \begin{cases} \sqrt{N\mathcal{E}}s + w, & \text{under } H_1 \\ w, & \text{under } H_0 \end{cases} \tag{1}$$

where: $\mathcal{E}$ is the received energy per pulse from the target; s and w are complex circularly-symmetric Gaussian random variables, the former, with unit-variance, representing the target response and the latter, with variance σ^2, representing the noise contribution; H_0 is the null hypothesis (the cell under test contains only noise); and H_1 is the alternative hypothesis (a target is present in the cell under test).

The likelihood ratio test for the cell under inspection is

$$\frac{|r|^2}{\sigma^2} \underset{H_0}{\overset{H_1}{\gtrless}} \gamma \tag{2}$$

where γ is the detection threshold. In general, σ^2 is not known, and it is common to employ a cell-averaging constant FAR (CA-CFAR) processor, so as to adapt the detector sensitivity to the variations in the noise level [3, 24, 25]. In particular, assuming that $L \geq 2$ independent and identically distributed secondary data samples are available, say $\{x_\ell\}_{\ell=1}^{L}$, with the same distribution as the noise in the cell under test, an estimate of the noise power is $\hat{\sigma}^2 = \frac{1}{L}\sum_{\ell=1}^{L} |x_\ell|^2$, and the test is expressed as in (2), once σ^2 is replaced with $\hat{\sigma}^2$, i.e.,

$$\frac{|r|^2}{\frac{1}{L}\sum_{\ell=1}^{L} |x_\ell|^2} \underset{H_0}{\overset{H_1}{\gtrless}} \gamma. \tag{3}$$

Denoting D the test statistic in (3) and $\rho = \mathcal{E}/\sigma^2$ the SNR per pulse, we have that $D \mid H_0$ and $(1+\rho N)^{-1} D \mid H_1$ follow an F-distribution with parameters 2 and $2L$ (since they are the ratio between two χ^2 random variables with 2 and $2L$ degrees of freedom [26]), whereby the probabilities of detection and false alarm are

$$P_\text{d} = \mathbb{P}(D \geq \gamma \mid H_1) = \left(1 + \frac{\gamma/L}{1+\rho N}\right)^{-L} \tag{4a}$$

$$P_\text{fa} = \mathbb{P}(D \geq \gamma \mid H_0) = (1 + \gamma/L)^{-L} \tag{4b}$$

respectively.

3. SYSTEM OPTIMIZATION

The probabilities of detection and false alarm are commonly used to evaluate the performance of a radar system. However, these metrics do not include any information about time and/or number of resolution elements. To overcome this limitation, radar engineers often consider the FAR, for it has a direct impact on the capacity of the operator in handling the hits produced by the detector. In the same way, DR is a relevant figure of merit: indeed, while P_d monotonically increases with the TOT for a fixed transmit power, it is questionable whether a series of tests with larger P_d but longer scan duration is desirable in surveillance operations [5, 7].

It is therefore meaningful to use FAR and DR for system optimization: specifically, DR can be maximized over the free system parameters, under a constraint on the maximum tolerable FAR level. Since there are K range bins, M azimuth bins, and N Doppler bins, the average number of false alarms in a scan is $KMNP_\text{fa}$, and

$$\text{FAR} = \frac{KMNP_\text{fa}}{T_\text{s}} = \frac{K}{T} P_\text{fa}. \tag{5}$$

This shows that constraining FAR is equivalent to constraining P_fa (for they are proportional to each other through the constant coefficient K/T), and the latter will be considered in the optimization problem. As to DR, instead, the average number of detections from the target in a scan is simply P_d, so that

$$\text{DR} = \frac{P_\text{d}}{T_\text{s}} = \frac{1}{MT}\frac{P_\text{d}}{N} \tag{6}$$

which shows that maximizing DR may be quite different from maximizing P_d: in particular, by adjusting the pulse train length N, we can now balance the contrasting needs of a short scan time and a long TOT (i.e., a large P_d).

The degrees of freedom for system optimization are the pulse train length N and the detection threshold γ: all the other parameters (number of range bins, number of azimuth bins, PRT, sample-size of the secondary data, transmit energy per pulse) are not adjusted. Therefore, letting α be the maximum tolerable level for the probability of false alarm, and denoting N_min and N_max, $N_\text{min} < N_\text{max}$, the minimum and maximum length of the pulse train,[1] the optimization problem tackled here is

$$\begin{aligned} &\max_{N \in \{N_\text{min}, \dots, N_\text{max}\}, \gamma \in \mathbb{R}} && \text{DR} \\ &\text{s.t.} && P_\text{fa} \leq \alpha \end{aligned} \tag{7}$$

which, from (4) and (6), can be restated as

$$\begin{aligned} &\max_{N \in \{N_\text{min}, \dots, N_\text{max}\}, \gamma \in \mathbb{R}} && \frac{1}{N}\left(1 + \frac{\gamma/L}{1+\rho N}\right)^{-L} \\ &\text{s.t.} && (1+\gamma/L)^{-L} \leq \alpha \end{aligned} \tag{8}$$

Since the objective function is decreasing with γ, the smallest threshold satisfying the constraint must be chosen,

[1] N_min is tied to the minimum accuracy level required for Doppler shift estimation, while N_max is dictated by the target velocity and is the largest value of N for which there is no range and/or azimuth and/or Doppler migration during the coherent processing inteval.

i.e.,

$$\gamma^* = L\big(\alpha^{-1/L} - 1\big) \tag{9}$$

and the optimum length of the pulse train can be computed as

$$N^* = \underset{N\in\{N_{\min},\dots,N_{\max}\}}{\arg\max}\ \frac{1}{N}\left(1+\frac{\alpha^{-1/L}-1}{1+\rho N}\right)^{-L}. \tag{10}$$

To solve (10), we introduce the continuously differentiable function

$$f(x) = \frac{1}{x}\left(1+\frac{\alpha^{-1/L}-1}{1+\rho x}\right)^{-L} \tag{11}$$

defined over $(0,+\infty)$, whose derivative is

$$f'(x) = \frac{-\left(1+\frac{\alpha^{-1/L}-1}{1+\rho x}\right)^{-L-1}}{x^2(1+\rho x)^2}\Big[\rho^2x^2 - \rho x \times \big((\alpha^{-1/L}-1)(L-1)-2\big) + \alpha^{-1/L}\Big]. \tag{12}$$

Observe that, if $\alpha \geq ((L-1)/(L+1))^{2L}$, $f'(x) \leq 0$ for any x and, if $\alpha < ((L-1)/(L+1))^{2L}$, $f'(x) < 0$ for $x \in (0,\bar{x}) \cup (x^*,\infty)$, $f'(x) = 0$ for $x \in \{\bar{x}, x^*\}$, and $f'(x) > 0$ for $x \in (\bar{x}, x^*)$, where

$$\bar{x} = \frac{(L-1)(\alpha^{-1/L}-1)-2}{2\rho} - \frac{\sqrt{(\alpha^{-1/L}-1)\big[(L-1)^2(\alpha^{-1/L}-1)-4L\big]}}{2\rho} > 0 \tag{13a}$$

$$x^* = \frac{(L-1)(\alpha^{-1/L}-1)-2}{2\rho} + \frac{\sqrt{(\alpha^{-1/L}-1)\big[(L-1)^2(\alpha^{-1/L}-1)-4L\big]}}{2\rho} > \bar{x}. \tag{13b}$$

This implies that the optimum length of the pulse train is

$$N^* = \begin{cases} N_{\min}, & \text{if } \alpha \geq \left(\frac{L-1}{L+1}\right)^{2L}, \text{ or if } \alpha < \left(\frac{L-1}{L+1}\right)^{2L} \\ & \text{and } x^* \leq N_{\min} \\ \underset{N\in\{N_{\min},N_{\max}\}}{\arg\max} \dfrac{\left(1+\frac{\alpha^{-1/L}-1}{1+\rho N}\right)^{-L}}{N}, & \text{if } \alpha < \left(\frac{L-1}{L+1}\right)^{2L} \\ & \text{and } x^* \geq N_{\max} \\ \underset{N\in\{N_{\min},\lfloor x^*\rfloor,\lceil x^*\rceil\}}{\arg\max} \dfrac{\left(1+\frac{\alpha^{-1/L}-1}{1+\rho N}\right)^{-L}}{N}, & \text{otherwise.} \end{cases} \tag{14}$$

It is worthwhile noticing that, unless the pair (α,ρ) requires $N^* = N_{\min}, N_{\max}$, the optimum pulse train length provides

$$P_{\mathrm{d}} \approx \left(1+\frac{\alpha^{-1/L}-1}{1+\rho x^*}\right)^{-L} = \left(1 - \frac{\frac{2}{1+\sqrt{\left(\frac{L-1}{L+1}\right)^2 - \frac{4L}{(L+1)^2(\alpha^{-1/L}-1)}}}}{L+1}\right)^{L}. \tag{15}$$

This value is increasing with α and spans the open interval $\big((1-2/(L+1))^L, (1-1/L)^L\big)$, so that $1/9 < P_{\mathrm{d}} < 1/e$ for any L and α.

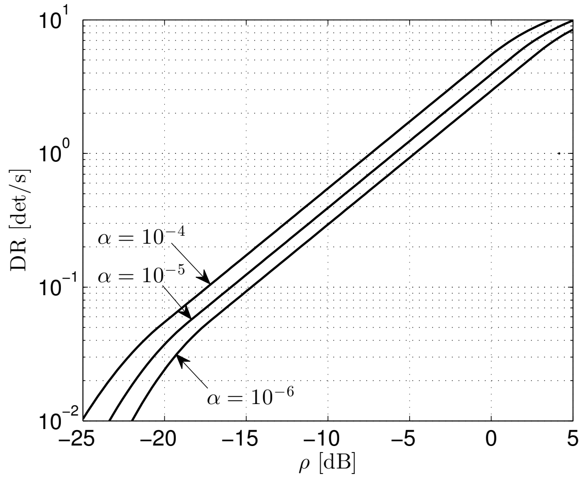

Fig. 1. Detection rate versus SNR per pulse for different values of α when $L = 15$, $\gamma = \gamma^*$, and $N = N^*$.

4. ANALYSIS

We consider a pulse radar where $K = 200$, $M = 60$, and $T = 100\,\mu$s. For this system, DR is optimized over N and γ according to Problem (7) when $N_{\min} = 10$ and $N_{\max} = 1000$.

Fig. 1 shows DR, measured in detections per seconds (det/s), versus ρ for different values of α when when $L = 15$, $\gamma = \gamma^*$, and $N = N^*$, while Figs. 2 and 3 report the corresponding values of P_{d} and T_{s}, respectively. Observe that for all inspected values of α, P_{d} is approximately constant and equal to the value in (15)—about 0.32—for a wide range of intermediate SNR's; at lower and higher SNR's, the probability of detection requires $N = N_{\max}$ and $N = N_{\min}$, respectively, and saturates towards 0 and 1, respectively. Fig. 4 shows DR versus ρ for different values of L when $\alpha = 10^{-6}$ $\gamma = \gamma^*$, and $N = N^*$. Clearly, when L increase, the estimate of the noise variance is more accurate, and the detection performance of the system improves.

Finally, the proposed optimization method is compared to the common procedure to choose N so as as to have P_{d} as close as possible to some desired value, say $\bar{P}_{\mathrm{d}}$, under the

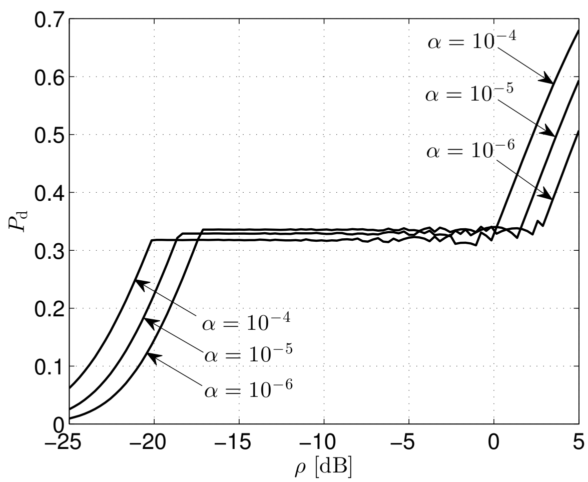

Fig. 2. Probability of detection versus SNR per pulse for different values of α when $L = 15$, $\gamma = \gamma^*$, and $N = N^*$.

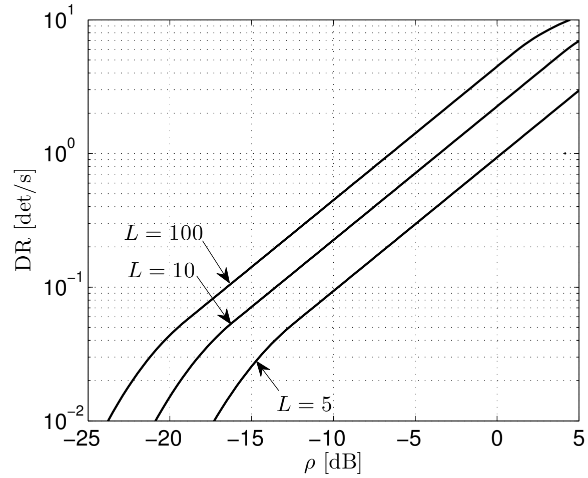

Fig. 4. Detection rate versus SNR per pulse for different values of L when $\alpha = 10^{-6}$, $\gamma = \gamma^*$, and $N = N^*$.

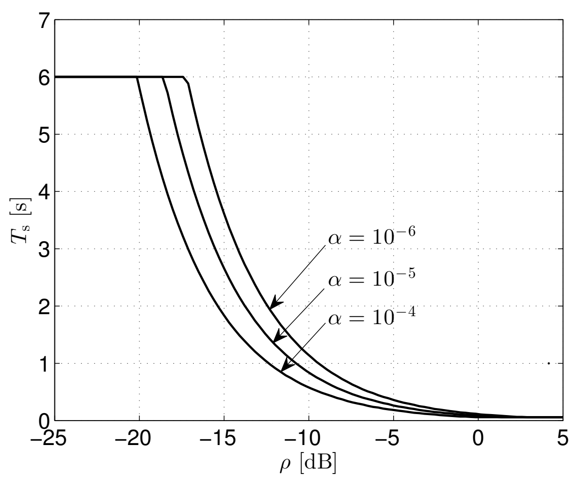

Fig. 3. Scan time versus SNR per pulse for different values of α when $L = 15$, $\gamma = \gamma^*$, and $N = N^*$.

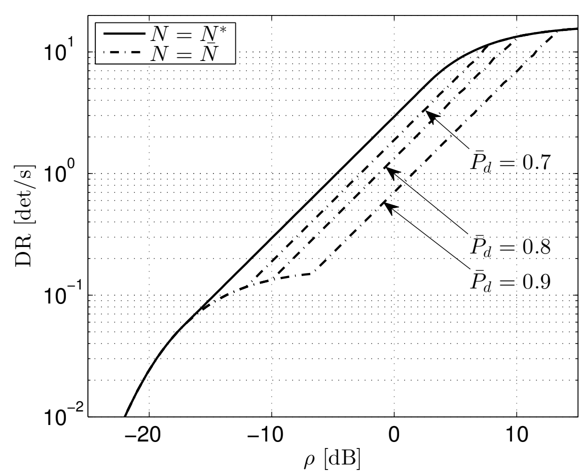

Fig. 5. Detection rate versus SNR per pulse for $N = N^*$, and for $N = \bar{N}$ with different values of $\bar{P}_\mathrm{d}$, when $L = 15$, $\alpha = 10^{-6}$, and $\gamma = \gamma^*$.

same constraint on P_fa. From (4), it can be easily seen that $\gamma = \gamma^*$, and that N must be set equal to

$$\bar{N} = \begin{cases} N_\mathrm{min}, & \text{if } \bar{y} \leq N_\mathrm{min} \\ N_\mathrm{max}, & \text{if } \bar{y} \geq N_\mathrm{max} \\ \underset{N \in \{\lfloor \bar{y} \rfloor, \lceil \bar{y} \rceil\}}{\arg\min} \left| \left(1 + \frac{\gamma/L}{1+\rho N}\right)^{-L} - \bar{P}_\mathrm{d} \right|, & \text{otherwise} \end{cases} \tag{16}$$

where $\bar{y} = \left((\bar{P}_\mathrm{d}/\alpha)^{1/L} - 1\right)/\rho$. Fig. 5 shows DR versus ρ for $N = N^*$, and for $N = \bar{N}$ with different values of $\bar{P}_\mathrm{d}$, when $L = 15$, $\alpha = 10^{-6}$, and $\gamma = \gamma^*$. For large and small values of ρ, both N^* and $\bar{N}$ saturate to N_min and N_max, respectively, so that DR is equal in both cases. At intermediate values of ρ, instead, requiring P_d as close as possible to 0.7, 0.8, and 0.9—notice that the optimal value in (15) is 0.34—results in a loss of about 2, 3.4, and 6.2 dB for DR $\in [0.15, 6.5]$ det/s.

5. CONCLUSION

In this work, the problem of maximizing the detection rate in radar systems with Gaussian observations and unknown noise power has been tackled. A closed-form solution for the optimum value of the pulse train length N has been provided, and the analysis has shown that it should be set so as to have a probability of detection approximately in the range $(1/9, 1/e)$ for a wide range of SNR, probability of false alarm, and sample-size of the secondary data.

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