APPLYING THE UNIT CIRCLE CONSTRAINT TO THE DIAGONALLY LOADED MINIMUM VARIANCE DISTORTIONLESS RESPONSE BEAMFORMER

*Colin Ryan   John R. Buck**

{cryan2, jbuck}@umassd.edu
Electrical and Computer Engineering Department
University of Massachusetts Dartmouth

ABSTRACT

Adaptive beamformers (ABF) are used when estimating the direction of arrival of a narrowband planewave with a uniform line array in an environment with loud, unwanted signals. Capon's minimum variance distortionless response (MVDR) suppresses interferers by steering beampattern nulls in their directions but maintains unity gain in the look direction. In practical situations, ABFs replace the ensemble covariance matrix (ECM) with a sample covariance matrix (SCM) estimated from array observations, or snapshots. If there are not enough snapshots available to average, the SCM is a poor estimate of the ECM, and may be rank deficient. Adding some diagonal loading (DL) to the SCM improves the beamformer's performance by increasing white noise gain, but may cost some in interferer suppression. Projecting the zeros of the beamformer's array polynomial onto the unit circle (UC) provides deeper nulls to suppress interferers but exhibits worse white noise behavior than DL. The UCDL beamformer adds DL to the SCM before applying UC constraint to calculate array weights that achieve better SINR than either the UC or DL beamformers alone. Even when the UCDL beamformer suffers mismatch on the DL level, the UCDL SINR rivals or betters the DL beamformer with the optimal DL level.

Index Terms— Adaptive Beamformer, MVDR, Diagonal Loading, Unit Circle, SINR

1. INTRODUCTION

Estimating the direction of arrival of a narrowband planewave using an N-element uniform line array (ULA) is a common array processing problem. In environments where loud interferers are present, adaptive beamformers (ABFs) use the noise only (or signal free) covariance matrix to calculate array weights that will attenuate interferers to minimize the output power of the system. A common ABF, proposed by Capon [2], maximizes SINR conditioned on keeping the mainlobe of the beampattern undistorted. Capon's beamformer, shown in (1) and (2), is often referred to as the minimum variance distortionless response (MVDR) beamformer where $\mathbf{R}$ is the noise only covariance matrix and $\mathbf{v}_0$ is the array manifold vector for the signal of interest.

$$\min_{\mathbf{w}} f(\mathbf{w}) = \mathbf{w}^H\mathbf{R}\mathbf{w}, \quad \text{while } \ \mathbf{w}^H\mathbf{v}_0 = 1 \tag{1}$$

$$\mathbf{w}_{MVDR} = \frac{\mathbf{R}^{-1}\mathbf{v}_0}{\mathbf{v}_0^H\mathbf{R}^{-1}\mathbf{v}_0} \tag{2}$$

*Supported by ONR Program grant N00014-15-1-2238.

The signal to interferer and noise ratio (SINR) is a common metric to evaluate an ABF's performance. The SINR is the ratio of the desired signal power present in the beamformer output to the total noise and interferer power in the beamformer output. Adaptive beamformers maximize SINR by attenuating loud interferers while reducing white noise power at the array's output. SINR can be written as:

$$\mathit{SINR} = \frac{\sigma_s^2|\mathbf{w}^H\mathbf{v}_0|^2}{\mathbf{w}^H\mathbf{R}_{Int}\mathbf{w} + \mathbf{w}^H\mathbf{R}_{Noise}\mathbf{w}} \tag{3}$$

For the case with a single interferer present, (3) simplifies to:

$$\mathit{SINR} = \frac{\sigma_s^2}{\sigma_{Int}^2 ND + \sigma_n^2/WNG} \tag{4}$$

where

$$\mathit{WNG} = \frac{1}{\mathbf{w}^2}, \quad \text{and} \quad ND = |\mathbf{w}^H\mathbf{v}_{Int}|^2, \tag{5}$$

where σ_s^2, σ_n^2, and σ_{Int}^2 are the power of the signal of interest, white noise, and interferer respectively, $\mathbf{R}_{Int}$ is the covariance matrix for the interferers, $\mathbf{R}_{Noise}$ is the covariance matrix for the background noise, and $\mathbf{v}_{Int}$ is the array manifold vector for the interferer. *ND* is the beampattern notch depth in the interferer direction and *WNG* is white noise gain from the array. Note that this equation can be expanded for scenarios with multiple interferers by adding the product of the input interferer power and the notch depth in that direction to the denominator.

This paper proposes a new adaptive beamformer that combines the common diagonal loading (DL) MVDR beamformer [1] with the recent unit circle (UC) MVDR beamformer proposed in [3] to design a beamformer whose interferer suppression is better than the DL beamformer and whose white noise gain is better than the UC beamformer. The resulting unit circle diagonally loaded (UCDL) MVDR beamformer achieves better SINR performance than either of those beamformers.

1.1. ABFs-Limitations

Calculating the optimal MVDR array weights requires theoretical knowledge about the environment, embodied in the ensemble covariance matrix (ECM). In practice, the ECM is not available, and is replaced in (2) by the sample covariance matrix (SCM). The SCM averages snapshots observed at the array to estimate the ECM. The beamformer substituting the SCM for the ECM in the MVDR equation is known as the Sample Matrix Inversion (SMI) beamformer [1]. As the number of snapshots (L) increases, the SCM becomes a more accurate estimate of the ECM, and the SMI beamformer's performance approaches the ideal ensemble performance. For this paper, we assume there is mismatch between the ECM and the SCM estimate from finite snapshots.

The SMI beamformer's performance is sensitive to the accuracy of the SCM's estimate of the ECM. This paper compares adaptive beamformers using SCMs built in three snapshot regimes: the snapshot deficient case assumes L<N, the snapshot starved case assumes L≈N, and the snapshot sufficient case implies L≥2N. Using a noise only covariance matrix improves the rate at which the SINR of an SMI beamformer converges to the ensemble SINR. Reed, Mallett, and Brennan [6] found the expected value of SINR for the SMI beamformer with 2N snapshots is 3 dB less than the SINR for the optimal MVDR with perfect knowledge of the ECM. This paper focuses on the scenario with signal free snapshots to form the SCM used to compute the beamformer weights. This is what Van Trees defines as the MVDR scenario, in contrast to the MPDR scenario including the desired signal in the training data [1, Sec. 6.2.1]. The SMI beamformer cannot be used in snapshot deficient scenarios because the SCM will not be full rank, a necessary property for matrix inversion.

The diagonally loaded MVDR beamformer is a more robust ABF that can be used in snapshot deficient scenarios [1]. DL beamformer adds a small diagonal component to the SCM to ensure it is full rank before inversion. In the extreme asymptotes, adding no diagonal loading to the SCM converges to the SMI beamformer and adding infinite diagonal loading to the SCM converges to the CBF. Mestre and Lagunas [4] proposed a search algorithm that uses the ECM and the ratio of sensors to snapshots to calculate the optimal diagonal loading factor to maximize expected SINR for a SMI beamformer. This optimal SINR comes with the cost of computational complexity in the search algorithm even with perfect knowledge of the covariance matrix.

The unit circle MVDR beamformer exploits properties of the array polynomial to achieve better interferer suppression than the DL and SMI beamformers [3]. The array polynomial is the z-transform of a beamformer's array weights. Steinhardt and Guerci found that for planewave beamforming using a ULA, the roots of the MVDR array polynomial are constrained to the unit circle whereas the SMI roots are not [5]. Tuladhar and Buck [3] proposed that radially projecting the zeros of the array polynomial for the SMI weights back to the unit circle improves planewave interferer suppression over SMI and DL beamformers. The UC beamformer requires a full rank SCM so the underdetermined matrix problem remains in a snapshot deficient case.

This paper proposes an adaptive beamformer combining the UC and DL beamformers to form a more robust beamformer called the unit circle diagonally loaded (UCDL) MVDR beamformer. For the DL and UC beamformers, the strength of one beamformer is the weakness of the other. The UC beamformer does well at suppressing interferers but has less white noise gain. The DL beamformer does well at suppressing white noise, but is less effective at suppressing interferers. The UCDL beamformer improves SINR by diagonally loading the SCM before projecting the array polynomial zeros back to the unit circle. The diagonal loading allows the UCDL beamformer to work in snapshot deficient scenarios where the UC beamformer cannot. Finally, the UCDL beamformer is robust, maintaining its high SINR even with mismatch within the ideal DL factor.

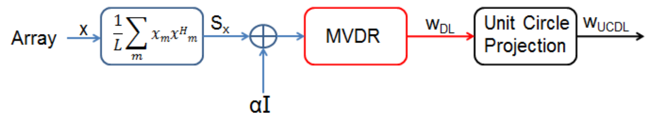

Fig. 1. UCDL MVDR block diagram.

2. UCDL ALGORITHM

Fig. 1 shows a block diagram of the UCDL beamformer from snapshots to array weights, and Algorithm 1 provides detailed pseudocode. The colors of the blocks in Fig. 1 correspond to the lines of the same color in Algorithm 1. Lines 1-2 compute the standard DL MVDR weights. Note the unity gain constraint is omitted for these initial weights as it does not impact the zero locations and the final steps impose the unity gain constraint after moving the zeros. Line 3 computes the z-transform of the weight vector $\mathbf{w}_{DL}$, and line 4 finds the zeros of that polynomial. Lines 5-11 project each zero to the unit circle (setting $r_n = 1$). Any zeros falling within the mainlobe CBF width are shifted to the edge of the mainlobe. Line 12 constructs the array polynomial $P_{UC}(z)$ guaranteeing unity gain in the look direction. Finally, line 13 computes the beamformer weights as the inverse z-transform of the UCDL array polynomial.

Algorithm 1 UCDL MVDR adaptive beamformer

Input: Snapshot vectors $\mathbf{x}_1, \ldots, \mathbf{x}_L$, diagonal loading α steering vector $\mathbf{v}_0$, with look direction u_0

Output: UCDL Beamformer weights $\hat{w}_1, \ldots, \hat{w}_N$

1: $\mathbf{S}_{DL} \leftarrow (1/L) \sum_{\ell=1}^{L} \mathbf{x}_\ell \mathbf{x}_\ell^H + \alpha \mathbf{I}$
2: $\mathbf{w}_{DL} \leftarrow \mathbf{S}_{DL}^{-1} \mathbf{v}_0$
3: $P_{DL}(z) \leftarrow \mathcal{Z}\{\mathbf{w}_{DL}\}$
4: $\xi_1, \ldots, \xi_{N-1} \leftarrow \text{roots}\{P_{DL}(z)\}$
5: **for all** $\xi_n = r_n \exp\{j\omega_n\}$ **do**
6:   **if** $|\omega_n - \pi u_0| > 2\pi/N$ **then**
7:     $\hat{\xi}_n \leftarrow \exp\{j\omega_n\}$
8:   **else**
9:     $\hat{\xi}_n \leftarrow \exp\{j(\pi u_0 + \text{sgn}(\omega_n - \pi u_0)(2\pi/N))\}$
10:   **end if**
11: **end for**
12: $P_{UC}(z) \leftarrow \prod_{n=1}^{N-1} \left(1 - \hat{\xi}_n z^{-1}\right) / \left(1 - \hat{\xi}_n \exp\{-j\pi u_0\}\right)$
13: $\hat{w}_1, \ldots, \hat{w}_N \leftarrow \mathcal{Z}^{-1}\{P_{UC}(z)\}$

3. RESULTS

The adaptive beamformers' performances were simulated with 1000 Monte Carlo trials of an N=11 sensor ULA with half-wavelength spacing. The SINR was computed for each Monte Carlo trial, as well as the interferer and white noise contributions to the ABF output power. The array was steered to broadside ($u = \cos\theta = 1$), while the interferer was located at ($u_{Int} = \cos\theta_{Int} = 3/N$) with power 40 dB above the background noise. The desired signal at broadside had 0 dB sensor level SNR. The simulations evaluated the SMI, DL, UC, and UCDL beamformers for the snapshot starved and sufficient scenarios (L=12 and 22 respectively) and the DL and UCDL beamformers for the snapshot deficient case (L=5). Both the DL and UCDL beamformers use the optimal diagonal loading value

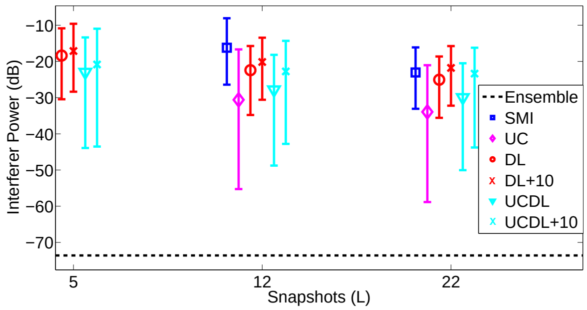

Fig. 2. Comparing various beamformer's ability to suppress a loud interferer as a function of snapshots available. Plot markers indicate median interferer power (in dB) with the spread indicating the 90% confidence interval. The black line represents the output interferer power with perfect knowledge of the ECM, shown for comparison.

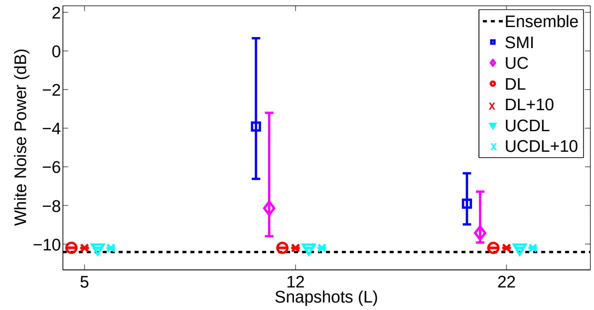

Fig. 3. Comparing various beamformer's white noise power output. Plot markers indicate median interferer power (in dB) with the spread indicating a 90% confidence interval. The black line represents the output white noise power for a CBF, shown for comparison.

α computed from the ECM with Mestre and Lagunas' algorithm [4] to provide the best case performance. Additionally, the simulations show results for the DL and UCDL beamformers when the diagonal loading factor is overestimated by 10 dB relative to the optimal DL to determine the beamformers' robustness to mismatch in DL level.

Figs. 2, 3 and 4 compare output power or SINR versus snapshots for the different ABFs studied. The horizontal axis of the graph groups the ABFs by number of snapshots in the SCM while the vertical axis shows output power in dB. The beamformers are distinguished by color and marker type, with the ABFs that have mismatched DL values sharing the same marker. The markers represent the median value for 1000 trials and the spread of the line shows the 90% confidence interval. The black dashed line is the optimal output power for the array computed from the ECM, providing a lower bound on output power or an upper bound on SINR.

3.1. Interferer Power

Fig. 2 compares the interferer power contribution to the ABF's outputs. Tuladhar and Buck [3] showed and Fig. 2 verifies that the UC beamformer has lowest median output interferer power for $L \geq N$. The UCDL beamformer has slightly higher interferer power than UC beamformer, but maintains a lower median value than DL and SMI beamformers. The advantage of the UC beamformer's lower median power is undermined somewhat by its larger confidence interval compared with the other beamformers.

3.2. White Noise Power

Fig. 3 compares the white noise power contribution to the beamformer's outputs. Reducing white noise power by as much as the number of sensors is an integral advantage to using sensor arrays. The SMI and UC beamformers struggle to reach the ideal white noise power output unlike the DL beamformer which maintains white noise power close to the lower bound regardless of snapshot availability. This is intuitively reasonable because the DL beamformer approaches the CBF as the DL factor approaches infinity. As the DL factor increases by 10 dB, the DL beamformer maintains its low white noise power, even seeing slight improvements. UCDL beamformer's white noise power follows the DL beamformer, remaining close to the lower bound. Even when the DL level is overestimated by 10 dB, the UCDL beamformer sustains low white noise power output.

3.3. SINR

Results for 1000 Monte Carlo simulations (see Fig. 4) show the UCDL beamformer has higher SINR than SMI, UC, and DL beamformers for all threes snapshot scenarios examined. This improvement is seen both in the median and spread of SINR values. UCDL's most visible SINR gains are seen when the beamformer is starved for snapshots which occurs in many practical applications. Diagonally loading the SCM with a factor that is chosen to maximize SINR helps the UCDL beamformer maintain stability while reducing white noise power. Radially projecting the zeros of this diagonally loaded array polynomial back to the unit circle minimizes the output interferer power.

The UCDL beamformer is also robust to mismatch in the DL factor chosen to maximize SINR. In practical situations where the ECM is not available, the DL factor would be based on the SCM and the estimate may be above or below the optimal level. Fig. 4 shows that the SINR for both DL and UCDL beamformers when the DL factor is 10 dB higher than the optimal factor calculated using the Mestre and Lagunas method in [4]. For the three snapshot scenarios, mismatch UCDL beamformer has higher median SINR than mismatch DL beamformer. Interestingly, the mismatched UCDL beamformer has higher median SINR than the DL beamformer with the optimal loading factor for low snapshot scenarios. At first glance, it appears impossible that another ABF could outperform the optimal DL beamformer. However, the algorithm in [4] optimizes only over the class of DL beamformers using the SCM, not over all possible beamformers. The UCDL falls outside of the class optimized over.

3.4. Mismatched Diagonal Loading Factor

The UCDL beamformer is more robust to mismatch in DL factor than the DL beamformer. Fig. 5 shows the effect of different degrees of DL level mismatch on median SINR over 1000 trials for the DL and UCDL beamformer algorithms. The horizontal of the figure shows the ratio, in dB, of the DL factor added to the SCM to the optimal DL factor. Note the optimal DL value α is different for each of the 3 snapshot scenarios portrayed, but this figure focuses on the relative mismatch. The dashed line shows the ideal

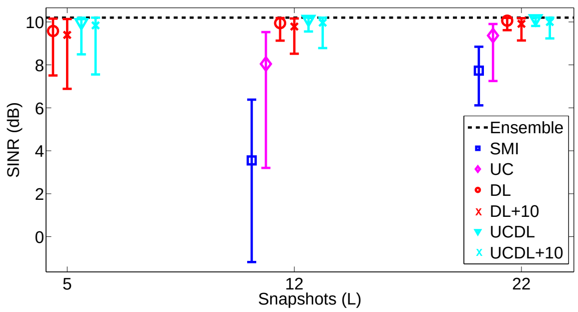

Fig. 4. Output SINR for various beamformers including new UCDL algorithm. UCDL has the highest median SINR for the three snapshot scenarios. UCDL also approaches the SINR found when using the ECM on the top end of its spread.

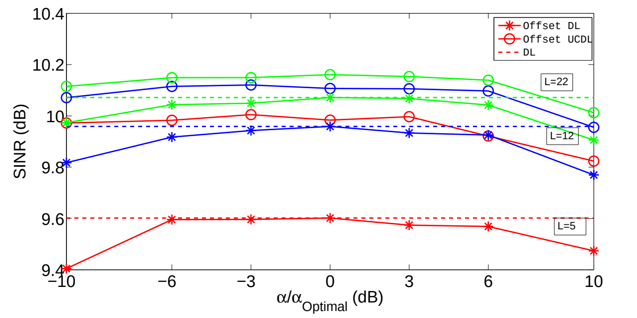

Fig. 5. Comparison of DL (stars) and UCDL (circles) output SINR for mismatch in DL level α for the snapshot deficient L=N/2 (red), snapshot starved L=N+1 (blue) and snapshot sufficient L=2N (green) scenarios. UCDL always outperforms DL with the same level of mismatch, and often mismatched UCDL even outperforms the optimal DL beamformer.

value for the optimal DL beamformer's SINR for the three snapshot scenarios. The different colors represent the snapshot deficient (red), snapshot starved (blue) and snapshot sufficient (green) scenarios, while symbol types represent the different beamformers, with DL (stars) and UCDL (circles). The DL beamformer's maximum SINR occurs when the offset is 0 dB as expected and decreases as the factor moves outwards in both directions (positive and negative) for the three snapshot cases.

The UCDL beamformer has higher SINR than the DL beamformer for the same mismatch factor in all cases from -10 dB to 10 dB mismatch. For the snapshot deficient (L=N/2) and snapshot starved (L=N+1) cases, the UCDL beamformer with mismatched DL from -10 to +10 dB performed better than the optimal DL beamformer. Similarly, even in the snapshot sufficient case (L=2N), the UCDL beamformer performed better for mismatch from -10 to 8.15 dB than the optimal DL beamformer. Finally, this figure shows that for this test scenario with one loud interferer at the peak sidelobe direction of the CBF, UCDL beamformer has higher SINR, even with some mismatch in DL factor, than optimal DL beamformer using twice as many snapshots. By inspection, UCDL for L=N/2 and mismatch values ranging from -10 dB to at least 3 dB has higher SINR than optimal DL beamformer for L=N+1 snapshots. This trend continues for the case where UCDL beamformer estimates the SCM from L=N+1 snapshots and DL beamformer uses L=2N snapshots.

4. CONCLUSION

This paper proposed the UCDL MVDR beamformer which combines diagonal loading with the unit circle constraint on MVDR array polynomial zeros to improve SINR. Numerical simulations verify that the UCDL beamformer has higher SINR than SMI, UC, and DL beamformers for the three snapshot scenarios analyzed, showing substantial improvement in the snapshot deficient and starved cases. Additionally, the UCDL beamformer is robust to mismatch in DL factor, maintaining higher SINR than the optimal DL beamformer when the UCDL beamformer's diagonal loading is off by up to 10 dB from the optimal factor.

5. REFERENCES

[1] Van Trees, H. L., *Estimation, and Modulation Theory, Optimum Array Processing,* John Wiley and Sons, 2002.

[2] Capon, J., “High-resolution frequency-wavenumber spectrum analysis,” *Proceedings of the IEEE,* vol. 57, no. 8, pp. 14081418, Aug. 196

[3] Tuladhar, S.R. and Buck, J.R. “Unit Circle MVDR Beamformer,” *2015 IEEE International Conference, Acoustics, Speech, and Signal Processing (ICASSP),* April 2015.

[4] X. Mestre and M. A. Lagunas, ”Finite Sample Size Effect on Minimum Variance Beamformers: Optimum Diagonal Loading Factor for Large Arrays,” in IEEE Transactions on Signal Processing, vol. 54, no. 1, pp. 69-82, Jan. 2006.

[5] A. Steinhardt and J. Guerci, ”STAP for RADAR: what works, what doesn't, and what's in store. IEEE radar conference, 2004,” Proceedings of the IEEE 2004 Radar Conference, 2004, pp. 469-473.

[6] I. S. Reed, J. D. Mallett and L. E. Brennan, ”Rapid Convergence Rate in Adaptive Arrays,” in IEEE Transactions on Aerospace and Electronic Systems, vol. AES-10, no. 6, pp. 853-863, Nov. 1974.