DIRECTION FINDING USING SPARSE LINEAR ARRAYS WITH MISSING DATA

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ABSTRACT

We investigate the problem of direction of arrival (DOA) estimation using sparse linear arrays, such as co-prime and nested arrays, in the case of missing data resulting from sensor failures. We introduce a signal model where sensor failures occur after taking certain number of snapshots. We formulate a structured covariance estimation problem by exploiting the special geometry of sparse linear arrays, which also provides enhanced degrees of freedom. Numerical examples show that, by utilizing the information in both complete measurements and incomplete measurements, our method achieves better estimation accuracy than the traditional method using only complete measurements.

***Index Terms*—** DOA estimation, missing data, maximum likelihood, coprime array, nested array

1. INTRODUCTION

Sparse linear arrays, such as minimum redundancy arrays (MRA) [1–3], nested arrays [4–6], and co-prime arrays [7–10], have the attractive property of providing $\mathcal{O}(M^2)$ degrees of freedom with only M sensors. The extra degrees of freedom are exploited by constructing an augmented covariance matrix from the difference coarray model [11, 12]. Due to their dependence on the coarray geometry, sparse linear arrays are more susceptible to sensor failures. If the measurements from one or more sensors are missing, the coarray structure will be partially destroyed, leading to performance degradation and loss of degrees of freedom.

Tackling missing data is important in robust DOA estimation, and many previous work has addressed the problem. Notably, in [13], Larsson et al. proposed a Cholesky parameterization based maximum likelihood estimator, and analyzed its asymptotic performance. However, their model is based on uniform linear arrays (ULAs), and requires a sequential failure pattern. In practice, any sensor may fail, so the sequential assumption may not be true. Recent advances in matrix completion [14, 15] and atomic norm minimization [16, 17] also bring new methods to tackle the missing data problem. By exploiting the low-rank property of the signal subspace, it is possible to extrapolate the missing data via semidefinite programming (SDP). However, when the number of measurements is large, the resulting SDP will be computationally expensive to solve. In this paper, we consider the direction finding problem using general sparse linear arrays with incomplete measurements. We do not assume a sequential failure pattern. We focus on deriving an algorithm that utilizes the information in both complete measurements and incomplete measurements based on the maximum-likelihood approach. We first estimate the augmented covariance matrix by exploiting its Toeplitz structure, and then apply the MUSIC algorithm [18] to obtain the DOA estimates. We derive the Cramér Rao bound (CRB) and confirm the efficacy of our algorithms via numerical examples.

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2. PROBLEM FORMULATION

We consider a sparse linear array whose sensors are located on a uniform grid. We represent the sensor locations by the integer set $\bar{\mathcal{D}} = \{\bar{d}_1, \bar{d}_2, \ldots, \bar{d}_M\}$, where M is the number of sensors. The actual sensor locations d_i are given by $\bar{d}_i d_0$ for $i = 1, 2, \ldots, M$, where d_0 denotes the grid size. Such an array can also be viewed as a thinned uniform linear array (ULA) that consists of $M_0 = \bar{d}_M + 1$ sensors. For example, a co-prime array whose sensors are located at $[0, 2, 3, 4, 6, 9]d_0$ can be viewed as a 10-sensor ULA with the 2nd, 6th, 8th, and 9th sensors removed.

We consider K far-field uncorrelated narrowband signals impinging on the array from directions $\boldsymbol{\theta} = [\theta_1, \theta_2, \cdots, \theta_K]^T$. The received the signal vectors are given by

$$\boldsymbol{y}(t) = \boldsymbol{S}\boldsymbol{A}_{\mathrm{U}}(\boldsymbol{\theta})\boldsymbol{x}(t) + \boldsymbol{n}(t),\ t = 1, 2, \ldots, N, \tag{1}$$

where $\boldsymbol{A}_{\mathrm{U}}(\boldsymbol{\theta}) = [\boldsymbol{a}_{\mathrm{U}}(\theta_1), \boldsymbol{a}_{\mathrm{U}}(\theta_2), \ldots, \boldsymbol{a}_{\mathrm{U}}(\theta_K)]$ is the steering matrix of a M_0-sensor ULA [19]. $\boldsymbol{S}$ is a $M \times M_0$ selection matrix, where S_{mn} is one if and only if the m-th sensor in the sparse linear array corresponds to the n-th sensor in the ULA, and otherwise zero. $x(t)$ is the source signal, and $n(t)$ is the additive complex white Gaussian noise. We assume that the source signals follow the unconditional model [19], and there is no temporal correlation between each snapshot.

With the above assumptions, the covariance matrix is given by $\boldsymbol{R} = \mathbb{E}[\boldsymbol{y}(t)\boldsymbol{y}^H(t)] = \boldsymbol{S}\boldsymbol{R}_{\mathrm{U}}\boldsymbol{S}^T$, where $\boldsymbol{R}_{\mathrm{U}} = \boldsymbol{A}_{\mathrm{U}}\boldsymbol{P}\boldsymbol{A}_{\mathrm{U}}^H + \sigma_{\mathrm{n}}^2\boldsymbol{I}$, $\boldsymbol{P} = \mathrm{diag}(p_1, p_2, \ldots, p_K)$, and p_k is the power of k-th source. Therefore the covariance matrix of a sparse linear array is a compressed version of the covariance matrix of a ULA. By vectorizing $\boldsymbol{R}$, we obtain

$$\boldsymbol{r} = (\boldsymbol{S} \otimes \boldsymbol{S})(\boldsymbol{A}_{\mathrm{U}}^* \odot \boldsymbol{A}_{\mathrm{U}})\boldsymbol{p} + \sigma_{\mathrm{n}}^2\boldsymbol{i}, \tag{2}$$

where $\boldsymbol{r} = \mathrm{vec}(\boldsymbol{R})$, $\boldsymbol{p} = [p_1, p_2, \cdots, p_K]^T$, and $\boldsymbol{i} = \mathrm{vec}(\boldsymbol{I})$, $\otimes$ denotes the Kronecker product, $\odot$ denotes the Khatri-Rao product (i.e., column-wise Kronecker product), and $\mathrm{vec}(\boldsymbol{A})$ converts $\boldsymbol{A}$ into a column vector by stacking the columns of $\boldsymbol{A}$ [20]. Model (2) resembles a measurement model with deterministic sources and noise, and $(\boldsymbol{S} \otimes \boldsymbol{S})(\boldsymbol{A}_{\mathrm{U}}^* \odot \boldsymbol{A}_{\mathrm{U}})$ embeds a steering matrix of a virtual array with enhanced degrees of freedom, whose sensor locations are given by $\bar{\mathcal{D}}_{\mathrm{co}} = \{(\bar{d}_m - \bar{d}_n) | \bar{d}_m, \bar{d}_n \in \bar{\mathcal{D}}\}$. If $\bar{\mathcal{D}}_{\mathrm{co}}$ consists of consecutive integers from $-M_0 + 1$ to $M_0 - 1$, we call the sparse linear array *complete*. If a sparse linear array is complete (e.g., minimum

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redundancy arrays and nested arrays), it is possible to estimate the elements in $\boldsymbol{R}_{\mathrm{U}}$ using rank enhanced spatial smoothing [4] or more sophisticated methods [21]. We are then able to identify more sources than the number of sensors through $\boldsymbol{R}_{\mathrm{U}}$. On the other hand, if a sparse linear array is incomplete (e.g., co-prime arrays), we define $\tilde{M}_0$ as the largest M such that $\{-M+1,\ldots,0,\ldots,M-1\} \subset \bar{\mathcal{D}}_{\mathrm{co}}$. In this case, we can recover only a $\tilde{M}_0 \times \tilde{M}_0$ submatrix of $\boldsymbol{R}_{\mathrm{U}}$ using similar methods. If $\tilde{M}_0 > M$, we again are able to identify more sources than the number of sensors.

We now consider the signal model with missing data. Without loss of generality, we consider L sampling periods. During the first period, we assume all the sensors are functioning normally. This assumption is reasonable because if some sensors fail from the beginning, we can simply remove them and form a new sparse linear array whose sensors are all functional during the first period. During the l-th ($2 \leq l \leq L$) period, some sensors fail and the measurement data from these sensors are missing. Let M_l be the number of working sensors during the l-th period. Let $\boldsymbol{T}_l$ be a selection matrix of size $M_l \times M$ such that the (i,j)-th element $\boldsymbol{T}_l$ is one if and only if the j-th sensor in the sparse linear array is the i-th working sensor during the l-th period, and otherwise zero. For notational simplicity, we define $\boldsymbol{T}_1 = \boldsymbol{I}_M$. After discarding the measurements from the malfunctioning sensors, the snapshots taken during the l-th period are given by

$$\boldsymbol{y}_l(t) = \boldsymbol{T}_l[\boldsymbol{S}\boldsymbol{A}_{\mathrm{U}}(\boldsymbol{\theta})\boldsymbol{x}(t) + \boldsymbol{n}(t)], \tag{3}$$

for $t = N_1 + \cdots + N_{l-1} + 1, \ldots, N_1 + \cdots + N_{l-1} + N_l$, where N_l is the number of snapshots collected during the l-th period. The total number of snapshots is denoted by $N = \sum_{l=1}^{L} N_l$. Correspondingly, we can collect L different sample covariance matrices $\hat{\boldsymbol{R}}_l = 1/N_l \sum_{t=N_1+\cdots+N_{l-1}+1}^{N_1+\cdots+N_{l-1}+N_l} \boldsymbol{y}_l(t)\boldsymbol{y}_l^H(t)$, $l = 1, 2, \ldots, L$. We also define their expectations as

$$\boldsymbol{R}_l = \mathbb{E}[\hat{\boldsymbol{R}}_l] = \boldsymbol{T}_l\boldsymbol{S}\boldsymbol{R}_{\mathrm{U}}\boldsymbol{S}^T\boldsymbol{T}_l^T + \sigma_{\mathrm{n}}^2\boldsymbol{I}, \tag{4}$$

whose vectorized versions are given by

$$\boldsymbol{r}_l = \mathrm{vec}(\boldsymbol{R}_l) = (\boldsymbol{T}_l\boldsymbol{S} \otimes \boldsymbol{T}_l\boldsymbol{S})(\boldsymbol{A}_{\mathrm{U}}^* \odot \boldsymbol{A}_{\mathrm{U}})\boldsymbol{p} + \sigma_{\mathrm{n}}^2\boldsymbol{i}. \tag{5}$$

Because of the missing data, $(\boldsymbol{T}_l\boldsymbol{S} \otimes \boldsymbol{T}_l\boldsymbol{S})(\boldsymbol{A}_{\mathrm{U}}^* \odot \boldsymbol{A}_{\mathrm{U}})$ no longer embeds a desired virtual array steering matrix and existing methods cannot be directly applied. If we use only $\hat{\boldsymbol{R}}_1$ for estimation, we lose much information provided in $\hat{\boldsymbol{R}}_l$ ($2 \leq l \leq L$). Therefore an estimator that utilizes all the information in $\hat{\boldsymbol{R}}_l$ ($1 \leq l \leq L$) is desired.

3. ESTIMATION IN THE PRESENCE OF MISSING DATA

3.1. Ad-hoc Estimator

The ad-hoc estimator for our signal model is inspired by redundancy averaging [22, 23], and is an extension of the ad-hoc estimator in [13]. Let $\mathcal{V}_k = \{(m,n) | \bar{d}_m - \bar{d}_n = k, \bar{d}_m, \bar{d}_n \in \bar{\mathcal{D}}\}$. Let $\mathcal{A}_{m,n}$ denote the set of snapshot indices when both the m-th and the n-th sensor are working. We define

$$u_k = \frac{\sum_{(m,n)\in\mathcal{V}_k} \sum_{t\in\mathcal{A}_{m,n}} y_m(t)y_n^*(t)}{\sum_{(m,n)\in\mathcal{V}_k} |\mathcal{A}_{m,n}|}, \tag{6}$$

where $\boldsymbol{y}(t) = [y_1(t), \cdots, y_M(t)]$ is the full measurement vector before discarding invalid data, $y_m(t)$ is the output of the m-th sensor, and $|\mathcal{A}|$ denotes the cardinality of $\mathcal{A}$. For complete arrays, we can obtain u_k for $k = -M_0+1, -M_0+2, \ldots, M_0-1$, and the ad-hoc estimate of $\boldsymbol{R}_{\mathrm{U}}$ is given by

$$\hat{\boldsymbol{R}}_{\mathrm{U}}^{(\mathrm{ad-hoc})} = \begin{bmatrix} u_0 & u_{-1} & \cdots & u_{-M_0+1} \\ u_1 & u_0 & \cdots & u_{-M_0+2} \\ \vdots & \vdots & \ddots & \vdots \\ u_{M_0} & u_{M_0-1} & \cdots & u_0 \end{bmatrix}. \tag{7}$$

We can then apply MUSIC or other DOA estimation methods to $\hat{\boldsymbol{R}}_{\mathrm{U}}^{(\mathrm{ad-hoc})}$ to obtain the DOA estimates.

For incomplete arrays, we can use a similar construction to obtain a $\tilde{M}_0 \times \tilde{M}_0$ matrix from u_k, $k = -\tilde{M}_0+1, \tilde{M}_0+2, \ldots, \tilde{M}_0-1$, which is the estimate of a submatrix of $\boldsymbol{R}_{\mathrm{U}}$.

It should be noted that while (6) and (7) are computationally efficient to evaluate, the resulting $\hat{\boldsymbol{R}}_{\mathrm{U}}^{(\mathrm{ad-hoc})}$ is not guaranteed to be positive definite, which may be undesired in some applications.

3.2. Maximum-Likelihood Based Estimators

Based on the results in [24], the negative log-likelihood function of our model is given by

$$L(\boldsymbol{R}_1, \ldots, \boldsymbol{R}_L) = \sum_{l=1}^{L} N_l[\log|\boldsymbol{R}_l| + \mathrm{tr}(\boldsymbol{R}_l^{-1}\hat{\boldsymbol{R}}_l)], \tag{8}$$

where we have omitted the constants.

Observe that $\boldsymbol{R}_{\mathrm{U}}$ is Hermitian Toeplitz. It is possible to reparameterize $\boldsymbol{R}_{\mathrm{U}}$ by exploiting the Toeplitz structure, and the estimation of $\boldsymbol{R}_{\mathrm{U}}$ becomes a structure covariance estimation problem. In the following discussion, we consider only complete arrays. Extension to non-restricted arrays will be discussed in the remarks.

Following the idea in [25], we construct the structured matrices as follows. Let $\boldsymbol{I}_M^{(i)}$ denotes the $M \times M$ matrix whose elements are zero except for the i-th upper diagonal (i.e., $\boldsymbol{I}_M^{(i)}(m,n) = \delta(n-m-i)$, where $\delta(n)$ is the Kronecker delta). For a given positive integer M, we define the matrices $\{\boldsymbol{Q}_M^{(i)}\}_{i=1}^{2M-1}$ as

$$\boldsymbol{Q}_M^{(i)} = \begin{cases} \boldsymbol{I}_M, & i = 1, \\ \boldsymbol{I}_M^{(i-1)} + (\boldsymbol{I}_M^{(i-1)})^T, & 2 \leq i \leq M, \\ -j\boldsymbol{I}_M^{(i-M)} + j(\boldsymbol{I}_M^{(i-M)})^T, & M+1 \leq i \leq 2M-1. \end{cases} \tag{9}$$

Then we are able to express $\boldsymbol{R}_{\mathrm{U}}$ as

$$\boldsymbol{R}_{\mathrm{U}} = \sum_{i=1}^{2M_0-1} c_i\boldsymbol{Q}_{M_0}^{(i)}, \tag{10}$$

where $\boldsymbol{c} = [c_1, c_2, \cdots, c_{2M_0-1}]^T \in \mathbb{R}^{2M_0-1}$ is the Hermitian Toeplitz parameterization of $\boldsymbol{R}_{\mathrm{U}}$. After obtaining its estimate, we can reconstruct $\boldsymbol{R}_{\mathrm{U}}$ from (10) and then perform DOA estimation. Substituting (10) into (8) and taking the derivative with respect to c_i, we obtain

$$\frac{\partial L(\boldsymbol{c})}{\partial c_i} = \sum_{l=1}^{L} N_l \,\mathrm{tr}\left[\boldsymbol{T}_l\boldsymbol{S}\boldsymbol{Q}_{M_0}^{(i)}\boldsymbol{S}^T\boldsymbol{T}_l^T\boldsymbol{R}_l^{-1}(\boldsymbol{R}_l - \hat{\boldsymbol{R}}_l)\boldsymbol{R}_l^{-1}\right]$$

for $i = 1, 2, \ldots, 2M_0-1$. Because $\mathrm{vec}(\boldsymbol{A}\boldsymbol{X}\boldsymbol{B}) = (\boldsymbol{B}^T \otimes \boldsymbol{A})\,\mathrm{vec}(\boldsymbol{X})$, and because $(\boldsymbol{A} \otimes \boldsymbol{B})^{-1} = \boldsymbol{A}^{-1} \otimes \boldsymbol{B}^{-1}$ for nonsingular $\boldsymbol{A}$, $\boldsymbol{B}$ [20], we have

$$\mathrm{vec}(\boldsymbol{T}_l\boldsymbol{S}\boldsymbol{Q}_{M_0}^{(i)}\boldsymbol{S}^T\boldsymbol{T}_l^T) = \boldsymbol{\Phi}_l\boldsymbol{q}_{M_0}^{(i)}, \tag{11}$$

where $\boldsymbol{q}_{M_0}^{(i)} = \mathrm{vec}(\boldsymbol{Q}_{M_0}^{(i)})$, and $\boldsymbol{\Phi}_l = \boldsymbol{T}_l\boldsymbol{S} \otimes \boldsymbol{T}_l\boldsymbol{S}$. We also have

$$\mathrm{vec}(\boldsymbol{R}_l^{-1}(\boldsymbol{R}_l - \hat{\boldsymbol{R}}_l)\boldsymbol{R}_l^{-1}) = \boldsymbol{W}_l^{-1}(\boldsymbol{\Phi}_l\boldsymbol{Q}_{M_0}\boldsymbol{c} - \hat{\boldsymbol{r}}_l), \tag{12}$$

where $\boldsymbol{W}_l = \boldsymbol{R}_l^T \otimes \boldsymbol{R}_l$, $\boldsymbol{Q}_{M_0} = [\boldsymbol{q}_{M_0}^{(1)}, \boldsymbol{q}_{M_0}^{(2)}, \cdots, \boldsymbol{q}_{M_0}^{(2M_0-1)}]$, and $\hat{\boldsymbol{r}}_l = \mathrm{vec}(\hat{\boldsymbol{R}}_l)$. Let all the partial derivatives with respect to c_i be zero. Then, we utilize (11) and (12) to obtain

$$\Big(\sum_{l=1}^{L} N_l\boldsymbol{G}_l\Big)\boldsymbol{c} = \sum_{l=1}^{L} N_l\boldsymbol{h}_l \tag{13}$$

where $\boldsymbol{G}_l = \boldsymbol{Q}_{M_0}^T\boldsymbol{\Phi}_l^T\boldsymbol{W}_l^{-1}\boldsymbol{\Phi}_l\boldsymbol{Q}_{M_0}$, and $\boldsymbol{h}_l = \boldsymbol{Q}_{M_0}^T\boldsymbol{\Phi}_l^T\boldsymbol{W}_l^{-1}\hat{\boldsymbol{r}}_l$. Note that if we have sufficient snapshots in each period, $\hat{\boldsymbol{R}}_l$ will be very close to $\boldsymbol{R}_l$, and we can replace $\boldsymbol{W}_l$ with its estimate $\hat{\boldsymbol{W}}_l = \hat{\boldsymbol{R}}_l^T \otimes \hat{\boldsymbol{R}}_l$. In this case the only unknown in (13) will be $\boldsymbol{c}$, whose estimate can be readily given by

$$\hat{\boldsymbol{c}}_{\mathrm{WLS}} = \Big[\sum_{l=1}^{L} N_l\hat{\boldsymbol{G}}_l\Big]^{-1}\Big[\sum_{l=1}^{L} N_l\hat{\boldsymbol{h}}_l\Big]. \tag{14}$$

where $\hat{\boldsymbol{G}}_l$ denotes $\boldsymbol{G}_l$ with $\boldsymbol{W}_l$ replaced by $\hat{\boldsymbol{W}}_l$, and $\hat{\boldsymbol{h}}_l$ denotes $\boldsymbol{h}_l$ with $\boldsymbol{W}_l$ replaced by $\hat{\boldsymbol{W}}_l$. Lemma 2 ensures that (14) produces real results.

Lemma 1. *Let $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}$ be Hermitian symmetric. Then $\mathrm{tr}(\boldsymbol{ABAC})$ is real.*

Proof. This can be shown by the fact that

$$\mathrm{tr}(\boldsymbol{ABAC})^* = \mathrm{tr}[(\boldsymbol{ABAC})^H] = \mathrm{tr}(\boldsymbol{CABA}) = \mathrm{tr}(\boldsymbol{ABAC}).$$

□

Lemma 2. *Both $\hat{\boldsymbol{G}}_l$ and $\hat{\boldsymbol{h}}_l$ are real.*

Proof. Through algebraic manipulations, the (m,n)-th element of $\hat{\boldsymbol{G}}_l$ can be rewritten as

$$\mathrm{tr}[\hat{\boldsymbol{R}}_l^{-1}\boldsymbol{T}_l\boldsymbol{S}\boldsymbol{Q}_{M_0}^{(m)}\boldsymbol{S}^T\boldsymbol{T}_l^T\hat{\boldsymbol{R}}_l^{-1}\boldsymbol{T}_l\boldsymbol{S}\boldsymbol{Q}_{M_0}^{(n)}\boldsymbol{S}^T\boldsymbol{T}_l^T].$$

By the definition of $\boldsymbol{Q}_{M_0}^{(m)}$ in (9), we know that $\boldsymbol{T}_l\boldsymbol{S}\boldsymbol{Q}_{M_0}^{(m)}\boldsymbol{S}^T\boldsymbol{T}_l^T$ is Hermitian symmetric. Because $\hat{\boldsymbol{R}}^{-1}$ is also Hermitian symmetric, we know that each element of $\hat{\boldsymbol{G}}_l$ is real by Lemma 2. The proof for the second claim follows the same idea. □

We call (14) the "weighted least squares" (WLS) estimate, because (14) is the solution to the weighted least squares problem: $\min_{\boldsymbol{c}} \sum_{l=1}^{L} N_l\|\boldsymbol{\Phi}_l\boldsymbol{Q}_{M_0}\boldsymbol{c} - \hat{\boldsymbol{r}}_l\|_{\hat{\boldsymbol{W}}_l^{-1}}^2$, where $\|\boldsymbol{x}\|_{\boldsymbol{W}} = \sqrt{\boldsymbol{x}^H\boldsymbol{W}\boldsymbol{x}}$.

We can also observe that (13) leads to the following fixed-point type iteration:

$$\hat{\boldsymbol{c}}_{\mathrm{FP}}^{(k)} = \Big[\sum_{l=1}^{L} N_l\boldsymbol{G}_l\big(\hat{\boldsymbol{c}}_{\mathrm{FP}}^{(k-1)}\big)\Big]^{-1}\Big[\sum_{l=1}^{L} N_l\boldsymbol{h}_l\big(\hat{\boldsymbol{c}}_{\mathrm{FP}}^{(k-1)}\big)\Big], \tag{15}$$

where $\boldsymbol{G}_l\big(\hat{\boldsymbol{c}}_{\mathrm{FP}}^{(k-1)}\big)$ and $\boldsymbol{h}_l\big(\hat{\boldsymbol{c}}_{\mathrm{FP}}^{(k-1)}\big)$ are constructed from $\hat{\boldsymbol{c}}_{\mathrm{FP}}^{(k-1)}$.

Remark 1. In practice, the computation of $\hat{\boldsymbol{G}}_l$ and $\hat{\boldsymbol{h}}_l$ can be efficiently implemented by exploiting the properties of Kronecker product and the fact that $\boldsymbol{\Phi}_l$ are Kronecker products of simple selection matrices. In our experiments, by setting the initial value as $\hat{\boldsymbol{c}}_{\mathrm{WLS}}$, $\{\hat{\boldsymbol{c}}_{\mathrm{FP}}^{(k)}\}$ showed good convergence in a few iterations.

Remark 2. When the signal-to-noise ratio (SNR) is very high, the conditional number of $\boldsymbol{R}_{\mathrm{U}}$ will be large, and the reconstructed $\hat{\boldsymbol{R}}_{\mathrm{U}}$ becomes indefinite. In this case, we project $\hat{\boldsymbol{R}}_{\mathrm{U}}$ onto the intersection of the positive semidefinite cone $\mathbb{PSD}$ and the Toeplitz subspace $\mathbb{T}$. This can be achieved via the alternating projections method. Because both $\mathbb{PSD}$ and $\mathbb{T}$ are convex and their $\mathbb{PSD} \cap \mathbb{T} \neq \emptyset$, the alternating projections method converges [26].

Remark 3. For incomplete arrays, not all elements in $\boldsymbol{R}_{\mathrm{U}}$ are present in $\boldsymbol{R}_l$. Therefore $\boldsymbol{Q}_{M_0}^T\boldsymbol{\Phi}_l$ is no longer full rank, and we cannot perform the matrix inversion in (14) or (15). In this case, we first delete the elements we cannot estimate from $\boldsymbol{c}$ and their corresponding basis matrices from $\{\boldsymbol{Q}_{M_0}^i\}_{i=1}^{2M_0-1}$ to form $\tilde{\boldsymbol{c}}$ and $\tilde{\boldsymbol{Q}}_{M_0}$. We then estimate $\tilde{\boldsymbol{c}}$ using (14) or (15), with $\boldsymbol{Q}_{M_0}$ replaced by $\tilde{\boldsymbol{Q}}_{M_0}$. Finally, we construct a submatrix of $\boldsymbol{R}_{\mathrm{U}}$ from the estimated $\tilde{\boldsymbol{c}}$.

4. PERFORMANCE BOUNDS

Because the measurements are assumed independent, the (m,n)-th element of the Fisher information matrix (FIM) for our signal model is given by [24, 27]:

$$\mathrm{FIM}_{mn} = \sum_{L=1}^{L} N_l \,\mathrm{tr}\left[\frac{\partial \boldsymbol{R}_l}{\partial \eta_m}\boldsymbol{R}_l^{-1}\frac{\partial \boldsymbol{R}_l}{\partial \eta_n}\boldsymbol{R}_l^{-1}\right].$$

Using the properties of the Kronecker product, we can express the FIM as

$$\mathrm{FIM}_{mn} = \sum_{L=1}^{L} N_l \left[\frac{\partial \boldsymbol{r}_{\mathrm{U}}}{\partial \eta_m}\right]^H \boldsymbol{\Phi}_l^H(\boldsymbol{R}_l^T \otimes \boldsymbol{R}_l)^{-1}\boldsymbol{\Phi}_l\frac{\partial \boldsymbol{r}_{\mathrm{U}}}{\partial \eta_n},$$

where $\boldsymbol{r}_{\mathrm{U}} = \mathrm{vec}(\boldsymbol{R}_{\mathrm{U}})$. Therefore, for complete arrays, the FIM for the Toeplitz parametrization is given by

$$\mathrm{FIM}_{\boldsymbol{c}} = \sum_{l=1}^{L} N_l\boldsymbol{Q}_{M_0}^H\boldsymbol{\Phi}_l^H(\boldsymbol{R}_l^T \otimes \boldsymbol{R}_l)^{-1}\boldsymbol{\Phi}_l\boldsymbol{Q}_{M_0}. \tag{16}$$

For incomplete arrays, as stated in Remark 3, not all elements in $\boldsymbol{c}$ is estimable. To compute the FIM of the estimable elements in $\boldsymbol{c}$, we need to replace $\boldsymbol{Q}_{M_0}$ by $\tilde{\boldsymbol{Q}}_{M_0}$ in a similar fashion.

For parameters $\boldsymbol{\eta} = [\boldsymbol{\theta}, \boldsymbol{p}, \sigma_{\mathrm{n}}^2]^T$, the FIM is given by

$$\mathrm{FIM}_{\boldsymbol{\eta}} = \sum_{l=1}^{L} N_l\boldsymbol{D}^H\boldsymbol{\Phi}_l^H(\boldsymbol{R}_l^T \otimes \boldsymbol{R}_l)^{-1}\boldsymbol{\Phi}_l\boldsymbol{D}, \tag{17}$$

where $\boldsymbol{D} = [\dot{\boldsymbol{A}}_{\mathrm{d}}\boldsymbol{P}\ \boldsymbol{A}_{\mathrm{d}}\ \boldsymbol{i}]$, and $\dot{\boldsymbol{A}}_{\mathrm{d}} = \dot{\boldsymbol{A}}_{\mathrm{U}}^* \odot \boldsymbol{A}_{\mathrm{U}} + \boldsymbol{A}_{\mathrm{U}}^* \odot \dot{\boldsymbol{A}}_{\mathrm{U}}$, $\dot{\boldsymbol{A}}_{\mathrm{U}} = [\partial \boldsymbol{a}_{\mathrm{U}}(\theta_1)/\partial\theta_1, \cdots, \partial \boldsymbol{a}_{\mathrm{U}}(\theta_K)/\partial\theta_K]$, $\boldsymbol{A}_{\mathrm{d}} = \boldsymbol{A}_{\mathrm{U}}^* \odot \boldsymbol{A}_{\mathrm{U}}$, and $\boldsymbol{i} = \mathrm{vec}(\boldsymbol{I}_{M_0})$. The corresponding CRBs can be obtained by inverting the FIMs in (16) and (17).

5. NUMERICAL EXAMPLES

We consider the following two sparse linear array configurations in the numerical examples:

- Nested array: $[0, 1, 2, 3, 7, 11, 15, 19]d_0$;
- Coprime array: $[0, 3, 5, 6, 9, 10, 12, 15, 20, 25]d_0$.

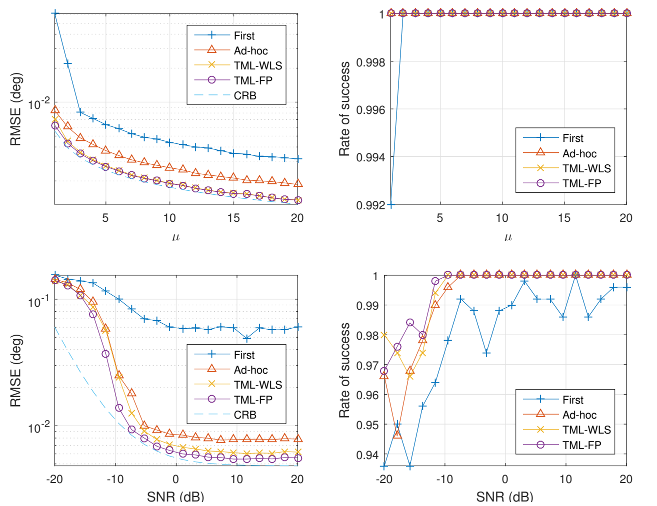

Fig. 1. Performance of different algorithms for the nested array configuration.

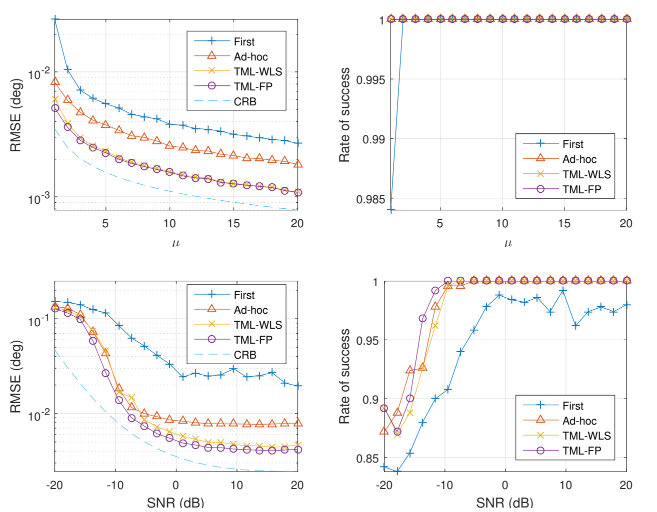

Fig. 2. Performance of different algorithms for the co-prime array configuration.

In all the experiments, we consider 12 sources uniformly distributed between $-\pi/3$ and $-\pi/3$. The number of sources is more than the number of sensors of either array. We set L to be 3. When $L = 2$ the last sensor of each array fails, and when $L = 3$, the last two sensors of each array fail. We set $N_1 = 50\mu$, $N_2 = 100\mu$, and $N_3 = 150\mu$, where μ is a tunable parameter. Hence we have more snapshots with missing data than those with complete data. When making comparisons under different numbers of snapshots, we fixed $\mathrm{SNR} = 0\mathrm{dB}$ and varied μ from 1 to 20. When making comparisons under different SNRs, we fixed $\mu = 1$ and varied SNR from -20dB to 20dB. The root-mean-square errors (RMSEs) were obtained from 500 trials, and the DOAs were estimated by MUSIC.

In all the figures, "First" denotes the results obtained using only $\hat{\boldsymbol{R}}_1$, while Ad-hoc, TML-WLS, and TML-FP denote the results obtained from (7), (14), and (15), respectively. We also include the CRB obtained from (17) for comparison.

Fig. 1 illustrates the performance of different algorithms for the nested array configuration. We observe that TML-FP achieves the best performance, and is very close to the CRB, while "First" results in the worst performance because it cannot utilize the information in $\hat{\boldsymbol{R}}_l$ ($l \geq 2$). We observe similar results for the co-prime configuration in Fig. 2. However, a gap exists between the RMSE of TML-FP and the CRB, which may be attributed to the fact that the co-prime array is incomplete.

6. CONCLUSION AND FUTURE WORK

In this paper, we discussed the problem of direction finding using sparse linear arrays with incomplete measurements. By exploiting the coarray structure, we proposed to reconstruct a covariance matrix with enhanced degrees of freedom using the Toeplitz parameterization. Specifically, by applying our method to co-prime and nested arrays, we can resolve more sources than the number of sensors in the missing data case. We used numerical examples to show that our method has better accuracy than the traditional method using only the complete measurements. Potential future work includes performance and identifiability analysis in the presence of missing data.

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