PERFORMANCE ANALYSIS OF AN AOA ESTIMATOR IN THE PRESENCE OF MORE MUTUAL COUPLING PARAMETERS

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ABSTRACT

The problem of Angle-of-Arrival estimation of multiple sources in the presence of mutual coupling is addressed. In this paper, we derive a Mean-Squared-Error (MSE) expression of a recently proposed algorithm that could estimate the Angles-of-Arrival of multiple sources in the presence of more mutual coupling parameters, compared to traditional methods. The MSE expression is compared with the MSE of MUSIC with known mutual coupling parameters and to the *Cramer-Rao* bound (CRB) of any unbiased estimator that estimates the Angles-of-Arrival in the presence of mutual coupling. It is shown that the proposed method is asymptotically unbiased. In addition, it is also shown that the method attains CRB for large number of antennas with fixed coupling parameters and uncorrelated sources. For high SNR, the CRB is not necessarily attained, however, we study the gap between the derived MSE and the CRB.

Index Terms— Angle-of-Arrival, Performance Analysis, Mutual Coupling, Mean-Squared-Error

1. INTRODUCTION

The presence of mutual coupling is a well-known problem in the context of array signal processing. This problem arises when antennas are close to each other [1], and thus the current developed in an antenna element depends on its own excitation and on the contributions from adjacent antennas. As a consequence, the performance of high resolution algorithms that perform Angle-of-Arrival (AoA) estimation, such as MUSIC [2], ESPRIT [3], etc., degrades in a significant manner. It is also worth mentioning other phenomena that perturb an ideal model, when not taken into account, such as different gain/phases [4] across antennas, synchronization and jitter effect [5], local scattering [6], etc. In the open literature, calibration methods that tackle the problem of mutual coupling are divided into two categories: Offline and Online. In an offline calibration approach, one estimates the mutual coupling parameters using known locations, such as the techniques in [7–9]. In contrast, online calibration consists of jointly estimating the coupling and AoA parameters. In this paper, our main focus is on the latter.

In the literature, several techniques deal with the online calibration problem. A RAnk-REduction estimator, known as RARE, was first proposed in [10] in the context of partly calibrated arrays. The same idea was used for totally uncalibrated Uniform Circular Arrays (UCA) in [11, 12] and Uniform Linear Arrays (ULA) in [13, 14]. This method makes use of the MUSIC algorithm to estimate AoAs in the presence of mutual coupling via rank reduction of an appropriate matrix. The method in [14] is a Recursive-RARE (R-RARE), which was shown to achieve better perform than the traditional RARE. Other techniques we could mention are those in [15–18]. From a performance analysis point of view, several authors have studied the effect of mutual coupling, or modelling errors in the more general case, such as [19, 20]. More specifically, these modeling errors induce a bias in the MUSIC estimator and could result in large mean-squared errors, when the errors are large enough [20].

In this paper, we derive a Mean-Squared-Error (MSE) expression of a recently proposed algorithm in [25] that could estimate the Angles-of-Arrival of multiple sources in the presence of more mutual coupling parameters, compared to the above mentioned methods. More specifically, let N and p denote the number of antennas and coupling parameters, respectively. It was shown in [23] that the above methods, except for [18], do not function properly when $p > \frac{N}{2}$. Furthermore, we have proposed an algorithm that is able to estimate the AoAs, even when $p > \frac{N}{2}$. After deriving the MSE of the proposed method, we study the performance loss, compared to the MUSIC algorithm with known mutual coupling parameters. Also, the MSE expression is compared with the *Cramer-Rao* bound of any unbiased estimator that estimates the Angles-of-Arrival in the presence of mutual coupling. It is shown that the proposed method is asymptotically unbiased and attains the *Cramer-Rao* bound for large number of antennas with fixed coupling parameters and uncorrelated sources. For high SNR, the *Cramer-Rao* bound is not necessarily attained, however, we study the gap between the between the MSE and the *Cramer-Rao* bound. Simulation results gave validated the derived MSE expression, as the experimental and theoretical (or derived) MSE quite agree for sufficiently high SNR.

Notations: Upper-case and lower-case boldface letters denote matrices and vectors, respectively. $(.)^{\mathrm{T}}$, $(.)^*$ and $(.)^{\mathrm{H}}$ represent the transpose, conjugate and the transpose-conjugate operators. The matrix $\mathbf{I}$ is the identity matrix with suitable dimensions. For any matrix $\mathbf{B}$, the $(i,j)^{th}$ entry of $\mathbf{B}$ is represented as $(\mathbf{B})_{i,j}$. The vector $\mathbf{e}_k$ is the k^{th} column of $\mathbf{I}$. For any matrix $\mathbf{B}$, the operator $\|\mathbf{B}\|$ denotes the *Frobenius* norm. Also $\mathbf{B}^+$ denotes pseudo-inverse of $\mathbf{B}$. The statement $X \implies Y$ means that "if statement X is true, then Y is true."

2. SYSTEM MODEL

Consider q narrowband sources impinging a Uniform Linear Array (ULA), composed of $N > q$ antennas. The angles are denoted as

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$\mathbf{\Theta} = [\theta_1 \dots \theta_q]$. Given L time snapshots, we can write

$$\mathbf{X} = \bar{\mathbf{A}}(\mathbf{\Theta})\mathbf{S} + \mathbf{W} \tag{1}$$

where $\mathbf{X} \in \mathbb{C}^{N\times L}$ is the data matrix with l^{th} time snapshot, $\mathbf{x}(t_l)$, stacked in the l^{th} column of $\mathbf{X}$. The matrix $\mathbf{S} \in \mathbb{C}^{q\times L}$ is the source matrix. Similar to $\mathbf{X}$, matrix $\mathbf{S}$ contains the l^{th} transmitted source vector $s(t_l)$ in its l^{th} column. The matrix $\mathbf{W} \in \mathbb{C}^{N\times L}$ is background noise. Moreover, the steering matrix $\bar{\mathbf{A}}(\mathbf{\Theta}) \in \mathbb{C}^{N\times q}$ is composed of q steering vectors, i.e. $\bar{\mathbf{A}}(\mathbf{\Theta}) = [\bar{\mathbf{a}}(\theta_1)\dots\bar{\mathbf{a}}(\theta_q)]$, where $\bar{\mathbf{a}}(\theta_i)$ is the array response in the presence of mutual coupling. Moreover, in the absence of mutual coupling, the response is $\mathbf{a}(\theta)$, where its k^{th} entry is z_θ^{k-1} and $z_\theta = e^{-j2\pi\frac{d}{\lambda}\sin(\theta)}$, d is the inter-element spacing and λ is the wavelength. Following [1,24], we can say

$$\bar{\mathbf{a}}(\theta) = \mathcal{T}_p(\mathbf{m})\mathbf{a}(\theta) \tag{2}$$

where $\mathcal{T}_p(\mathbf{m}) \in \mathbb{C}^{N\times N}$ is a banded symmetric Toeplitz matrix, i.e.

$$\big(\mathcal{T}_p(\mathbf{m})\big)_{i,j} = \begin{cases} m_{|i-j|} & \text{if } |i-j| < p \\ 0 & \text{else} \end{cases} \tag{3}$$

Note that the matrix $\mathcal{T}_p(\mathbf{m})$ is independent from the AoAs. The model in equations (2) and (3) suggest that antennas that are placed at least p inter-element spacings apart do not interfere, i.e. $m_i = 0$ for all $i \geq p$.

Throughout the paper, we assume the following:

- **A1**: $\bar{\mathbf{A}}(\mathbf{\Theta})$ is full column rank.
- **A2**: The transmitted signals $\mathbf{s}(t_l)$ are fixed within a snapshot. The signals are allowed to be highly, but not fully, correlated.
- **A3**: The number of sources is known.
- **A4**: The vector $\mathbf{w}(t_l)$ is Gaussian noise with zero mean and covariance $\sigma^2\mathbf{I}$ and independent from the sources.

We are now ready to address our problem:

"Given $\mathbf{X}$, p and q, estimate the AoAs $\mathbf{\Theta}$ in the presence of mutual coupling $\mathcal{T}_p(\mathbf{m})$."

3. PROPOSED ALGORITHM

This section makes use of the MUSIC algorithm in order to estimate the angles of arrivals $\mathbf{\Theta}$ in the presence of mutual coupling. We start by exploiting the structure of the received signal covariance matrix. Under assumption **A4**, we have

$$\mathbf{R_{xx}} = \mathrm{E}\{\mathbf{x}(\mathrm{t})\mathbf{x}^{\mathrm{H}}(\mathrm{t})\} = \bar{\mathbf{A}}(\mathbf{\Theta})\mathbf{R_{ss}}\bar{\mathbf{A}}^{\mathrm{H}}(\mathbf{\Theta}) + \sigma^2\mathbf{I} \tag{4}$$

By spectral decomposition, we can write $\mathbf{R_{xx}}$ as

$$\begin{aligned}\mathbf{R_{xx}} &= \left[\begin{array}{c|c} \mathbf{U_s} & \mathbf{U_n}\end{array}\right]\left[\begin{array}{c|c}\mathbf{\Sigma_s} & \mathbf{0}\\ \hline \mathbf{0} & \mathbf{\Sigma_n}\end{array}\right]\left[\begin{array}{c|c} \mathbf{U_s} & \mathbf{U_n}\end{array}\right]^{\mathrm{H}} \\ &= \mathbf{U_s\Sigma_sU_s^{\mathrm{H}}} + \mathbf{U_n\Sigma_nU_n^{\mathrm{H}}}\end{aligned} \tag{5}$$

Matrix $\mathbf{R_{xx}}$ is composed of two major parts: The signal part, namely $\bar{\mathbf{A}}(\mathbf{\Theta})\mathbf{R_{ss}}\bar{\mathbf{A}}^{\mathrm{H}}(\mathbf{\Theta})$, which under assumptions **A1** and **A2**, is rank q. Therefore, due to $\sigma^2\mathbf{I}$ in equation (4), the matrix $\mathbf{\Sigma_s}$ is a $q\times q$ diagonal matrix composed of eigenvalues strictly greater than σ^2. The corresponding eigenvectors of $\mathbf{\Sigma_s}$, which are the columns of $\mathbf{U_s}$ is called the signal subspace. The second part is the noise subspace, whose eigenvalues are easily verified to be $\mathbf{\Sigma_n} = \sigma^2\mathbf{I}$, and eigenvectors are the columns of $\mathbf{U_n}$. The MUSIC algorithm is based on the orthogonality between the two subspaces, viz.

$$\bar{\mathbf{a}}^{\mathrm{H}}(\theta_i)\mathbf{U_n}\mathbf{U_n^{\mathrm{H}}}\bar{\mathbf{a}}(\theta_i) = 0, \qquad \text{for all } i = 1\dots q. \tag{6}$$

In practical scenarios, one estimates the covariance matrix thru sample averaging, i.e. $\widehat{\mathbf{R}}_{\mathbf{xx}} = \frac{1}{L}\mathbf{X}\mathbf{X}^{\mathrm{H}}$. The MUSIC algorithm estimates $\mathbf{\Theta}$ as follows

$$\{\widehat{\theta}_i\}_{i=1}^q = \arg\max_\theta \frac{1}{\bar{\mathbf{a}}^{\mathrm{H}}(\theta)\widehat{\mathbf{U}}_{\mathbf{n}}\widehat{\mathbf{U}}_{\mathbf{n}}^{\mathrm{H}}\bar{\mathbf{a}}(\theta)} \tag{7}$$

where $\widehat{\mathbf{U}}_{\mathbf{n}}$ is an estimate of $\mathbf{U_n}$ and is extracted from $\widehat{\mathbf{R}}_{\mathbf{xx}}$ by eigendecomposition. Unfortunately, MUSIC doesn't directly apply as in equation (7) because the functional form of $\bar{\mathbf{a}}(\theta)$ is not known. Fortunately, the following theorem turns out to be useful:

Lemma: *Let $\boldsymbol{\alpha} = [\alpha_0, \alpha_1 \dots \alpha_{p-1}]^T$ and $\mathbf{a} \in \mathbb{C}^{N\times 1}$. Define the corresponding matrix $\mathcal{T}_p(\boldsymbol{\alpha})$, then*

$$\mathcal{T}_p(\boldsymbol{\alpha})\mathbf{a} = \mathbf{B}_p\boldsymbol{\alpha} \tag{8}$$

where

$$\mathbf{B}_p = \left[\begin{array}{c|c|c|c} \mathbf{a} & \mathbf{\Pi}_1\mathbf{a} & \dots & \mathbf{\Pi}_{p-1}\mathbf{a}\end{array}\right] \tag{9}$$

where $\mathbf{\Pi}_k \in \mathbb{C}^{N\times N}$ is an all-zero matrix except at the k^{th} sub- and super-diagonals, which are set to 1.

Proof. See [17,24]. □

Therefore, we can say $\bar{\mathbf{a}}(\theta) = \mathcal{T}_p(\mathbf{m})\mathbf{a}(\theta) = \mathbf{B}(\theta)\mathbf{m}$, where $\mathbf{B}(\theta)$ is defined as in equation (9). Note the equation (6) could be re-written as

$$\mathbf{z}^{\mathrm{H}}\mathbf{K}(\theta)\mathbf{z} = 0 \Longrightarrow \{\theta \in \mathbf{\Theta} \text{ and } \mathbf{z} = \mathbf{m}\} \tag{10}$$

where $\mathbf{K}(\theta) = \mathbf{B}^{\mathrm{H}}(\theta)\mathbf{U_n}\mathbf{U_n^{\mathrm{H}}}\mathbf{B}(\theta)$. Therefore, one way to formulate the problem of estimating the AoAs in the presence of mutual coupling is to

$$\begin{cases} \underset{\mathbf{z},\theta}{\text{minimise}} & \mathbf{z}^{\mathrm{H}}\widehat{\mathbf{K}}(\theta)\mathbf{z} \\ \text{subject to} & \mathbf{e_1^{\mathrm{H}}}\mathbf{B}(\theta)\mathbf{z} = 1 \end{cases} \tag{11}$$

where $\widehat{\mathbf{K}}(\theta) = \mathbf{B}^{\mathrm{H}}(\theta)\widehat{\mathbf{U}}_{\mathbf{n}}\widehat{\mathbf{U}}_{\mathbf{n}}^{\mathrm{H}}\mathbf{B}(\theta)$. The constraint prevents the vector $\mathbf{z}$ to fall in the null-space of $\mathbf{B}(\theta)$. The solution to the above optimisation problem is given as [23]

$$\big\{\widehat{\theta}_k\big\}_{k=1}^q = \arg\min_\theta \frac{1}{f(\theta)} \tag{12a}$$

where

$$f(\theta) = \mathbf{a}_p^{\mathrm{T}}(\theta)\widehat{\mathbf{K}}^{-1}(\theta)\mathbf{a}_p^*(\theta) \tag{12b}$$

where $\mathbf{a}_p(\theta)$ is a $p\times 1$ vector defined as in equation (2). For more information on why the proposed method could estimate AoAs in the presence of more coupling parameters given that $p+q \leq N$, the reader is referred to [23, 25].

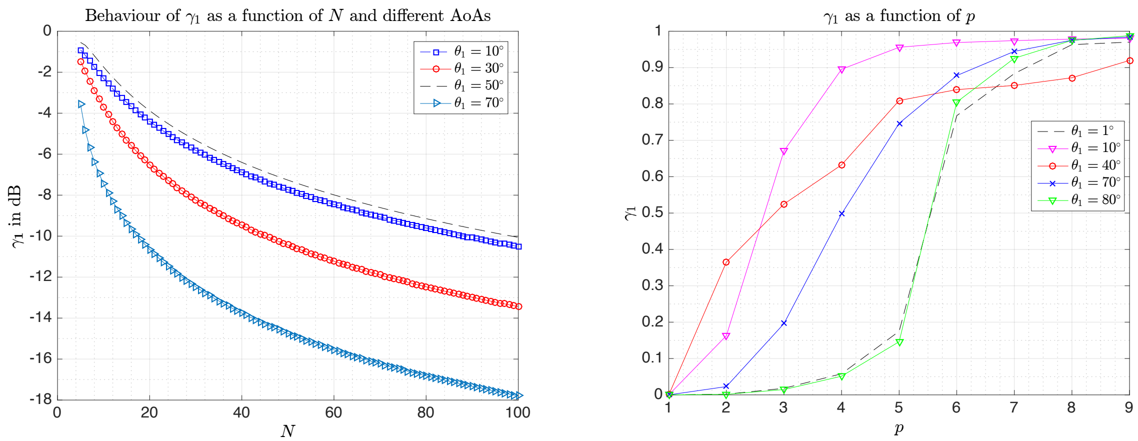
4. PERFORMANCE ANALYSIS AND COMPARISON WITH CRAMER-RAO BOUND

Theorem 1: *The estimates $\big\{\widehat{\theta}_k\big\}_{k=1}^q$ estimated as in equation (12) are asymptotically unbiased. Furthermore, the MSE expression $\mathbb{E}\big\{(\widetilde{\theta}_k)^2\big\}$, where $\widetilde{\theta}_k = \theta_k - \widehat{\theta}_k$ is the estimation error on θ_k, is given as*

$$\mathbb{E}\big\{(\widetilde{\theta}_k)^2\big\} \triangleq \mathrm{var}_f^{(p)}(\widehat{\theta}_k) = \frac{\sigma^2}{2L}\frac{\bar{\mathbf{a}}^{\mathrm{H}}(\theta_k)\mathbf{U}\bar{\mathbf{a}}(\theta_k)}{\bar{\mathbf{d}}^{\mathrm{H}}(\theta_k)\mathbf{U_n}\mathbf{P_k^{\perp}}\mathbf{U_n^{\mathrm{H}}}\bar{\mathbf{d}}(\theta_k)} \tag{13}$$

where

$$\mathbf{U} = \mathbf{U_s}\mathbf{\Sigma_s}\big(\mathbf{\Sigma_s} - \sigma^2\mathbf{I}\big)^{-2}\mathbf{U_s^{\mathrm{H}}} \tag{14}$$



(a) $q = 1$ (b) $N = 10$

Fig. 1: The behaviour of γ_k

and $\mathbf{P}_{\mathbf{k}}^{\perp} = \mathbf{I} - \mathbf{P}_{\mathbf{k}}$ *with*

$$\mathbf{P}_{\mathbf{k}} = \mathbf{U}_{\mathbf{n}}^{\mathrm{H}}\mathbf{B}(\theta_k)\mathbf{K}^{+}(\theta_k)\mathbf{B}^{\mathrm{H}}(\theta_k)\mathbf{U}_{\mathbf{n}} \tag{15}$$

Also $\bar{\mathbf{d}}(\theta_k) = \frac{\partial}{\partial \theta_k}\bar{\mathbf{a}}(\theta_k)$.

Proof. See [23]. □

It is interesting to see that when $p = 1$, the above MSE expression coincides with the MSE expression of MUSIC derived in [21],

$$\mathrm{var}_f^{(1)}(\widehat{\theta}_k) = \frac{\sigma^2}{2L}\frac{\mathbf{a}^{\mathrm{H}}(\theta_k)\mathbf{U}\mathbf{a}(\theta_k)}{\mathbf{d}^{\mathrm{H}}(\theta_k)\mathbf{U}_{\mathbf{n}}\mathbf{U}_{\mathbf{n}}^{\mathrm{H}}\mathbf{d}(\theta_k)} = \mathrm{var}_{\mathrm{MU}}(\widehat{\theta}_k;\mathbf{a}) \tag{16}$$

where $\mathrm{var}_{\mathrm{MU}}(\widehat{\theta}_k;\mathbf{a})$ is the variance (or MSE) of $\widehat{\theta}_k$ obtained by MUSIC by utilising a steering vector $\mathbf{a}(\theta)$. We adopt this notation because the MSE expression, $\mathrm{var}_f^{(p)}(\widehat{\theta}_k)$, could be also written as

$$\mathrm{var}_f^{(p)}(\widehat{\theta}_k) = \left(\frac{1}{1-\gamma_k}\right)\mathrm{var}_{\mathrm{MU}}(\widehat{\theta}_k;\bar{\mathbf{a}}) \tag{17}$$

where

$$0 \leq \gamma_k = \frac{\bar{\mathbf{d}}^{\mathrm{H}}(\theta_k)\mathbf{U}_{\mathbf{n}}\mathbf{P}_{\mathbf{k}}\mathbf{U}_{\mathbf{n}}^{\mathrm{H}}\bar{\mathbf{d}}(\theta_k)}{\bar{\mathbf{d}}^{\mathrm{H}}(\theta_k)\mathbf{U}_{\mathbf{n}}\mathbf{U}_{\mathbf{n}}^{\mathrm{H}}\bar{\mathbf{d}}(\theta_k)} = \mathcal{R}\Big(\mathbf{P}_{\mathbf{k}}, \mathbf{U}_{\mathbf{n}}^{\mathrm{H}}\bar{\mathbf{d}}(\theta_k)\Big) < 1 \tag{18}$$

where the bounds in equation (18) are due to the fact that γ_k is a *Rayleigh quotient*, which is always bounded between the minimum and maximum eigenvalues of $\mathbf{P}_{\mathbf{k}}$. Since $\mathbf{P}_{\mathbf{k}}$ is a projector matrix, then the eigenvalues are either 0 or 1.

Observation: It is very important to observe that $\mathrm{var}_{\mathrm{MU}}(\widehat{\theta}_k;\bar{\mathbf{a}})$ is, indeed, the MSE of $\widehat{\theta}_k$ estimated thru MUSIC with *known mutual coupling parameters*. Therefore, the quantity $\frac{1}{1-\gamma_k}$ quantifies the loss of performance between the proposed method in equation (12) and the MUSIC algorithm with *known mutual coupling parameters*. The *Cramer-Rao* Bound (CRB) of any unbiased estimator of $\widehat{\theta}_k$ in the presence of mutual coupling was given in [22], i.e.

$$\mathrm{var}_{\mathrm{CRB}}(\widehat{\theta}_k) = \frac{\sigma^2}{2L}\Big(\big[\bar{\mathbf{D}}^{\mathrm{H}}\mathbf{P}_{\bar{\mathbf{A}}}^{\perp}\bar{\mathbf{D}} \odot \mathbf{R}_{\mathbf{ss}}\big]^{-1}\Big)_{k,k} \tag{19}$$

where $\mathbf{P}_{\bar{\mathbf{A}}}^{\perp} = \mathbf{I} - \mathbf{P}_{\bar{\mathbf{A}}}$ and $\mathbf{P}_{\bar{\mathbf{A}}}$ is given by

$$\mathbf{P}_{\bar{\mathbf{A}}} = \bar{\mathbf{A}}(\boldsymbol{\Theta})\big(\bar{\mathbf{A}}^{\mathrm{H}}(\boldsymbol{\Theta})\bar{\mathbf{A}}(\boldsymbol{\Theta})\big)^{-1}\bar{\mathbf{A}}^{\mathrm{H}}(\boldsymbol{\Theta}) \tag{20}$$

and

$$\bar{\mathbf{D}} = \left[\begin{array}{c|c|c} \frac{\partial \bar{\mathbf{a}}(\theta_1)}{\partial \theta_1} & \dots & \frac{\partial \bar{\mathbf{a}}(\theta_q)}{\partial \theta_q} \end{array}\right] \tag{21}$$

Following similar steps as in [21], we re-write the MSE equation, $\mathrm{var}_f^{(p)}(\widehat{\theta}_k)$, in a way that turns out to be useful when comparing to the CRB

$$\mathrm{var}_f^{(p)}(\widehat{\theta}_k) = \frac{\sigma^2}{2L}\frac{\big(\mathbf{R}_{\mathbf{ss}}^{-1}\big)_{k,k} + \sigma^2\big(\mathbf{R}_{\mathbf{ss}}^{-1}(\bar{\mathbf{A}}^{\mathrm{H}}\bar{\mathbf{A}})^{-1}\mathbf{R}_{\mathbf{ss}}^{-1}\big)_{k,k}}{\bar{\mathbf{d}}^{\mathrm{H}}(\theta_k)\mathbf{U}_{\mathbf{n}}\mathbf{P}_{\mathbf{k}}^{\perp}\mathbf{U}_{\mathbf{n}}^{\mathrm{H}}\bar{\mathbf{d}}(\theta_k)} \tag{22}$$

4.1. Large Number of Antennas

We study the performance of the proposed algorithm in the asymptotic regime when $\frac{p}{N} \to 0$, i.e. $N \to \infty$ for fixed p. We have the following Theorem, which is a generalisation of the case with no mutual coupling in [21]:

Theorem 2: *The limits of* $\mathrm{var}_{\mathrm{CRB}}(\widehat{\theta}_k)$, $\mathrm{var}_f^{(p)}(\widehat{\theta}_k)$, *and* γ_k *are given as follow*

$$\mathrm{var}_{\mathrm{CRB}}(\widehat{\theta}_k) \xrightarrow[\frac{p}{N}\to 0]{} \frac{6\sigma^2}{N^3 L|\mathbf{h}_k^{\mathrm{H}}\mathbf{m}|^2}\frac{1}{\big(\mathbf{R}_{\mathbf{ss}}\big)_{k,k}} \tag{23}$$

$$\mathrm{var}_f^{(p)}(\widehat{\theta}_k) \xrightarrow[\frac{p}{N}\to 0]{} \frac{6\sigma^2}{N^3 L|\mathbf{h}_k^{\mathrm{H}}\mathbf{m}|^2}\big(\mathbf{R}_{\mathbf{ss}}^{-1}\big)_{k,k} \tag{24}$$

$$\gamma_k \xrightarrow[\frac{p}{N}\to 0]{} 0 \tag{25}$$

where

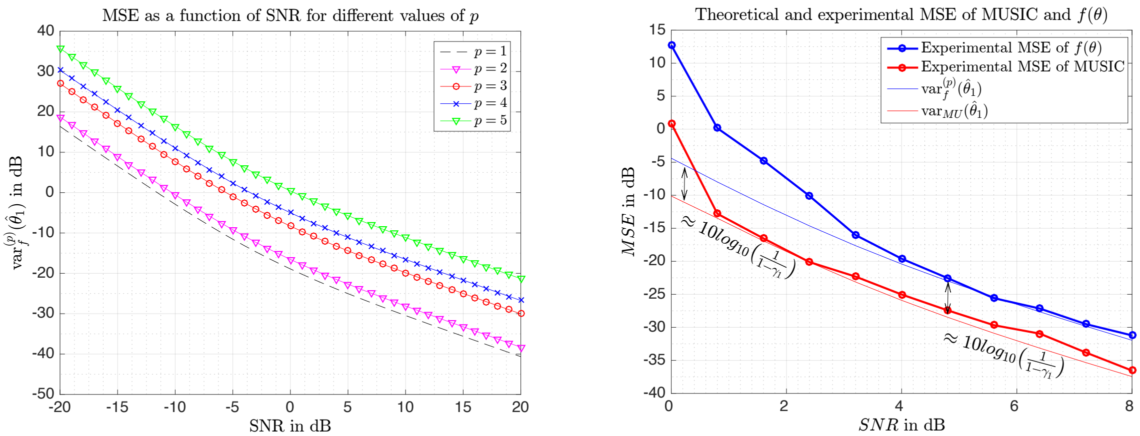
$$\mathbf{h}_k = \mathbf{a}_p(\theta_k) + \mathbf{a}_p^{*}(\theta_k) - \mathbf{e}_1 \tag{26}$$

Proof. See [23] □

Note that the factor $\gamma_k \to 0$ as $\frac{p}{N} \to 0$. Using this theorem, we have that

$$\frac{\mathrm{var}_f^{(p)}(\widehat{\theta}_k)}{\mathrm{var}_{\mathrm{CRB}}(\widehat{\theta}_k)} = \big(\mathbf{R}_{\mathbf{ss}}\big)_{k,k}\big(\mathbf{R}_{\mathbf{ss}}^{-1}\big)_{k,k} \tag{27}$$

Hence the CRB is attained for uncorrelated signals, when $\frac{p}{N} \to 0$.



(a) $N = 6$, $q = 1$, and $\theta_1 = 50^\circ$

(b) $N = 6$, $q = 1$, $\theta_1 = 10^\circ$, and $p = 3$

Fig. 2: MSE of the proposed method as a function of SNR.

4.2. High SNR

For high SNR, the analysis here is similar to [21]. For the case of uncorrelated signals, one could show the following relation

$$\frac{\mathrm{var}_f^{(p)}(\widehat{\theta}_k)}{\mathrm{var}_{\mathrm{CRB}}(\widehat{\theta}_k)} = \left(1 + \frac{\left((\bar{\mathbf{A}}^{\mathrm{H}}\bar{\mathbf{A}})^{-1}\right)_{k,k}}{\mathrm{SNR}_k}\right)\left(\frac{1}{1-\gamma_k}\right) \tag{28}$$

where $\mathrm{SNR}_k = \frac{(\mathbf{R_{ss}})_{k,k}}{\sigma^2}$. For high SNR, the ratio in equation (28) is controlled by the factor $\frac{1}{1-\gamma_k}$, which is not necessarily equal to 1 for any p and N. However, as discussed in the case of large N with fixed p, if the ratio $\frac{p}{N}$ is relatively small, then the above ratio could be argued to be close to 1.

5. SIMULATION RESULTS

In Fig. 1, we study the behaviour of γ_k given in equation (18) as a function of p and N. In Fig. 1a, we set the coupling vector $\mathbf{m}$ to

$$\mathbf{m} = \left[1;\ -0.08 + 0.5j;\ -0.14 - 0.3j\right]^{\mathrm{T}} \tag{29}$$

Fig. 1a plots γ_1 for one source $q = 1$, but different AoAs. We see that $\gamma_k \to 0$ as $\frac{p}{N} \to 0$. Furthermore, the rate of decay depends on the AoA. It should also be noted that this rate depends on other factors such as the number of sources q and the coupling vector $\mathbf{m}$. In Fig. 1b, we study the behaviour of γ_k by fixing N and increasing p. The coupling parameters are generated by first forming a vector $\bar{\mathbf{m}}$, where $\left\{\bar{m}_k = \frac{1}{k+1}e^{j2\pi\phi_k}\right\}_{k=1}^{N}$, where ϕ_k is randomly chosen. Then, in order to compute γ_k, for $p = p_0$, we choose the first p_0 elements of $\bar{\mathbf{m}}$ to form the vector $\mathbf{m} \in \mathbb{C}^{p_0 \times 1}$. We have set $N = 10$. We also observe that γ_k is increasing as p increases for fixed N. This results in an increase of the MSE, $\mathrm{var}_f^{(p)}(\widehat{\theta}_k)$ given in equation (13), when p increases due to the factor $\left(\frac{1}{1-\gamma_k}\right)$.

The MSE of the proposed algorithm in equation (13), namely $\mathrm{var}_f^{(p)}(\widehat{\theta}_k)$, is simulated in Fig. 2. In Fig. 2a, we set $N = 6$, $q = 1$, and $\theta_1 = 50^\circ$. The number of snapshots is $L = 10^3$. The coupling parameters are chosen from vector

$$\bar{\mathbf{m}} = [1;\ -0.08+0.5j;\ -0.14-0.3j;\ -0.04+0.04j;\ 0.03-0.02j]^{\mathrm{T}} \tag{30}$$

as done in the case of Fig. 1b. This figure tells us that a higher MSE is obtained as p increases. In Fig. 2b, we quantify this loss of performance. We have $q = 1$ source impinging an array of $N = 6$ at $\theta_1 = 10^\circ$. The number of snapshots is $L = 10^2$. The number of coupling parameters is $p = 3$ with $\mathbf{m}$ equal to that in the scenario depicted in Fig. 1. We have plotted the experimental and theoretical MSE of MUSIC and the proposed algorithm. For the experimental MSE, we have averaged over 10^3 Monte-Carlo simulations. A nice observation is to see that that the gap between the MSE of MUSIC of the propsed algorithm is about $\frac{1}{1-\gamma_1}$, for sufficiently high SNR. Computing γ_1 using the related parameters, we get $10\log_{10}\left(\frac{1}{1-\gamma_1}\right) \simeq 6$dB. This factor is the loss of performance compared to the *Coupling-free* MUSIC. Furthermore, we could also observe that the experimental and theoretical MSE curves are in agreement, for sufficiently high SNR.

6. CONCLUSIONS

We have first revised a recently proposed method that could estimate AoAs in the presence of more mutual coupling parameters, compared to the mentioned methods in the Introduction. We have first derived the MSE expression of the proposed method, which is an unbiased estimator. Moreover, we have compared it with the MSE of the MUSIC estimator with known coupling parameters and with the CRB of any unbiased estimator that estimates the AoAs in the presence of mutual coupling (at high SNR and high number of antennas). It was shown that the MSE of the proposed method could attain the CRB, for uncorrelated sources, for high number of antennas at fixed number of coupling parameters, but not at high SNR.

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