AVERAGE SCR LOSS ANALYSIS FOR POLARIMETRIC STAP WITH KRONECKER STRUCTURED COVARIANCE MATRIX

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ABSTRACT

The paper presents the average signal-to-clutter loss (SCRL) analysis for polarimetric space-time adaptive processing by exploiting the Kronecker structure of the clutter covariance matrix (CM). An expression for the average SCRL as a function of the mean square error of the corresponding CM estimator is derived. Based on that expression, one can determine how many samples are required in order to achieve a desired SCRL. The proposed average SCRL analysis methodology can be extended to more general scenarios, where closed-form CM estimates are not available. Simulations indicate that even in the non-asymptotic regime, the proposed method can provide a good prediction of the average SCRL.

Index Terms— Average SCR Loss, Polarimetric STAP, Cramér-Rao Bound, Kronecker Structure.

1. INTRODUCTION

It is well-known that polarization diversity [1] is useful in improving the performance of radar detection, estimation, and tracking [2, 3]. In polarimetric radars [4, 5, 6], the clutter covariance matrix (CM) can be expressed as the Kronecker product of the polarization covariance matrix and the spacetime covariance matrix. Exploiting such Kronecker structure may significantly reduce the number of unknown clutter parameters to be estimated and achieve considerable estimation accuracy. Kronecker structured CM estimation is considered in Gaussian [7] and Compound Gaussian cases [8].

When the clutter CM is replaced by the CM estimate which is obtained using a set of training samples, the adaptive version of the minimum variance distortionless response space-time adaptive processing (MVDR-STAP) filter is referred to as the sample matrix inversion STAP (SMI-STAP) filter [9]. The performance of the SMI-STAP technique highly depends on the signal-to-clutter ratio loss (SCRL) and the number of training samples [10]. In the Gaussian case, for the unstructured CM (positive definite Hermitian), 2N training samples are required for 3 dB SCRL [10], where N is the signal length. For the persymmetric structured CM, the entries of which are Hermitian symmetric with respect to the diagonal and cross diagonal, N training samples are required to achieve 3 dB SCRL [11]. For the Compound Gaussian clutter, or other CM structures [8], such as the Kronecker and Toeplitz structure, it is difficult to analyze the statistical properties of the SCRL. This is mainly due to the implicit form of the corresponding estimators, which are defined through fixed point equations [8]. The goal of this paper is to derive an approximate expression of the average SCRL of the SMI-STAP filter, built with the aforementioned estimators. In particular, we exploit the Kronecker structure of the clutter CM for polarimetric STAP in Compound Gaussian clutter. We analyze the mean square error (MSE) for the Kronecker maximum likelihood estimator (KMLE), and then derive an approximate expression of the average SCRL as a function of the MSE. Finally, we extend this method of SCRL analysis in more general scenarios. Numerical simulations validate the effectiveness of the proposed method.

The paper is organized as follows. The problem formulation is given in Section 2. The Kronecker structured CM estimation is given in Section 3. The proposed method of average SCRL analysis and the extensions are given in Section 4 and 5. Simulations and conclusion are respectively provided in Sections 6 and 7.

2. PROBLEM FORMULATION

2.1. Signal Model

Consider the following polarimetric STAP signal model of the received signal:

$$\mathbf{y} = \beta \mathbf{t} + \mathbf{n} \in \mathbb{C}^{N \times 1},\tag{1}$$

where β is the unknown deterministic amplitude of the target, $\mathbf{t} \in \mathbb{C}^{N \times 1}$ is the polarization-space-time steering vector [6], given by

$$\mathbf{t} = \mathbf{a}_p \otimes \mathbf{a}_s \tag{2}$$

where $\mathbf{a}_p \in \mathbb{C}^{N_p \times 1}$, $\mathbf{a}_s \in \mathbb{C}^{N_s \times 1}$ denote the polarization and space-time steering vectors respectively. N_p takes the values

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1,2,3,4 for different polarization radar schemes [1, 2]. In (1), the term **n** denotes clutter returns, obeys the Compound Gaussian (CG) distribution, and can be expressed in terms of a real positive texture component, τ , and a speckle component, **c**, as $\mathbf{n} = \sqrt{\tau} \mathbf{c}$. τ may be unknown, deterministic or random. In this paper, we assume that τ obeys the inverse Gamma (IG) distribution. However, the main result corresponding to other textures can be obtained similarly. The probability density function (PDF) of the IG distribution with shape parameter v is given by

$$p_{\mathcal{IG}}(\tau; v) = \frac{v^v}{\Gamma(v)\tau^{v+1}} \exp(-\frac{v}{\tau}), \tau > 0$$
(3)

where $\Gamma(\cdot)$ is the gamma function. c is a complex Gaussian random vector with zero mean and Kronecker structured covariance matrix **R**, i.e.,

$$\mathbf{R} = \mathbf{R}_p \otimes \mathbf{R}_s. \tag{4}$$

 $\mathbf{R}_p \in \mathbb{C}^{N_p \times N_p}$ is the polarization covariance matrix, which is closely related to the terrain of the clutter [13]. $\mathbf{R}_s \in \mathbb{C}^{N_s \times N_s}$ is the space-time covariance matrix [9]. Both \mathbf{R}_p and \mathbf{R}_s are assumed to be positive Hermitian matrices.

2.2. Problem Formulation

The SCR loss (SCRL) is defined as the ratio of the output SCR of the SMI-STAP filter using the true covariance matrix, \mathbf{R} , and an estimate $\hat{\mathbf{R}}$ [10, 11]

$$\rho \stackrel{\Delta}{=} \frac{|\mathbf{t}^H \hat{\mathbf{R}}^{-1} \mathbf{t}|^2}{(\mathbf{t}^H \mathbf{R}^{-1} \mathbf{t})(\mathbf{t}^H \hat{\mathbf{R}}^{-1} \mathbf{R} \hat{\mathbf{R}}^{-1} \mathbf{t})},\tag{5}$$

where $(\cdot)^H$ denotes the Hermitian transpose operator.

Exploiting the structural information of the CM can significantly improve the CM estimation accuracy and the corresponding SCRL. In the Gaussian case with sample covariance matrix (SCM) $\hat{\mathbf{R}} = \frac{1}{K} \sum_{k=1}^{K} \mathbf{y}_k \mathbf{y}_k^H$, where $\mathbf{y}_k \in \mathbb{C}^{N \times 1}$ is the *k*th training sample, the average SCRL is given by $\mathbb{E}\{\rho_{\text{SCM}}\} = \frac{K-N+2}{K+1}$ [10]. For the persymmetric structured CM and the corresponding average SCRL, it holds that $\mathbb{E}\{\rho_{\text{pers}}\} = \frac{2K-N+2}{2K+1}$ [11].

However, for the Compound Gaussian case and for other structured covariance matrix, the average SCRL is not tractable. This motivates us to develop an approximate expression that may be useful in more general scenarios.

3. KRONECKER STRUCTURED CM ESTIMATION

3.1. Kronecker MLE

The general maximum likelihood estimate (MLE) of the CM, also known as the fixed-point estimator, in Compound Gaussian clutter with IG texture is given by [12]

$$\hat{\mathbf{R}}_{\text{MLE}} = \frac{v+N}{K} \sum_{k=1}^{K} \frac{\mathbf{y}_k \mathbf{y}_k^H}{v + \mathbf{y}_k^H \hat{\mathbf{R}}_{\text{MLE}}^{-1} \mathbf{y}_k}$$
(6)

The Kronecker MLE (KMLE) estimator in Compound Gaussian clutter with deterministic texture is obtained in [8]. The estimate in [8], i.e., the Kronecker version of Tyler's robust estimator, is obtained through majorization minimization (MM) algorithm. This method can be readily extended to random textures.

The KMLE is the Kronecker product of the MLE of the sub-matrices \mathbf{R}_s and \mathbf{R}_p . For Compound Gaussian case with IG texture, the MLE of the sub-matrices \mathbf{R}_s and \mathbf{R}_p can be obtained by solving the fixed point equations [8]

$$\hat{\mathbf{R}}_{s} = \frac{v+N}{KN_{p}} \sum_{k=1}^{K} \frac{\mathbf{Y}_{k} \hat{\mathbf{R}}_{p}^{-T} \mathbf{Y}_{k}^{H}}{v + \mathbf{y}_{k}^{H} (\hat{\mathbf{R}}_{p}^{-1} \otimes \hat{\mathbf{R}}_{s}^{-1}) \mathbf{y}_{k}}$$
(7)

$$\hat{\mathbf{R}}_{p} = \frac{v+N}{KN_{s}} \sum_{k=1}^{K} \frac{\mathbf{Y}_{k}^{T} \hat{\mathbf{R}}_{s}^{-T} \mathbf{Y}_{k}^{*}}{v + \mathbf{y}_{k}^{H} (\hat{\mathbf{R}}_{p}^{-1} \otimes \hat{\mathbf{R}}_{s}^{-1}) \mathbf{y}_{k}}$$
(8)

where $(\cdot)^*$ denotes conjungate operator, $(\cdot)^T$ denotes transpose operator, $\mathbf{y}_k = \begin{bmatrix} \mathbf{y}_{k1}^T & \mathbf{y}_{k2}^T & \cdots & \mathbf{y}_{kN_p}^T \end{bmatrix}^T$ with $\mathbf{y}_{ki} \in \mathbb{C}^{N_s \times 1}$ being the received data in the *i*th polarization channel, $i = 1, \cdots, N_p$, and $\mathbf{Y}_k = \begin{bmatrix} \mathbf{y}_{k1}, \cdots, \mathbf{y}_{kN_p} \end{bmatrix} \in \mathbb{C}^{N_s \times N_p}$. Therefore, the KMLE is given by

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$$\mathbf{R}_{\mathrm{KMLE}} = \mathbf{R}_p \otimes \mathbf{R}_s. \tag{9}$$

The MM method can iteratively increase the value of the likelihood function until convergence. Also, the global optimum can be ensured by the geodesic convexity property [16] of the likelihood function with respect to the covariance matrix **R**. In general, the MM method will convergent within a small number of iterations. The KMLE given by (9) will be the Kronecker version of Tyler's estimator [8] for v = 0, and will be the Flip-Flop estimator in Gaussian case for $v \to \infty$ [7].

3.2. Cramér-Rao Bound

It is well known that the MLE is an unbiased, asymptotically efficient estimator. Thus its mean square error (MSE) will asymptotically achieve the Cramér-Rao bound (CRB) for large K. Let us denote the unknown parameter vector of a Hermitian matrix \mathbf{R} as $\boldsymbol{\theta} = [\Re \{ \operatorname{vech}\{\mathbf{R}\} \}^T, \Im \{ \underbrace{\operatorname{vech}\{\mathbf{R}\} }^T]^T \in \mathbb{R}^{N^2 \times 1}$, where $\Re \{ \operatorname{vech}\{\cdot\} \}$ and $\Im \{ \underbrace{\operatorname{vech}\{\cdot\} \} \}$ stack the real and imaginary parts below the main diagonal columnwise with "vech" including the main diagonal whereas "<u>vech</u>" not including the diagonal entries [14]. For a Kronecker structured covariance matrix $\mathbf{R} = \mathbf{R}_p \otimes \mathbf{R}_s$, let us denote the unknown parameter vector as $\boldsymbol{\theta}_{\kappa} \stackrel{\Delta}{=} [\boldsymbol{\theta}_p^T \quad \boldsymbol{\theta}_s^T]^T \in \mathbb{R}^{(N_p^2 + N_s^2) \times 1}$ with $\boldsymbol{\theta}_p$ and $\boldsymbol{\theta}_s$ stacking respectively N_p^2 and N_s^2 real components of the Hermitian sub-matrices \mathbf{R}_p and \mathbf{R}_s .

The Fisher information matrix (FIM) \mathcal{I} for θ_{κ} is given by

$$\mathcal{I} = \mathbf{H}_{\kappa}^{H} \mathbf{\Sigma} \mathbf{H}_{\kappa} \tag{10}$$

where

$$\mathbf{H}_{\kappa} = \frac{\partial \mathrm{vec}\{\mathbf{R}\}}{\partial \boldsymbol{\theta}_{\kappa}^{T}} \in \mathbb{C}^{N^{2} \times (N_{p}^{2} + N_{s}^{2})}, \tag{11}$$

and Σ is the FIM for θ in Compound Gaussian clutter. It holds that [14]

$$\boldsymbol{\Sigma} = v_1 (\mathbf{R}^{-T} \otimes \mathbf{R}^{-1}) + v_2 \operatorname{vec}(\mathbf{R}^{-1}) \operatorname{vec}(\mathbf{R}^{-1})^H \quad (12)$$

where v_1 and v_2 is related to the distribution of texture. For IG texture, it holds that

$$v_1 \stackrel{\scriptscriptstyle \Delta}{=} \frac{v+N}{v+N+1}, \quad v_2 \stackrel{\scriptscriptstyle \Delta}{=} \frac{1}{v+N+1}.$$
 (13)

Based on the results in [15], the MSE of the CM estimate has the lower bound

$$\boldsymbol{\mathcal{E}} \stackrel{\Delta}{=} \mathbb{E}\{\operatorname{vec}\{\hat{\mathbf{R}} - \mathbf{R}\}\operatorname{vec}\{\hat{\mathbf{R}} - \mathbf{R}\}^{H}\} \geq \boldsymbol{\Xi} \qquad (14)$$

where $\mathbf{A} \geq \mathbf{B}$ means that $\mathbf{A} - \mathbf{B}$ is positive definite. The CRB matrix Ξ for $\boldsymbol{\theta}_{\kappa}$ is given by [17]

$$\boldsymbol{\Xi} \stackrel{\Delta}{=} \boldsymbol{\Xi}(\mathbf{R}) = \mathbf{H}_{\kappa} \boldsymbol{\mathcal{I}}^{\dagger} \mathbf{H}_{\kappa}^{H}$$
$$= c_{1} (\mathbf{R}^{T} \otimes \mathbf{R})^{\frac{1}{2}} \mathbf{P}_{1} (\mathbf{R}^{T} \otimes \mathbf{R})^{\frac{1}{2}} + c_{2} \operatorname{vec}(\mathbf{R}) \operatorname{vec}(\mathbf{R})^{H}$$
(15)

where $[\cdot]^{\dagger}$ is the generalized inversion, $\mathbf{P}_1 = \mathbf{H}_1 \mathbf{H}_1^{\dagger}$ is the projection matrix of $\mathbf{H}_1 = (\mathbf{R}^T \otimes \mathbf{R})^{-\frac{1}{2}} \mathbf{H}_{\kappa}$, and

$$c_1 = \frac{1}{v_1}, \quad c_2 = \frac{v_2}{v_1(v_1 - v_2N)}.$$
 (16)

4. AVERAGE SCRL ANALYSIS

In this section, we derive an asymptotic expression of the average SCRL as a function of the MSE of the estimator.

Theorem 1 Given a covariance matrix estimate $\hat{\mathbf{R}}$ with sufficiently small error, the average SCRL of the SMI-STAP filter can be approximately expressed as

$$\bar{\rho} \stackrel{\Delta}{=} \mathbb{E}\{\rho(\hat{\mathbf{R}})\} \approx 1 - \operatorname{tr}\{\boldsymbol{\Delta}\bar{\boldsymbol{\mathcal{E}}}\}$$
(17)

where $\mathbf{\Delta} \stackrel{\Delta}{=} (\mathbf{I} - \mathbf{e}_1 \mathbf{e}_1^T) \otimes \mathbf{e}_1 \mathbf{e}_1^T$, \mathbf{I} is the identity matrix, $\mathbf{e}_1 = [1, 0, \cdots, 0]^T \in \mathbb{R}^{N \times 1}$,

$$\boldsymbol{\mathcal{E}} \stackrel{\scriptscriptstyle \Delta}{=} \boldsymbol{\mathcal{E}}(\mathbf{R}) = \left((\mathbf{R}^{-\frac{1}{2}} \mathbf{U})^T \otimes \mathbf{U}^H \mathbf{R}^{-\frac{1}{2}} \right) \boldsymbol{\mathcal{E}} \left((\mathbf{R}^{-\frac{1}{2}} \mathbf{U})^T \otimes \mathbf{U}^H \mathbf{R}^{-\frac{1}{2}} \right)^H,$$
(18)

 \mathcal{E} is defined in (14), and the unitary matrix U satisfies $\mathbf{U}^{H}\mathbf{R}^{-\frac{1}{2}}\mathbf{t} = (\mathbf{t}^{H}\mathbf{R}^{-1}\mathbf{t})^{\frac{1}{2}}\mathbf{e}_{1}.$

Proof: The proof is based on the second order Taylor expansion of ρ at $\hat{\mathbf{R}} = \mathbf{R}$ [17]. Then, the first order term equals 0 and the second order term equals $-\text{tr}\{\Delta \bar{\mathcal{E}}\}$.

According to Theorem 1, the average SCRL is related to the sum of the corresponding diagonal entries of $\bar{\boldsymbol{\mathcal{E}}}$ which can be regarded as the whitened MSE of the CM estimate. For the KMLE, it is easy to show that $(\mathbf{U}^{H}\mathbf{R}^{-\frac{1}{2}})\hat{\mathbf{R}}_{\text{KMLE}}(\mathbf{U}^{H}\mathbf{R}^{-\frac{1}{2}})^{H}$ is the KMLE of the identity matrix according to (7) and (8), with $\mathbf{U} = \mathbf{U}_{p} \otimes \mathbf{U}_{s}$, where the first columns of the unitary matrices \mathbf{U}_{p} and \mathbf{U}_{s} are $(\mathbf{a}_{p}^{H}\mathbf{R}_{p}^{-1}\mathbf{a}_{p})^{-\frac{1}{2}}\mathbf{R}_{p}^{-\frac{1}{2}}\mathbf{a}_{p}$ and $(\mathbf{a}_s^H \mathbf{R}_s^{-1} \mathbf{a}_s)^{-\frac{1}{2}} \mathbf{R}_s^{-\frac{1}{2}} \mathbf{a}_s$ respectively, and the other columns are in the orthogonal subspace of the first column. Thus we have

$$\bar{\boldsymbol{\mathcal{E}}}(\mathbf{R}) = \boldsymbol{\mathcal{E}}(\mathbf{I}). \tag{19}$$

Corollary 1 In Compound Gaussian clutter, for a sufficiently large K, the average SCRL corresponding to the KMLE is given by

$$\mathbb{E}\{\rho(\hat{\mathbf{R}}_{\mathrm{KMLE}})\} \approx 1 - \frac{c_1 M_{\kappa}}{K},\tag{20}$$

where

$$M_{\kappa} = \frac{N_s - 1}{N_p} + \frac{N_p - 1}{N_s}.$$

Proof: This proof is followed by substituting the CRB matrix in (15) into Theorem 1. Based on (19), we just need to evaluate the CRB when $\mathbf{R} = \mathbf{I}$. It can be verified that $tr{\{\Delta vec\{\mathbf{I}\}^H\}} = 0$ and the *n*th diagonal entries of \mathbf{P}_1 at $\mathbf{R} = \mathbf{I}$ with $n = (i-1)N_s + (j-1)NN_s + k + (l-1)N, i, j = 1, \dots, N_p, k, l = 1, \dots, N_s$, is given by

$$[\mathbf{P}_{1}]_{nn} = \begin{cases} \frac{N_{p} + N_{s} - 1}{N_{p}} & i = j, k = l \\ \frac{1}{N_{p}} & i = j, k \neq l \\ \frac{1}{N_{s}} & i \neq j, k = l \\ 0 & i \neq j, k \neq l \end{cases}$$
(21)

According to Corollary 1, when $N_p = 1$, the KMLE is reduced to the MLE, and the corresponding average SCRL is $\mathbb{E}\{\rho(\hat{\mathbf{R}}_{\text{MLE}})\} \approx 1 - \frac{c_1(N-1)}{K}$. For large N_p and N_s , we have $c_1 \approx 1$ and $M_{\kappa} \approx \frac{N_s}{N_p} + \frac{N_p}{N_s} \geq 2$, where the equality holds if $N_p = N_s$. This implies that the least amount of the training samples for 3 dB average SCRL is only $K = 2c_1M_{\kappa} \approx 4$. Roughly speaking, this is because polarization channels provide extra training samples (totally KN_p) for the estimation of space-time covariance matrix \mathbf{R}_s , while the space-time channels provide extra samples (totally KN_s) for the estimation of polarization covariance matrix \mathbf{R}_p .

5. EXTENSIONS

Theorem 1 and Corollary 1 can be suitable for other scenarios as well. In general, the parameters c_1 and M_{κ} in Corollary 1 are related to the texture and the structure, respectively. To be specific, we have $c_1 = 1$ for Gaussian case; $c_1 = \frac{N+1}{N}$ for Tyler's fixed point estimator (deterministic texture); $c_1 = \frac{v+N+1}{v+N}$ for the IG texture. For other random textures, c_1 may be obtained by the method in [14]. For the SCM which obeys the complex Wishart distribution, we have $M_{\text{SCM}} = N - 1$, with the corresponding MSE given by $\mathcal{E} = \frac{1}{K} \mathbf{R}^T \otimes \mathbf{R}$. For the complex-valued persymmetric structured covariance matrix, according to the real Wishart distribution [11], we have $M_{\text{pers}} = \frac{N-1}{2}$ since the covariance matrix can be equivalently transformed to a real SCM. Therefore, the approximate average SCRLs for SCM and persymmetric structured CM in the Gaussian case are $\mathbb{E}\{\rho_{\text{SCM}}\} \approx \frac{K-N+1}{K}$ and $\mathbb{E}\{\rho_{\text{pers}}\} \approx \frac{2K-N+1}{2K}$ respectively. They are very close to the results given in Sec. 2.2.

It needs to be pointed out that for the Toeplitz structure, unlike the persymmetric structure and the unstructured case (SCM), its average SCRL is related to the steering vector \mathbf{t} and the covariance matrix \mathbf{R} , i.e., $\bar{\boldsymbol{\mathcal{E}}}(\mathbf{R}) \neq \boldsymbol{\mathcal{E}}(\mathbf{I})$. This is because the Toeplitz structure implies a specific clutter spectral structure according to the Vandermonde decomposition $\mathbf{R} = \mathbf{A}\mathbf{\Lambda}\mathbf{A}^{H}$ [18], where \mathbf{A} is the Fourier matrix, and $\mathbf{\Lambda}$ is a positive definite diagonal matrix.

MLE is an asymptotically efficient estimator and its MSE will approximately achieve the CRB for large K. The MLE of a structured CM can be easily obtained by the MM method for a general linear model with $\mathbf{R} = \sum_{k=1}^{r} \sigma_i \boldsymbol{\Sigma}_i$ [8]. $\boldsymbol{\Sigma}_i$ are the matrix basis related to the corresponding CM structure. The CRB of this CM model can be obtained by replacing \mathbf{H}_{κ} by $\mathbf{H} = [\text{vec}\{\boldsymbol{\Sigma}_1\}, \cdots, \text{vec}\{\boldsymbol{\Sigma}_r\}]$ in (15). However, the MM method may not work well for some non-convex cases, e.g., negative unknown parameters $\sigma_i < 0$. In such cases, we may consider an alternative way to obtain the asymptotically efficient estimators, namely, the generalized least square (GLS) [19],

$$\min_{\boldsymbol{\sigma}} \|\mathbf{H}\boldsymbol{\sigma} - \hat{\mathbf{r}}_{\text{MLE}}\|_{\mathbf{W}}^2 = \min_{\boldsymbol{\sigma}} (\mathbf{H}\boldsymbol{\sigma} - \hat{\mathbf{r}}_{\text{MLE}}) \mathbf{W}^{-1} (\mathbf{H}\boldsymbol{\sigma} - \hat{\mathbf{r}}_{\text{MLE}})$$
(22)

where $\boldsymbol{\sigma} = [\sigma_1, \cdots, \sigma_r]^T$, $\mathbf{W} = \hat{\mathbf{R}}^T \otimes \hat{\mathbf{R}}$ with the consistent estimator $\hat{\mathbf{R}}$, $\hat{\mathbf{r}}_{\text{MLE}} = \text{vec}\{\hat{\mathbf{R}}_{\text{MLE}}\}$ is the vectorization of the unstructured MLE. A good initial choice of $\hat{\mathbf{R}}$ for GLS may be given by the LS method, i.e., the solution of (22) with $\mathbf{W} = \mathbf{I}$, which is more asymptotically efficient (approaching CRB with a smaller K) than $\hat{\mathbf{R}} = \hat{\mathbf{R}}_{\text{MLE}}$.

Further, Theorem 1 may be suitable for covariance estimation with some constraints by using the constrained CRB (CCRB) [20].

6. SIMULATION RESULTS

In this section, we validate the theoretical results of the average SCRL for polarimetric STAP by exploiting the Kronecker structure. We consider $N_p = 2, 3$ and $N_s = 4$, a polarization matrix with elements $[\mathbf{R}_p]_{ij} = \epsilon_p^{i-j}$ and a space-time covariance matrix with elements $[\mathbf{R}_s]_{ij} = \epsilon_s^{i-j}$. For simplicity, we assume $\epsilon = \epsilon_p = \epsilon_s$.

The average SCRL as a function of the number of training samples is compared in Fig. 1 and Fig. 2. It is observed that, the numerical results are accurately predicted by the theoretical expressions (20), and that the prediction is efficient only when K = 4 for the KMLE from Fig. 2. The average SCRLs of the KMLE and the MLE are independent of the correlation coefficient ϵ except for the deviation when K < N where the MLE is singular. With $N_s = 4$, according to (20), in order to achieve the same average SCRL, the MLE needs $\frac{N-1}{M_{\kappa}} = 4$ times more training samples than the KMLE for $N_p = 2$, and 7.33 times more samples for $N_p = 3$. This implies that



Fig. 1. The average SCRL as a function of the number of training samples with $N_p = 2$, $N_s = 4$, v = 4.



Fig. 2. The average SCRL as a function of the number of training samples with $N_p = 3$, $N_s = 4$, v = 4.

exploiting the *a priori* knowledge of the Kronecker structure can save a large amount of training samples, especially with more polarimetric channels.

7. CONCLUSION

We exploited the *a priori* knowledge of the Kronecker structure of CM in the polarimetric STAP problem. We analyzed how many training samples are required to achieve an SCRL by exploiting Kronecker structure. This methodology can also be extended to other scenarios with structured CM. Simulations have shown that significant savings in number of training samples can be achieved by exploiting structural information.

8. REFERENCES

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