AN ACCURATE PERTURBATION ANALYSIS ALGORITHM FOR MUSIC WITH TOEPLITZ COVARIANCE MATRIX

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ABSTRACT

In this paper, a new perturbation analysis algorithm for the MUltiple SIgnal Classification (MUSIC) estimator applied to a Hermitian Toeplitz covariance matrix is presented. Inspired by the perspective that the MUSIC algorithm can be recognized as a structured matrix approximation, the perturbation of parameter estimates can be predicted more accurately by seeking the minimum of a Frobenius norm. The prediction results are analytically expressed through a weighted least squares method. The performance of the MUSIC estimators can also be predicted using our algorithm.

Index Terms— Toeplitz, MUSIC, Perturbation analysis.

1. INTRODUCTION

MUltiple SIgnal Classification (MUSIC) is a subspace-based algorithm with high estimation accuracy and super resolution capability. The algorithm is widely used in direction of arrival (DOA) estimation, spectral analysis and system identification [1]. The core step of MUSIC is eigen-decomposing the signal covariance matrix (CVM) [2]. In practice, the exact CVM is unknown. An estimated CVM is a perturbed version of the exact CVM and is drawn from sample data, which leads to the perturbation of estimated parameters. Therefore, the perturbation analysis of MUSIC, which is used to predict the perturbation of signal parameters given the estimation error of the CVM, is directly related to the performance analysis of the MUSIC estimator. For the spectrum-MUSIC estimator [3], the minimum point shift of the objective function caused by subspace perturbation was analyzed in [4]. In [5], an operator-based approach was studied to predict the perturbation of the root-MUSIC algorithm. Other results were also presented in [6][7]. These approaches were based on a first-order approximation of the objective function. However, the expressions of the perturbation in the previous works were complicated for discussion, and the first-order approximation was not accurate enough for the perturbation predictions [8].

The performance of MUSIC is strongly dependent on the quality of CVM estimation. Although the sample CVM is often considered as the maximum likelihood estimator, a Hermitian Toeplitz constraint for the CVM estimator is quite helpful for improving the performance of MUSIC if the underlying signal is stationary [9]. The result in [10] showed that the Toeplitz rectification of the CVM can improve the estimation accuracy in low SNR situations. Another example is that when the auto-covariance function (ACF) of continuous lags can be estimated from sample data, a Hermitian Toeplitz CVM can be built with sample ACF by exploring the stationary feature of the underlying signal. Nonuniform arrays, such as minimum redundancy arrays [11] and coprime arrays [12][13], essentially construct a virtual co-array that can fully measure all the ACFs on the consecutive lags [14]. For the performance analysis of MUSIC with Hermitian Toeplitz CVM such as [15], a more accurate perturbation analysis is helpful in this context.

Inspired by the perspective that the MUSIC estimation can be recognized as a structured matrix approximation, we propose a new perturbation analysis algorithm for the MUSIC estimator that can avoid the complicated procedures adopted in the previous approaches. For a Hermitian Toeplitz CVM, the parameter perturbation can be predicted by finding the minimum of a Frobenius norm. Using weighted least squares, the prediction results are provided alongside concise analytical expressions. We demonstrate that the new algorithm provides higher prediction accuracy compared to previous results. Moreover, the mean square error (MSE) of the MUSIC estimator can be predicted given the second-order statistics of the ACF estimate. As an example, the performance analysis of a DOA estimator is given for a coprime array.

2. PROBLEM FORMULATION

Let $\boldsymbol{a}(\omega) = \left[1, e^{j\omega}, \dots, e^{j(M-1)\omega}\right]^T$ be a steering vector for the frequency ω. $(\cdot)^T$ denotes vector or matrix transposition. A white-noise-corrupted stationary signal with K uncorrelated sinusoids has the following CVM of an $M \times M$ Hermitian Toeplitz structure [3]:

$$\mathbf{R}(\boldsymbol{\omega}, \mathbf{p}, \sigma^2) = \sum_{k=1}^{K} p_k \boldsymbol{a}(\omega_k) \boldsymbol{a}^H(\omega_k) + \sigma^2 \mathbf{I}_M. \qquad (1)$$

In (1), $(\cdot)^H$ denotes the conjugate transpose, $\boldsymbol{\omega} = [\omega_1, \dots, \omega_K]$, $\mathbf{p} = [p_1, \dots, p_K]$ are the frequencies and powers of the K

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sinusoids, σ^2 is the noise power, and $\mathbf{I}_M$ is the identity matrix of size M. Denote $\mathbf{R} = \mathcal{T}(\mathbf{c})$ to indicate that $\mathbf{R}$ is Hermitian Toeplitz with its first column equals to $\mathbf{c} = [c_0, \ldots, c_{M-1}]^T$, where c_r is the signal ACF of lag r:

$$c_r = \sum_{k=1}^{K} p_k e^{j\omega_k r} + \sigma^2 \delta_r, \quad |r| \le M-1. \tag{2}$$

In (2), δ_r is the discrete Dirac function. Due to the conjugate symmetry of the ACF for a stationary signal, $c_{-m} = c_m^*$.

In this paper, we assume that the estimated CVM is Hermitian Toeplitz constrained:

$$\hat{\mathbf{R}} = \mathcal{T}(\hat{\mathbf{c}}), \tag{3}$$

where $\hat{\mathbf{c}} = [\hat{c}_0, \hat{c}_1, \ldots, \hat{c}_{M-1}]^T$ represents the estimation of the ACF. The ACF estimation error is $\Delta c_r = \hat{c}_r - c_r$ for all lags $|r| \le M-1$. From a perturbed CVM $\hat{\mathbf{R}}$, a MUSIC estimator derives the perturbed estimate $\hat{\omega}_k, 1 \le k \le K$. Then, the corresponding power $\hat{p}_k$ can be calculated given the frequency estimation. The exact parameter perturbation of the kth signal is

$$\Delta\hat{\omega}_k = \hat{\omega}_k - \omega_k, \qquad \Delta\hat{p}_k = \hat{p}_k - p_k. \tag{4}$$

In the perturbation analysis of MUSIC, one needs to predict the above perturbations, given the true CVM $\mathbf{R}$ and its perturbed estimation $\hat{\mathbf{R}}$ without applying MUSIC directly. Suppose that $\Delta\bar{\omega}_k$ and $\Delta\bar{p}_k$ are predicted perturbations and that

$$\Delta\bar{\omega}_k = \bar{\omega}_k - \omega_k, \qquad \Delta\bar{p}_k = \bar{p}_k - p_k, \tag{5}$$

where $\bar{\omega}_k, \bar{p}_k$ are the predicted frequency and power, respectively. A more accurate perturbation analysis should yield a smaller difference between $\Delta\bar{\omega}_k$ and $\Delta\hat{\omega}_k$.

3. PERTURBATION ANALYSIS OF MUSIC

Caratheodory's theorem [3] states that a unique parameter set $(\boldsymbol{\omega}, \mathbf{p}, \sigma^2)$ can be derived from a structured CVM. Combining this with Eq.(1), a one-to-one mapping exists between a parameter set $(\boldsymbol{\omega}, \mathbf{p}, \sigma^2)$ and a structured CVM, $\mathbf{R}$. The arithmetic steps in MUSIC can be interpreted from a constrained CVM approximation perspective.

- *Step 1*: Low rank matrix approximation.

  An estimated CVM $\hat{\mathbf{R}}$ is eigendecomposed as

  $$\hat{\mathbf{R}} = \hat{\mathbf{S}}\hat{\Lambda}_S\hat{\mathbf{S}}^H + \hat{\mathbf{G}}\hat{\Lambda}_G\hat{\mathbf{G}}^H + \hat{\sigma}^2\mathbf{I}_M, \tag{6}$$

  where $\hat{\Lambda}_S$ and $\hat{\Lambda}_G$ are diagonal matrices containing the large and small eigenvalues, respectively, $\hat{\mathbf{S}}$ and $\hat{\mathbf{G}}$ are their corresponding eigenvectors. The noise power $\hat{\sigma}^2$ is estimated that the average of the diagonal elements in $\hat{\Lambda}_G$ equals zero. Following Eckart-Young's theorem [16], the best rank-K matrix approximation to $\hat{\mathbf{R}} - \hat{\sigma}^2\mathbf{I}_M$ in the Frobenius norm sense is

  $$\mathbf{M}_K = \hat{\mathbf{S}}\hat{\Lambda}_S\hat{\mathbf{S}}^H. \tag{7}$$

- *Step 2*: Structured matrix approximation.

  The sinusoid frequencies $\hat{\boldsymbol{\omega}}$ are determined at K local minima of the following function.

  $$\hat{\omega}_k = \arg\min_{\omega} \ \boldsymbol{a}^H(\omega)\hat{\mathbf{G}}\hat{\mathbf{G}}^H\boldsymbol{a}(\omega). \tag{8}$$

  Due to the orthogonality between $\hat{\mathbf{S}}$ and $\hat{\mathbf{G}}$, (8) attempts to find K vectors $\boldsymbol{a}(\hat{\omega}_k)$ from the array manifold which most closely fit the signal subspace $\operatorname{span}\{\hat{\mathbf{S}}\}$.

  The sinusoid powers $\hat{\mathbf{p}}$ are determined via minimization of the Frobenius norm of matrix differences as

  $$\hat{\mathbf{p}} = \arg\min_{\mathbf{p}} \left\| \sum_{k=1}^{K} p_k \boldsymbol{a}(\hat{\omega}_k)\boldsymbol{a}^H(\hat{\omega}_k) - \mathbf{M}_K \right\|_F^2. \tag{9}$$

  Because $\hat{\mathbf{S}}$ span the column space of $\mathbf{M}_K$, the matrix

  $$\mathbf{Q}_K = \sum_{k=1}^{K} \hat{p}_k \boldsymbol{a}(\hat{\omega}_k)\boldsymbol{a}^H(\hat{\omega}_k) \tag{10}$$

  is a rank-K Hermitian Toeplitz matrix that is closest to $\mathbf{M}_K$ in the Frobenius norm sense.

To summarize, the procedure for MUSIC can be divided into two parts. First, a rank-K matrix $\mathbf{M}_K$ that is closest to $\hat{\mathbf{R}} - \hat{\sigma}^2\mathbf{I}_M$ is found. Second, a rank-K Hermitian Toeplitz matrix $\mathbf{Q}_K$ that is closest to $\mathbf{M}_K$ is found. Combining these two parts, the MUSIC estimator can be recognized as finding a structured matrix $\mathbf{Q}_K + \hat{\sigma}^2\mathbf{I}_M$, which is a structured approximation to $\hat{\mathbf{R}}$ in the Frobenius norm sense.

Suppose that $\bar{\boldsymbol{\omega}} = [\bar{\omega}_1, \bar{\omega}_2, \ldots, \bar{\omega}_K]$ are predicted sinusoid frequencies, $\bar{\mathbf{p}} = [\bar{p}_1, \bar{p}_2, \ldots, \bar{p}_K]$ are their respective powers, and $\bar{\sigma}^2$ is the predicted noise power. The parameter set $(\bar{\boldsymbol{\omega}}, \bar{\mathbf{p}}, \bar{\sigma}^2)$ corresponds to a structured CVM, given by

$$\mathbf{R}(\bar{\boldsymbol{\omega}}, \bar{\mathbf{p}}, \bar{\sigma}^2) = \sum_{k=1}^{K} \bar{p}_k \boldsymbol{a}(\bar{\omega}_k)\boldsymbol{a}^H(\bar{\omega}_k) + \bar{\sigma}^2\mathbf{I}_M. \tag{11}$$

Then, a MUSIC estimator, which is considered as a constrained CVM approximation, would minimize

$$\mathcal{F} = \left\| \mathbf{R}\left(\bar{\boldsymbol{\omega}}, \bar{\mathbf{p}}, \bar{\sigma}^2\right) - \hat{\mathbf{R}} \right\|_F^2. \tag{12}$$

The prediction of perturbations can be performed by seeking the minimum of the above equation, which is detailed in the following subsections.

3.1. Factorization of the structured CVM Approximation

By the Hermitian Toeplitz structure, we can write $\mathbf{R}(\bar{\boldsymbol{\omega}}, \bar{\mathbf{p}}, \bar{\sigma}^2) = \mathcal{T}(\bar{\mathbf{c}})$, where $\bar{\mathbf{c}} = [\bar{c}_0, \bar{c}_1, \ldots, \bar{c}_{M-1}]^T$ is the first column. The difference between $\mathbf{R}\left(\bar{\boldsymbol{\omega}}, \bar{\mathbf{p}}, \bar{\sigma}^2\right)$ and $\hat{\mathbf{R}}$ is

$$\begin{aligned} \mathbf{R}\left(\bar{\boldsymbol{\omega}}, \bar{\mathbf{p}}, \bar{\sigma}^2\right) - \hat{\mathbf{R}} &= \mathcal{T}(\bar{\mathbf{c}}) - \mathcal{T}(\hat{\mathbf{c}}) \\ &= \mathcal{T}(\bar{\mathbf{c}} - \mathbf{c}) - \mathcal{T}(\hat{\mathbf{c}} - \mathbf{c}), \end{aligned} \tag{13}$$

where $\hat{\mathbf{c}}-\mathbf{c}$ is the perturbation of the ACF. Using the Toeplitz property, equation (12) becomes

$$\mathcal{F}=\sum_{r=-M+1}^{M-1}(M-|r|)\left((\bar{c}_r-c_r)-\Delta\hat{c}_r\right)^2. \quad (14)$$

Assuming that $\Delta\bar{\omega}_k, \Delta\bar{p}_k$ and $\bar{\mathbf{c}}-\mathbf{c}$ are sufficiently small, we can approximate $\bar{\mathbf{c}}-\mathbf{c}$ by the first-order of Taylor expansion:

$$\begin{aligned}\bar{c}_r-c_r&=\sum_{k=1}^{K}\left[\bar{p}_k e^{j\bar{\omega}_k r}-p_k e^{j\omega_k r}\right]+\left(\bar{\sigma}^2-\sigma^2\right)\delta_r\\ &\simeq\sum_{k=1}^{K}\left[e^{j\omega_k r}\Delta\bar{p}_k+jre^{j\omega_k r}\bar{p}_k\Delta\bar{\omega}_k\right]+\Delta\bar{\sigma}^2\delta_r.\end{aligned} \quad (15)$$

Substituting $\bar{c}_r-c_r$ in equation (14) by its approximation in (15), $\mathcal{F}$ is factorized into a combination of the perturbation predictions $\Delta\bar{\omega}_k, \Delta\bar{p}_k, \Delta\bar{\sigma}^2$ and ACF perturbations $\Delta\hat{c}_r$.

3.2. F-Norm Minimization and Perturbation Analysis

For $r=0$, one can always find a $\bar{\sigma}^2$ that satisfies $\bar{c}_0-c_0=\Delta\hat{c}_0$. The diagonal elements in $\mathbf{R}\left(\bar{\boldsymbol{\omega}},\bar{\mathbf{p}},\bar{\sigma}^2\right)-\hat{\mathbf{R}}$ are zero when $\mathcal{F}$ reaches the minimum. Therefore, only the off-diagonal elements contribute to the minimum of $\mathcal{F}$. Denote

$$\Delta\bar{\mathbf{p}}=\left[\Delta\bar{p}_1,\ldots,\Delta\bar{p}_K\right]^T, \quad (16a)$$
$$\bar{\mathbf{p}}\Delta\bar{\boldsymbol{\omega}}=\left[\bar{p}_1\Delta\bar{\omega}_1,\ldots,\bar{p}_K\Delta\bar{\omega}_K\right]^T, \quad (16b)$$
$$\Delta\hat{\mathbf{C}}=\left[\Delta\hat{c}_{1-M},\ldots,\Delta\hat{c}_{-1},\Delta\hat{c}_1,\ldots,\Delta\hat{c}_{M-1}\right]^T. \quad (16c)$$

Equation (14) can be written as

$$\mathcal{F}=\left(\mathbf{J}\begin{bmatrix}\Delta\bar{\mathbf{p}}\\ \bar{\mathbf{p}}\Delta\bar{\boldsymbol{\omega}}\end{bmatrix}-\Delta\hat{\mathbf{C}}\right)^H\mathbf{W}\left(\mathbf{J}\begin{bmatrix}\Delta\bar{\mathbf{p}}\\ \bar{\mathbf{p}}\Delta\bar{\boldsymbol{\omega}}\end{bmatrix}-\Delta\hat{\mathbf{C}}\right). \quad (17)$$

In (17), $\mathbf{W}=\mathrm{diag}\left[1,2,\ldots,M-1,M-1,M-2,\ldots,1\right]$ is a weighting matrix, $\mathbf{J}=[\mathbf{J}_p,\mathbf{J}_\omega]$, and $\mathbf{J}_p,\mathbf{J}_\omega$ are $2(M-1)\times K$ coefficient matrices, detailed as

$$\mathbf{J}_p=\begin{bmatrix}e^{(1-M)j\omega_1} & \cdots & e^{(1-M)j\omega_K}\\ \vdots & \ddots & \vdots\\ e^{-j\omega_1} & \cdots & e^{-j\omega_K}\\ e^{j\omega_1} & \cdots & e^{j\omega_K}\\ \vdots & \ddots & \vdots\\ e^{(M-1)j\omega_1} & \cdots & e^{(M-1)j\omega_K}\end{bmatrix}, \quad (18)$$

$$\mathbf{J}_\omega=\left[\frac{d}{d\omega_1}\mathbf{J}_p(1),\ldots,\frac{d}{d\omega_K}\mathbf{J}_p(K)\right], \quad (19)$$

where $\mathbf{J}_p(k)$ is the kth column vector in $\mathbf{J}_p$, and $\frac{d}{d\omega_k}$ is the derivative with respect to ω_k.

As discussed above, a MUSIC estimator can be regarded as a minimization of $\mathcal{F}$ in (17), which can be solved using weighted least squares:

$$\begin{bmatrix}\Delta\bar{\mathbf{p}}\\ \bar{\mathbf{p}}\Delta\bar{\boldsymbol{\omega}}\end{bmatrix}=\mathbf{L}(\omega)\Delta\hat{\mathbf{C}}, \quad (20)$$

where $\mathbf{L}(\omega)=\left(\mathbf{J}^H\mathbf{W}\mathbf{J}\right)^{-1}\mathbf{J}^H\mathbf{W}$ is a $2K\times 2(M-1)$ matrix. Using $\{\cdot\}_k$ to represent the kth element of a column vector, $\Delta\bar{p}_k$ and $\Delta\bar{\omega}_k$ can be represented as

$$\Delta\bar{p}_k=\left\{\mathbf{L}(\omega)\Delta\hat{\mathbf{C}}\right\}_k, \quad (21a)$$
$$\Delta\bar{\omega}_k=\frac{\left\{\mathbf{L}(\omega)\Delta\hat{\mathbf{C}}\right\}_{k+K}}{p_k+\Delta\bar{p}_k}. \quad (21b)$$

3.3. Performance of the MUSIC Estimator

If the ACF estimator is unbiased, i.e., $E\left(\hat{c}_r\right)=c_r$ for all $|r|\leq M-1$, the minimizer when (17) is approximated to the first order is also unbiased due to the linearity in (20). For the second-order statistics of $\Delta\bar{\boldsymbol{\omega}}$ and $\Delta\bar{\mathbf{p}}$, consider the variance-covariance matrix of $[\Delta\bar{\mathbf{p}},\bar{\mathbf{p}}\Delta\bar{\boldsymbol{\omega}}]^T$, which is presented in the following $2K\times 2K$ matrix:

$$\begin{aligned}\boldsymbol{\Theta}&=E\left(\begin{bmatrix}\Delta\bar{\mathbf{p}}\\ \bar{\mathbf{p}}\Delta\bar{\boldsymbol{\omega}}\end{bmatrix}\begin{bmatrix}\Delta\bar{\mathbf{p}}\\ \bar{\mathbf{p}}\Delta\bar{\boldsymbol{\omega}}\end{bmatrix}^T\right)\\ &=\mathbf{L}(\omega)E\left(\Delta\hat{\mathbf{C}}\Delta\hat{\mathbf{C}}^T\right)\mathbf{L}^T(\omega).\end{aligned} \quad (22)$$

In (22), $E\left(\Delta\hat{\mathbf{C}}\Delta\hat{\mathbf{C}}^T\right)$ is the second-order statistics of the ACF estimation, whose detailed expression is determined by the considered application. Under the circumstances as in [17][18], the second-order statistics are readily given. The MSE of the MUSIC estimator can be drawn from the diagonal elements of $\boldsymbol{\Theta}$ directly:

$$\mathrm{var}\left(\bar{p}_k\right)=\left\{\boldsymbol{\Theta}\right\}_{k,k}, \quad (23a)$$
$$\mathrm{var}\left(\bar{\omega}_k\right)=\frac{\left\{\boldsymbol{\Theta}\right\}_{k+K,k+K}}{p_k^2}. \quad (23b)$$

An intuitive explanation of (23) can be provided. First, the MSE of $\hat{\omega}_k$ is in general inversely proportional to the square of the sinusoid power. Second, $\mathbf{L}(\omega)$ influences the performance as well. For example, when two distinct frequencies are similar, the condition number of $\mathbf{L}(\omega)$ becomes large, which increases the MSE of the estimator.

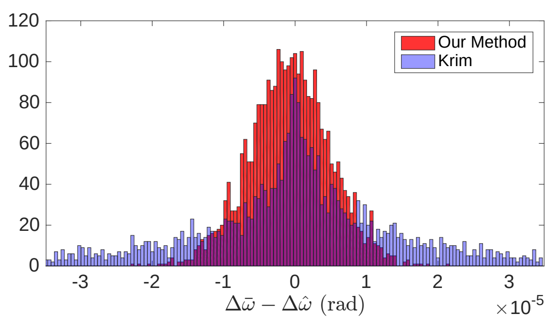
4. NUMERICAL RESULTS AND DISCUSSIONS

The validity of the new algorithm is verified by comparing the theoretical predictions to the simulation results. For illustration purposes, we consider a coprime array with $N_1=5, N_2=7$ sensors, and the unit inter-element spacing is half of a wavelength. In [14], it is shown that when using MUSIC for the DOA estimation, an empirical Hermitian Toeplitz CVM would be equivalent to the CVM obtained by the spatial smoothing in [12]. In our simulations, a Hermitian Toeplitz CVM of size $N_1N_2+1=36$ is generated. In one experiment, the CVM $\hat{\mathbf{R}}$ is estimated using 300 snapshots.

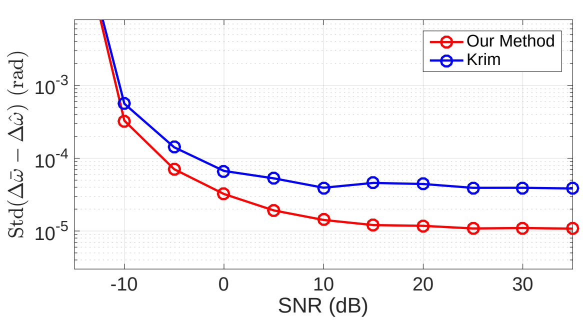
4.1. Perturbation Prediction

Two uncorrelated narrowband signals with zero-mean, complex circular Gaussian amplitudes and equal power arrive at the arrays from normalized DOA $\omega_1 = -0.1$ and $\omega_2 = 0.1$. The noise on the array elements is assumed Gaussian white, SNR=10dB (SNR is defined at one sensor as the ratio between the power of one incident signal to the noise power). The predicted DOA perturbation of the first source $\Delta\bar{\omega}_1$ from our approach is compared with the result given by Krim [5]. A total of 3,000 Monte Carlo trials were performed, and we utilize histograms to present the distribution of $\Delta\bar{\omega}_1 - \Delta\hat{\omega}_1$, i.e., the difference between the predicted and true perturbations of the signal angular frequency. In Fig.1(a), the differences obtained by the proposed method are more concentrated about zero, which implies a more accurate prediction.

Moreover, we compare the prediction accuracy for SNRs ranging from 0 dB to 30 dB, where each SNR is assigned 500 trials to calculate the standard deviation of $\Delta\bar{\omega}_1 - \Delta\hat{\omega}_1$. In Fig.1(b), $\mathrm{Std}\,(\Delta\bar{\omega}_1 - \Delta\hat{\omega}_1)$ by our prediction is consistently lower, showing that our method obtains a higher accuracy.



(a) Histogram of $\Delta\bar{\omega}_1 - \Delta\hat{\omega}_1$



(b) Comparison of $\mathrm{Std}\,(\Delta\bar{\omega}_1 - \Delta\hat{\omega}_1)$

Fig. 1. A comparison of the distribution of $\Delta\bar{\omega}_1 - \Delta\hat{\omega}_1$. The result when SNR=10dB is shown by histogram in (a), and the standard deviation under various SNRs is shown in (b).

4.2. Performance Analysis of coprime array

To evaluate the predicted performance of the MUSIC estimator in coprime array, we maintain the parameters as in the previous subsection except that the powers of the sources are $p_1 : p_2 = 3 : 1$, that Source 1 is 5dB stronger than Source 2. The MSEs of the DOA estimator for SNRs between -20 dB and 30 dB are derived (SNR=p_1/σ^2). The theoretical MSE of the DOA estimator is given by (22), where the second-order statistics of the co-array signal are computed by the coprime parameters. The result is compared to empirical experiment with 500 trials.

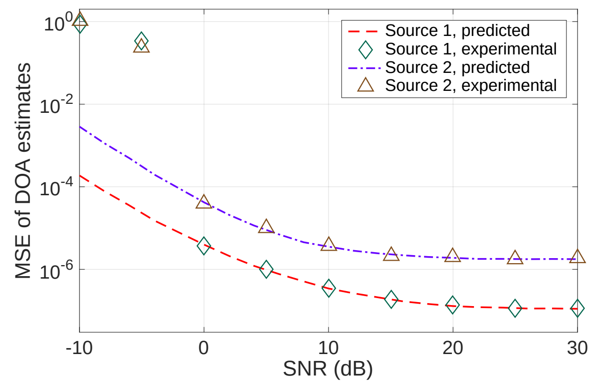


Fig. 2. Predicted vs. experimental MSE of the MUSIC DOA estimator of both sources. The predicted MSEs are plotted with dashed lines, and the experimental MSEs are indicated by $\Diamond$ and $\triangle$ for the two sources.

Fig.2 shows the comparison between the predicted and the experimental performance of the DOA estimator. Because the two sources have different powers, the MSE of the stronger source is lower than that of the weaker source. Under high SNRs, the experimental MSE of the MUSIC estimator matches the predicted MSE very well. When the SNR is low, the DOA estimator fails due to subspace swap or false detection. It is therefore verified that the performance of coprime arrays can be predicted using our method.

5. CONCLUSION

We presented a new perturbation analysis algorithm for the MUSIC estimator in which the estimated CVM is Hermitian Toeplitz. Analytic expressions for perturbation predictions were given, and the simulation results demonstrated that the new algorithm obtained a higher accuracy than the previous methods. The new algorithm can also be used for the performance analysis of the Hermitian-Toeplitz-covariance-based applications with high accuracy, such as coprime arrays.

6. REFERENCES

[1] Hamid Krim and Mats Viberg, "Two decades of array signal processing research: The parametric approach," *IEEE Signal Process. Mag.*, vol. 13, no. 4, pp. 67–94, jul 1996.

[2] R. Schmidt, "Multiple emitter location and signal parameter estimation," *IEEE Trans. Antennas Propag.*, vol. 34, no. 3, pp. 276–280, 1986.

[3] Petre Stoica and Randolph L. Moses, *Spectral Analysis of Signals*, Prentice Hall, Upper Saddle River, NJ, 2005.

[4] Petre Stoica and Arye Nehorai, "MUSIC, maximum likelihood, and Cramer-Rao bound," *IEEE Trans. Acoust.*, vol. 37, no. 5, pp. 720–741, 1989.

[5] H. Krim, P. Forster, and J.G. Proakis, "Operator approach to performance analysis of root-MUSIC and root-min-norm," *IEEE Trans. Signal Process.*, vol. 40, no. 7, 1992.

[6] Fu Li, Hui Liu, and Richard J. Vaccaro, "Performance analysis for DOA estimation algorithms: unification, simplification, and observations," *IEEE Trans. Aerosp. Electron. Syst.*, vol. 29, no. 4, pp. 1170–1184, 1993.

[7] Jun Liu, Xiangqian Liu, and Xiaoli Ma, "First-order perturbation analysis of singular vectors in singular value decomposition," *IEEE Trans. Signal Process.*, vol. 57, no. 7, pp. 3044–3049, 2008.

[8] Zhengyuan Xu, "Perturbation analysis for subspace decomposition with applications in subspace-based algorithms," *IEEE Trans. Signal Process.*, vol. 50, no. 11, pp. 2820–2830, 2002.

[9] H Li and P Stoica, "Computationally efficient maximum likelihood estimation of structured covariance matrices," *IEEE Trans. Signal Process.*, vol. 47, no. 5, pp. 1314–1323, 1999.

[10] P Vallet and P Loubaton, "Toeplitz rectification and DOA estimation with music," in *2014 IEEE Int. Conf. Acoust. Speech Signal Process.*, 2014, pp. 2237–2241.

[11] James Martin Nohrden, *Spectral Estimation Using Nonuniform Sampling*, Ph.D. thesis, Massachusetts Institute of Technology, 1995.

[12] Piya Pal and P. P. Vaidyanathan, "Coprime sampling and the music algorithm," *Digit. Signal Process. Work. IEEE Signal Process. Educ. Work. (DSP/SPE), 2011 IEEE*, vol. 0, no. 1, pp. 289–294, 2011.

[13] Zhao Tan, Yonina C. Eldar, and Arye Nehorai, "Direction of arrival estimation using co-prime arrays: A super resolution viewpoint," *IEEE Trans. Signal Process.*, vol. 62, no. 21, pp. 5565–5576, 2014.

[14] Chun-lin Liu and P P Vaidyanathan, "Remarks on the spatial smoothing step in coarray MUSIC," *Signal Process. Lett. IEEE*, vol. 22, no. 9, pp. 1438–1442, 2015.

[15] Mianzhi Wang and Arye Nehorai, "Coarrays, MUSIC, and the Cram\'er Rao Bound," pp. 1–13, 2016.

[16] Carl Eckart and Gale Young, "The approximation of one matrix by another of lower rank," *Psychometrika*, vol. 1, no. 3, pp. 211–218, 1936.

[17] R.J Marshall, "Autocorrelation estimation of time series with randomly missing observations," *Biometrika*, vol. 67, no. 3, pp. 567–570, 1980.

[18] Yonina Rosen and Boaz Porat, "Optimal ARMA parameter estimation based on the sample covariances for data with missing observations," *IEEE Trans. Inf. Theory*, vol. 35, no. 2, pp. 342–349, 1989.