ATOMIC NORM MINIMIZATION FOR MODAL ANALYSIS WITH RANDOM SPATIAL COMPRESSION

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ABSTRACT

Identifying characteristic vibrational modes and frequencies is of great importance for monitoring the health of structures such as buildings and bridges. In this work, we address the problem of estimating the modal parameters of a structure from small amounts of vibrational data collected from wireless sensors distributed on the structure. We consider a randomized spatial compression scheme for minimizing the amount of data that is collected and transmitted by the sensors. Using the recent technique of atomic norm minimization, we show that under certain conditions exact recovery of the mode shapes and frequencies is possible. In addition, in a simulation based on synthetic data, our method outperforms a singular value decomposition (SVD) based method for modal analysis that uses the uncompressed data set.

***Index Terms*—** Modal analysis, spectral analysis, structural health monitoring, atomic norm minimization

1. INTRODUCTION

Due to the considerable time and expense required to perform manual inspections of physical structures such as buildings and bridges, and the difficulty of repeating these inspections frequently, there is a growing interest in developing automated techniques for structural health monitoring (SHM) [1, 2, 3] based on data collected in a wireless sensor network. For example, one could envision a collection of battery-operated wireless sensors deployed across a structure that record vibrational displacements over time and then transmit this information to a central node for analysis [4, 5].

Modal analysis is an analytical technique for assessing the health of a structure in terms of estimating the modal parameters such as mode shapes and the corresponding frequencies. As explained in Section 2, the vibration response of a linear time-invariant (LTI) system can be expressed as a linear combination of the natural modes which are inherent to the system and determined completely by its physical properties such as mass, damping and stiffness [6]. Each mode shape corresponds to a certain natural frequency. A structure can be characterized by these modal parameters which will change when the structure is damaged.

It is said that atomic norm, which generalizes the ℓ_1 norm for sparse recovery and the nuclear norm for low-rank matrix completion, is the best convex heuristic for underdetermined linear inverse problems [7]. Recently, atomic norm minimization (ANM) based approaches which deal with continuous valued frequencies have been shown to be efficient and powerful for exactly recovering the unobserved samples and identifying the unknown frequencies in both single measurement vector (SMV) [8] and multiple measurement vector (MMV) [9, 10] problems. With sufficient random samples and well separated frequencies, exact frequency localization can be guaranteed. Moreover, these kinds of approaches can also eliminate the basis mismatch problems [11, 12, 13] that plague existing compressed sensing techniques.

In this work, we consider a particular random spatial compression protocol for data collection and transmission. In this protocol, at each time sample, each sensor modulates its sample value by a random number, and the sensors transmit these values coherently to the central node, where the modulated values add to result in a single measurement. As proposed in [4], such randomized spatial aggregation of the measurements can be achieved as part of using phase-coherent analog transmissions to the base station. We highlight the fact that, based on this compressed data, modal analysis can be formulated as an ANM problem. The ANM problem can be solved efficiently and in some cases perfectly recover a structure's mode shapes and frequencies. In contrast to an alternative singular value decomposition (SVD) based approach, the ANM method can succeed even when the mode shapes are not orthogonal.

2. PREVIOUS WORK

For an N degree-of-freedom LTI system [14], the second-order equations of motion which represent the dynamic behavior of the system can be formulated as

$$[\mathbf{M}]\{\ddot{\boldsymbol{x}}(t)\} + [\mathbf{C}]\{\dot{\boldsymbol{x}}(t)\} + [\mathbf{K}]\{\boldsymbol{x}(t)\} = \{\boldsymbol{f}(t)\} \quad (1)$$

where $[\mathbf{M}]$, $[\mathbf{C}]$ and $[\mathbf{K}]$ denote the mass, damping and stiffness matrix, respectively. $\{\boldsymbol{x}(t)\} \in \mathbb{R}^N$ and $\{\boldsymbol{f}(t)\} \in \mathbb{R}^N$ are the displacement data and the excitation force.

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According to the modal expansion theorem [15], the system response $\{\boldsymbol{x}(t)\}$ can be expressed uniquely as a linear combination of the system modes:[1]

$$\{\boldsymbol{x}(t)\} = [\boldsymbol{\Psi}]\{\boldsymbol{q}(t)\} = \sum_{k=1}^{K}\{\boldsymbol{\psi}_k\}q_k(t), \tag{2}$$

where $[\boldsymbol{\Psi}]$ is an $N \times K$ matrix containing the structure's K active mode shapes $\{\boldsymbol{\psi}_k\}$ as columns, and $\{\boldsymbol{q}(t)\}$ is a time-varying length-K vector of modal responses $q_k(t) = A_k e^{j2\pi f_k t}$ for some complex amplitudes A_k and frequencies f_k.

In recent years, blind source separation (BSS) based methods have been widely used in modal analysis. The authors in [17] and [18] propose a new modal identification algorithm based on sparse component analysis (SCA) to deal with even the underdetermined case where sensors may be highly limited compared to the number of active modes. In [19], a novel decentralized modal identification method termed PARAllel FACtor (PARAFAC) based sparse BSS (PSBSS) method is proposed. Independent component analysis (ICA) is a powerful method to solve the BSS problem. Yang et al. present an ICA-based method to identify the modal parameters of lightly and highly damped systems, even in the case with heavy noise and nonstationarity [20]. Another work [21] also presents a new method based on a combination of CS and complexity pursuit (CP) to solve the modal identification problem.

In [16], Park et al. characterize the performance of a standard modal analysis algorithm, the Singular Value Decomposition (SVD), which enables the number of samples to be minimized without sacrificing estimation performance. They also consider the randomized compression of the data at each sensor before transmission to the central node. However, the SVD-based algorithm will typically only return an approximate estimate of a structure's mode shapes instead of exactly recovering them. Moreover, the applicability of the SVD-based algorithm is limited to orthogonal or nearly orthogonal mode shapes.

3. MAIN RESULTS

3.1. Problem formulation

In this paper, we work on an ideal model in free vibration with no damping. Our goal is to recover the natural frequencies and mode shapes from the compressed measurements. We consider the following analytic signal [16]

$$\{\boldsymbol{x}(t)\} = \sum_{k=1}^{K}\{\boldsymbol{\psi}_k\}A_k e^{j2\pi f_k t}, \tag{3}$$

[1] Similar to [16], we assume that real-valued displacement recordings have been converted to their complex analytic equivalents.

where $\{\boldsymbol{\psi}_k\} = \begin{bmatrix} \{\boldsymbol{\psi}_k\}(1) \; \{\boldsymbol{\psi}_k\}(2) \; \cdots \; \{\boldsymbol{\psi}_k\}(N) \end{bmatrix}^\top$ with "$\top$" denoting the non-conjugate transpose. The analytic signal in (3) can also be rewritten as

$$\{\boldsymbol{x}(t)\} = \begin{bmatrix} \sum_{k=1}^{K}\{\boldsymbol{\psi}_k\}(1)A_k e^{j2\pi f_k t} \\ \sum_{k=1}^{K}\{\boldsymbol{\psi}_k\}(2)A_k e^{j2\pi f_k t} \\ \vdots \\ \sum_{k=1}^{K}\{\boldsymbol{\psi}_k\}(N)A_k e^{j2\pi f_k t} \end{bmatrix}$$

with $\sum_{k=1}^{K}\{\boldsymbol{\psi}_k\}(n)A_k e^{j2\pi f_k t}$ denoting the signal from the nth sensor. Taking regularly spaced Nyquist samples at

$$T = \{t_1,\ t_2,\ \cdots,\ t_M\} = \{0, T_s,\ \cdots,\ (M-1)T_s\}$$

with $T_s \leq \frac{1}{2\max\{f_k\}}$ being the sampling interval, we obtain the data matrix $[\mathbf{X}] = [\boldsymbol{x}(t_1)\ \cdots\ \boldsymbol{x}(t_M)]$, which can be written as

$$[\mathbf{X}] = \sum_{k=1}^{K} A_k \begin{bmatrix} \{\boldsymbol{\psi}_k\}(1)e^{j2\pi f_k t_1} & \cdots & \{\boldsymbol{\psi}_k\}(1)e^{j2\pi f_k t_M} \\ \{\boldsymbol{\psi}_k\}(2)e^{j2\pi f_k t_1} & \cdots & \{\boldsymbol{\psi}_k\}(2)e^{j2\pi f_k t_M} \\ \vdots & \ddots & \vdots \\ \{\boldsymbol{\psi}_k\}(N)e^{j2\pi f_k t_1} & \cdots & \{\boldsymbol{\psi}_k\}(N)e^{j2\pi f_k t_M} \end{bmatrix} \tag{4}$$

$$= \sum_{k=1}^{K} A_k\{\boldsymbol{\psi}_k\}\boldsymbol{a}(f_k)^T \in \mathbb{C}^{N\times M}, \tag{5}$$

where $\boldsymbol{a}(f_k) := [e^{j2\pi f_k t_1}\ \cdots\ e^{j2\pi f_k t_M}]^\top$.

Frequency estimation from a mixture of complex sinusoids is a classical problem in signal processing. Conditions for solving the SMV frequency estimation problem—analogous to (5) with $N = 1$—using ANM have recently been established [8]. As detailed in [9, 10], the MMV problem (i.e., (5) with $N \geq 1$) can also be formulated using ANM, specifically over a continuous dictionary of atoms taking the form $\boldsymbol{h}\boldsymbol{a}(f)^T$ with $\|\boldsymbol{h}\|_2 = 1$. However, the works [9, 10] differ from ours in that they consider random temporal compression (effectively compressing $[\mathbf{X}]$ along its rows), whereas in this work we consider random spatial compression (compressing $[\mathbf{X}]$ along its columns).

3.2. Randomized spatial compression

In order to conserve power and extend battery life, it is desirable to minimize the amount of data that must be collected and transmitted in the sensor network. We discuss the possibility of compressing the sensor readings by random spatial modulation of the time series data.

Define the inner product of $[\mathbf{X}]$ and $[\mathbf{Y}]$ as

$$\langle[\mathbf{X}], [\mathbf{Y}]\rangle = \mathrm{Tr}([\mathbf{Y}]^H[\mathbf{X}]).$$

We compress each column of $[\mathbf{X}]$ by computing its correlation with a single random vector:

$$y_m = \langle[\mathbf{X}](:,m), \boldsymbol{b}_m\rangle = \langle[\mathbf{X}]\boldsymbol{e}_m, \boldsymbol{b}_m\rangle$$
$$= \langle[\mathbf{X}], \boldsymbol{b}_m\boldsymbol{e}_m^H\rangle, \qquad 1 \leq m \leq M,$$

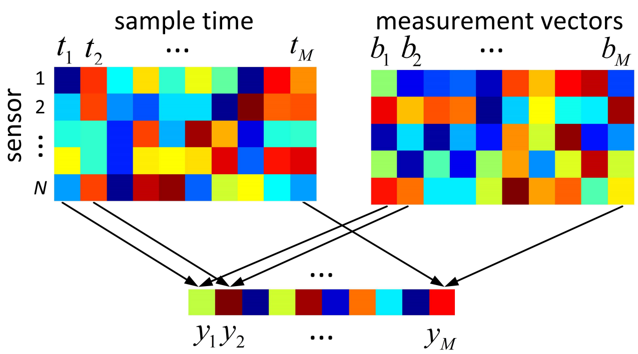

Fig. 1. Random spatial compression: the compressed measurements are obtained by computing the correlation between the data samples and the random vectors.

where $\boldsymbol{b}_m \in \mathbb{C}^{N\times 1}$ is a random vector and $\boldsymbol{e}_m \in \mathbb{R}^{M\times 1}$ is the mth canonical basis vector. As proposed in [4], such randomized spatial aggregation of the measurements can be achieved as part of the transmission process using phase-coherent analog transmissions to the base station; see also Fig. 1. Randomized gossiping algorithms [22] can also be used for aggregation of such measurements in a network.

Inspired by [8], we can formulate the modal analysis problem as an ANM problem:

$$\begin{aligned} &\min_{[\hat{\mathbf{X}}]} \|[\hat{\mathbf{X}}]\|_{\mathcal{A}} \\ &\text{s. t. } y_m = \langle [\hat{\mathbf{X}}], \boldsymbol{b}_m \boldsymbol{e}_m^H \rangle, \ \ 1 \le m \le M. \end{aligned} \quad (6)$$

Here, the atomic set $\mathcal{A}$ is defined as

$$\mathcal{A} = \{\boldsymbol{h}\boldsymbol{a}(f)^T : \ \|\boldsymbol{h}\|_2 = 1\},$$

and the atomic norm $\|[\mathbf{X}]\|_{\mathcal{A}}$ is defined as

$$\begin{aligned} \|[\mathbf{X}]\|_{\mathcal{A}} &= \inf\{t > 0 : [\mathbf{X}] \in \mathrm{t\,conv}(\mathcal{A})\} \\ &= \inf\left\{\sum_k c_k : [\mathbf{X}] = \sum_k c_k \boldsymbol{h}_k \boldsymbol{a}(f_k)^T, \ c_k \ge 0\right\}. \end{aligned}$$

This ANM problem, which coincides with the one in [23], is equivalent to a Semi-Definite Program (SDP) and can be solved efficiently. From the dual polynomial that results from the optimization problem, one can extract the modal frequencies f_k and subsequently use least-squares to recover the mode shapes $\{\boldsymbol{\psi}_k\}$ up to a phase ambiguity.

Using machinery we have recently developed [23] for super-resolution of complex exponentials from modulations with unknown waveforms, the follow theorem shows that the recovery will be successful with high probability if the number of time samples M is proportional to KN.

Theorem 3.1. *Suppose we observe the data matrix $[\mathbf{X}]$ with the above random spatial compression scheme. Assume that the random vectors $\mathbf{b}_m$ are i.i.d samples from an distribution with the isotropic and μ-incoherent properties defined in [23]. Assume that the signs $\frac{\{\boldsymbol{\psi}_k\}(n)A_k}{|\{\boldsymbol{\psi}_k\}(n)A_k|}$ are drawn i.i.d. from the uniform distribution on the complex unit circle, and assume the minimum separation $\Delta_f = \min_{k\neq j} |(f_k - f_j)T_s| \ge \frac{4}{M-1}$ [8, 24]. Then there exists a numerical constant C such that*

$$M \ge C\mu KN \log\left(\frac{MKN}{\delta}\right)\log^2\left(\frac{MN}{\delta}\right)$$

is sufficient to guarantee that we can recover $[\mathbf{X}]$ via (6) and localize the frequencies with probability at least $1-\delta$.

In this case, the total number of measurements received at the base station (one per sampling time, for each of M sampling times) is essentially proportional to KN, the number of degrees of freedom in the problem.

4. SIMULATIONS

In this section, we demonstrate the performance of estimating frequencies and mode shapes with ANM under the context of random spatial compression. In the following experiments, we use a modal assurance criterion (MAC)

$$\mathrm{MAC}(\{\boldsymbol{\psi}_k\}, \{\hat{\boldsymbol{\psi}}_k\}) = |\langle \{\boldsymbol{\psi}_k\}, \{\hat{\boldsymbol{\psi}}_k\}\rangle|$$

to evaluate the correlation between the normalized estimated mode shape $\{\hat{\boldsymbol{\psi}}_k\}$ and the normalized true mode shape $\{\boldsymbol{\psi}_k\}$. $\mathrm{MAC} = 1$ indicates perfect recovery of the true mode shape.

4.1. ANM-based strategy vs. SVD-based strategy

In the first experiment, we set the true frequencies as $f = 2,\ 3$ and 10 Hz. The number of sensors is $N = 10$ and the number of samples is $M = 100$. We also compare the random spatial compression strategy with an SVD-based strategy which was proposed in [16]. We allow the SVD-based strategy to use the full, uncompressed data matrix of size 10×100. As explained in [16] and as shown in Table 1, Fig. 2 and Fig. 3, the SVD-based method performs poorly when the mode shapes are correlated. However, the proposed ANM-based random spatial compression strategy (using a total of only $M = 100$ compressed measurements) performs very well both when the mode shapes are orthogonal and when they are correlated.

4.2. M vs. K and N

In the second experiment, we verify that the minimal number of samples needed for perfect recovery is approximately linearly proportional to the number of active modes K and the number of sensors N. The true mode shapes are generated randomly. In the first part of this experiment, we fix $N = 5$ and change K from 2 to 10. In the second part of this experiment, we fix $K = 3$ and set the true frequencies as $2,\ 3,$ and 10 Hz. The number of sensors N ranges from 2 to 10. It is shown in both Fig. 4 and Fig. 5 that there does exist a nearly linear transition from complete failure to perfect recovery.

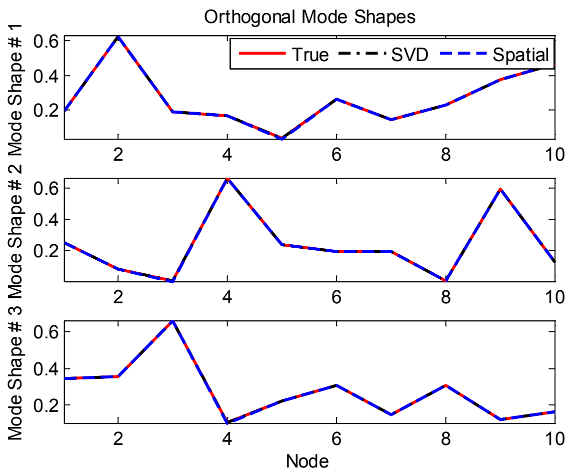

Fig. 2. The recovered mode shapes.

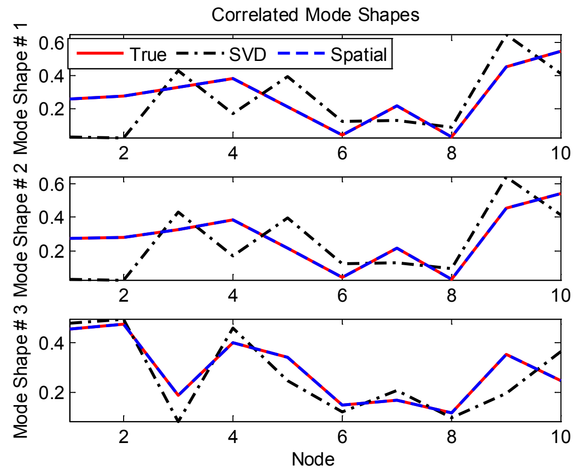

Fig. 3. The recovered mode shapes.

5. CONCLUSIONS

In this work, we highlight the fact that modal analysis in free vibration and with no damping under the context of random spatial compression can be formulated as an ANM problem, which is equivalent to an SDP and can be solved efficiently. The recovery will be successful with high probability if the number of time samples M is proportional to KN. We also compare this proposed strategy with an SVD-based strategy. In future, we will take damping and noise into consideration, which will increase the practical relevance of this technique.

f_{true}	f_{P}^{O}	f_{P}^{C}	$\text{MAC}_{\text{P}}^{\text{O}}$	$\text{MAC}_{\text{S}}^{\text{O}}$	$\text{MAC}_{\text{P}}^{\text{C}}$	$\text{MAC}_{\text{S}}^{\text{C}}$
2	2	2	1.0000	1.0000	1.0000	0.8632
3	3	3	1.0000	1.0000	1.0000	0.8593
10	10	10	1.0000	1.0000	1.0000	0.9678

Table 1. The true frequencies are set as $f = 2,\ 3$ and 10 Hz. "P", "S", "O", and "C" stand for "Spatial", "SVD", "Orthogonal", and "Correlated", respectively.

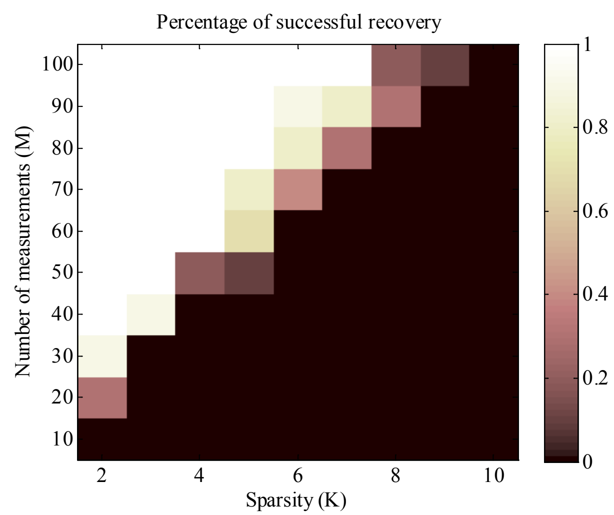

Fig. 4. Phase transition: number of sensors is set as $N = 5$.

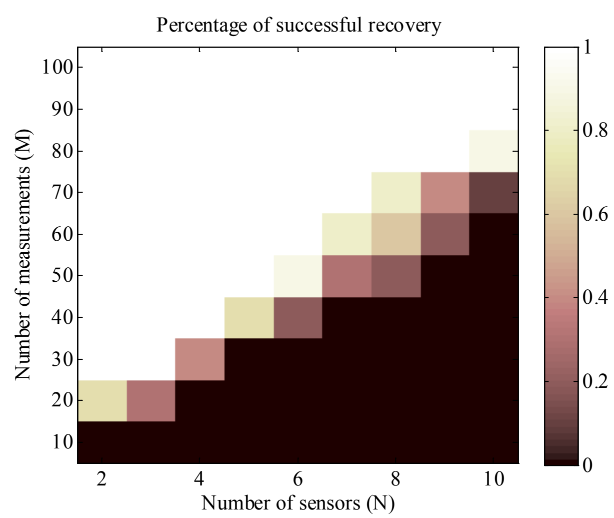

Fig. 5. Phase transition: number of active modes is set as $K = 3$.

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