UNIFIED ANALYSIS OF CO-ARRAY INTERPOLATION FOR DIRECTION-OF-ARRIVAL ESTIMATION

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ABSTRACT

This paper considers the problem of co-array interpolation for direction-of-arrival (DOA) estimation with sparse non-uniform arrays. By utilizing the much longer difference co-array associated with these arrays, it is possible to perform DOA estimation of more sources than sensors. Since the co-array may contain holes (or missing lags), interpolation algorithms have been proposed to fully utilize the remaining elements of the co-array beyond that captured in the contiguous ULA segment. However, the quality and stability of interpolation performed by such algorithms (especially in presence of modeling errors) have not been analyzed. This paper provides a unified analysis of co-array interpolation algorithms to bound the interpolation error in terms of modeling errors. The results are universal in the sense that they can be applied to analyze any algorithm that utilizes the positive semidefinite (PSD) structure of the interpolated covariance matrix. The general framework is then applied to analyze specific algorithms and simulations are conducted to study their interpolation errors. [1]

***Index Terms*—** Co-array, array interpolation, Toeplitz completion, nuclear norm minimization, DOA estimation.

1. INTRODUCTION

Direction-of-arrival (DOA) estimation of energy-emitting sources is a central problem arising in diverse applications such as radar, sonar, medical imaging and communications [1, 2]. Sparse non-uniform arrays such as nested, coprime and minimum redundancy arrays are known to offer distinct advantages over traditionally used Uniform Linear Arrays (ULA) owing to their ability to resolve more sources than sensors [3, 4, 5]. The basic idea is to create a longer virtual difference co-array [4] by judicious array design, whose degrees-of-freedom (DOF) can be exploited by well-designed algorithms such as Co-array MUSIC [6, 7].

For many non uniform arrays (such as coprime arrays), the difference co-array is not continuous and has holes or missing lags. Since co-array MUSIC algorithms are capable of only exploiting the DOF of a continuous ULA segment of the co-array, several array interpolation techniques such as positive definite Toeplitz completion [8], co-array interpolation/extrapolation [9, 10], and nuclear norm minimization [11] have been proposed to interpolate the correlation values at the missing lags and use the interpolated co-array for DOA estimation.

In this paper, we propose a unified framework for analyzing co-array interpolation algorithms by developing an explicit upper bound on the interpolation error, in terms of the measurement error. Our analysis framework is very generic and can be applied to *any algorithm that utilizes the positive semidefinite (PSD) structure* of the interpolated covariance matrix. As special cases, we use this general framework to develop algorithm-specific error bounds for the algorithms in [8, 11]. Our results establish stability of these interpolation algorithms with respect to modeling errors (such as that due to finite snapshot averaging) and demonstrate that perfect interpolation is possible as the error decays to zero.

Related Work. While the performance of traditional array interpolation techniques have been analyzed in the past in terms of bias and mean squared error [12, 13], these methods are primarily based on interpolating the *physical array* using linear transforms, and cannot be used for *co-array interpolation* since the co-array is a non-linear function (Kronecked product) of the physical array. On the other hand, interpolation algorithms that directly work in the co-array domain [8, 11], have not been analyzed. In this paper, we bridge this gap by providing a unified analysis of co-array interpolation algorithms. Our analysis is based on recently developed tools from super resolution theory and positive semidefiniete Toeplitz covariance compression [14, 15, 16, 17, 18].

2. CO-ARRAY BASED SIGNAL MODEL AND NEED FOR INTERPOLATION

Consider D narrowband statistically uncorrelated sources impinging on a linear sensor array from directions $\bar{\theta}_i, 1 \leqslant i \leqslant D$. The array contains K sensors with the kth sensor located at $z_k d$, where z_k is an integer and $d = \lambda/2$ (λ being the carrier wavelength of the narrowband sources). The signals received at the K sensors are given by

$$\mathbf{x} = \sum_{i=1}^{D} c_i \mathbf{a}_{\mathbb{S}}(\theta_i) + \mathbf{n}_{\mathbb{S}} \tag{1}$$

where c_i denotes the amplitude of each source (assumed to be zero-mean random variables) and $\mathbf{a}_{\mathbb{S}}(\theta_i) \in \mathbb{C}^K$ represents the steering vector corresponding to the normalized DOA $\theta_i \in \mathbb{T} = [-1/2, 1/2]$, which is given by $\theta_i = (d/\lambda)\sin\bar{\theta}_i$ ($\bar{\theta}_i$ being the DOA satisfying $\bar{\theta}_i \in [-\pi/2, \pi/2]$). The steering

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vector satisfies $[\mathbf{a}_{\mathbb{S}}(\theta_i)]_k = [e^{j2\pi z_k \theta_i}]$. Here, $\mathbf{n}_{\mathbb{S}}$ represents zero-mean additive noise at K sensors, statistically uncorrelated with the source amplitudes c_i. The statistical assumptions on source signal and noise are summarized as [2, 11]

$$\mathbb{E}[c_i^* c_j] = \sigma_i^2 \delta_{i,j}, \mathbb{E}[c_i^* \mathbf{n}_{\mathbb{S}}] = \mathbf{0}, \mathbb{E}[\mathbf{n}_{\mathbb{S}} \mathbf{n}_{\mathbb{S}}^H] = \sigma^2 \mathbf{I}$$

Under the above assumptions, the correlation matrix $\mathbf{R}_{\mathbb{S}} \in \mathbb{C}^{K \times K}$ of the received signals is given by

$$\mathbf{R}_{\mathbb{S}} = \sum_{i=1}^{D} \sigma_i^2 \mathbf{a}_{\mathbb{S}}(\theta_i) \mathbf{a}_{\mathbb{S}}^H(\theta_i) + \sigma^2 \mathbf{I} \tag{2}$$

Denoting $\mathbb{S} = \{z_k, 1 \leqslant k \leqslant K\}$ as the set of sensor positions (normalized with respect to d), its difference co-array is defined as [3]

$$\mathbb{D} \triangleq \{z_k - z_j | z_k, z_j \in \mathbb{S}\}$$

Let us associate a vector $\mathbf{a}_{\mathbb{D}}$ with this difference set, such that $[\mathbf{a}_{\mathbb{D}}(\theta_i)]_m = [e^{j2\pi n_m \theta_i}], n_m \in \mathbb{D}$. The vectorized version of (2), after removal of repeated rows, is given by

$$\mathbf{r}_{\mathbb{D}} = \sum_{i=1}^{D} \sigma_i^2 \mathbf{a}_{\mathbb{D}}(\theta_i) + \sigma^2 \mathbf{e}_0 \tag{3}$$

where $\mathbf{e}_0$ has zero entries everywhere except at the location corresponding to the lag 0 [11, 7]. Due to similarities between (3) and (1), we can treat (3) as the signal received at *a virtual sensor array with sensors positions given by* $\mathbb{D}$.

Depending on the geometry of $\mathbb{S}$, the difference co-array $\mathbb{D}$ may be continuous (i.e. it can itself be a uniform linear array or a ULA, consisting of only consecutive integers), or it may contain holes. The former is known as a fully augmentable array and the latter is called a partially augmentable array [8]. Following [11], we associate the following two uniform linear arrays $\mathbb{U}$ and $\mathbb{V}$ with $\mathbb{D}$ as follows:

Definition 1. *[11] Let $\mathbb{U}$ be the maximum ULA contained in $\mathbb{D}$ such that $\mathbb{U} = \{m | \{-|m|, \cdots, -1, 0, 1, \cdots, |m|\} \subseteq \mathbb{D}\}$ and $\mathbb{V}$ be the smallest ULA containing $\mathbb{D}$ such that $\mathbb{V} = \{m | \min(\mathbb{D}) \leqslant m \leqslant \max(\mathbb{D})\}$.*

As an example, let $\mathbb{S} = \{0, 1, 2, 6\}$, then $\mathbb{D} = \{-6, -5, -4, -2, -1, 0, 1, 2, 4, 5, 6\}$, $\mathbb{U} = \{-2, -1, 0, 1, 2\}$ and $\mathbb{V} = \{-6, -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, 6\}$. For fully augmentable arrays, we have $\mathbb{U} = \mathbb{D} = \mathbb{V}$. Examples include ULA, nested array [3] and minimum redundancy array [5]. On the other hand, for partially augmentable arrays, we have $\mathbb{U} \subset \mathbb{D} \subset \mathbb{V}$. Coprime array [4] is an example of partially augmentable array.

2.1. Co-Array MUSIC for Partially Augmentable Arrays

Co-array based DOA estimation algorithms (such as co-array MUSIC [6]) can utilize the degrees of freedom (given by the cardinality) in the virtual ULA segment $\mathbb{U}$ contained in $\mathbb{D}$, and for well-designed arrays, it is possible to resolve more sources than sensors. For nested and coprime arrays with K sensors, $|\mathbb{U}| = O(K^2)$ and hence it is possible to resolve $O(K^2)$ sources using only K sensors.

For partially augmentable arrays, the virtual ULA $\mathbb{V}$ is strictly larger than $\mathbb{U}$. However, co-array MUSIC [6] cannot directly utilize the DOF in $\mathbb{V}$ since certain entries of $\mathbb{V}$ do not appear in $\mathbb{D}$. To address this issue, *a preprocessing step based on interpolation* has been suggested [8, 11]. Similar to $\mathbf{r}_{\mathbb{D}}$, let $\mathbf{r}_{\mathbb{U}}$ be the sub-vector of $\mathbf{r}_{\mathbb{D}}$, containing the correlation values evaluated at lags given by $\mathbb{U}$, and $\mathbf{r}_{\mathbb{V}}$ be a vector that consists of correlation values at lags given by the set $\mathbb{V}$. Co-array MUSIC can be applied on $\mathbf{r}_{\mathbb{V}}$ to fully exploit the DOF of partially augmentable arrays (provided $\mathbf{r}_{\mathbb{V}}$ can be estimated using $\mathbf{r}_{\mathbb{D}}$).

2.2. Interpolation Algorithms

Let $\tilde{\mathbf{r}}_{\mathbb{D}}$ and $\tilde{\mathbf{r}}_{\mathbb{U}}$ be the corresponding estimates of $\mathbf{r}_{\mathbb{D}}$ and $\mathbf{r}_{\mathbb{U}}$ computed using finite number of snapshots. In particular, $\tilde{\mathbf{r}}_{\mathbb{U}}$ is a subvector of $\tilde{\mathbf{r}}_{\mathbb{D}}$ and

$$\tilde{\mathbf{r}}_{\mathbb{D}} = \mathbf{r}_{\mathbb{D}} + \mathbf{w}_{\mathbb{D}} \tag{4}$$

where $\mathbf{w}_{\mathbb{D}}$ captures the finite-snapshot estimation error. We now briefly describe two algorithms, one based on maximum entropy method [8], and the other based on nuclear norm minimization [11] that aim to estimate $\mathbf{r}_{\mathbb{V}}$ from $\tilde{\mathbf{r}}_{\mathbb{D}}$. For convenience, we denote $\mathbb{U}^+, \mathbb{V}^+, \mathbb{D}^+$ as the non-negative subsets of $\mathbb{U}, \mathbb{V}, \mathbb{D}$ respectively, and let $\mathcal{T}(\mathbf{v})$ be the Hermitian symmetric Toeplitz matrix with $\mathbf{v}$ as the first column.

(a) Maximum Entropy Method:

In [8], the authors used maximum entropy (ME) as a criterion for extrapolation of correlation at lags in $\mathbb{V}$ outside the range of $\mathbb{D}$. The algorithm consists of two steps. Firstly, given $\tilde{\mathbf{r}}_{\mathbb{U}^+}$, it aims to find the closest positive semidefinite (PSD) Toeplitz matrix $\mathcal{T}(\mathbf{r}_{\mathbb{U}^+}^{ME})$ fitting the data as follows:

$$\begin{aligned} \mathbf{r}_{\mathbb{U}^+}^{ME} &= \arg\min_{\mathbf{x}_{\mathbb{U}^+}} \|\mathbf{x}_{\mathbb{U}^+} - \tilde{\mathbf{r}}_{\mathbb{U}^+}\|_2 \quad (\textbf{ME-1}) \\ s.t. \quad & \mathcal{T}(\mathbf{x}_{\mathbb{U}^+}) \succeq 0 \end{aligned} \tag{5}$$

In the next step, the vector of autocorrelation values $\mathbf{r}_{\mathbb{V}^+}^{ME}$ (extrapolated at lags in $\mathbb{V}^+$), is computed as

$$\begin{aligned} \mathbf{r}_{\mathbb{V}^+}^{ME} &= \arg\max_{\mathbf{x}_{\mathbb{V}^+}} \det(\mathcal{T}(\mathbf{x}_{\mathbb{V}^+})) \quad (\textbf{ME-2}) \\ s.t. \quad & [\mathcal{T}(\mathbf{x}_{\mathbb{V}^+})]_{n,1} = [\mathbf{r}_{\mathbb{U}^+}^{ME}]_n, n \in \mathbb{U}^+ \\ & \|[\mathcal{T}(\mathbf{x}_{\mathbb{V}^+})]_{\mathbb{D}^+ \setminus \mathbb{U}^+, 1} - \tilde{\mathbf{r}}_{\mathbb{D}^+ \setminus \mathbb{U}^+}\|_2 \leqslant \epsilon_1, \\ & \mathcal{T}(\mathbf{x}_{\mathbb{V}^+}) \succeq 0 \end{aligned} \tag{6}$$

Here ϵ_1 is a parameter that can be tuned to ensure non-empty feasible set. In particular it can be made equal to $\|\mathbf{w}_{\mathbb{D}^+ \setminus \mathbb{U}^+}\|_2$. Co-array MUSIC can be finally applied on $\mathcal{T}(\mathbf{r}_{\mathbb{V}^+}^{ME})$ to perform DOA estimation using the DOF of $\mathbb{V}$. Notice that the ME method utilizes PSD constraint in both steps so that the Toeplitz matrix constructed using the extrapolated values continues to be PSD.

(b) Nuclear Norm Minimization:

In [11], the authors assumed that the desired covariance matrix $\mathcal{T}(\mathbf{r}_{\mathbb{V}^+})$ exhibits low rank and proposed to minimize its nuclear norm (as a surrogate for rank) to perform interpolation. In its original form, the algorithm does not impose any PSD constraint on the solution. Since our analysis framework will explicitly use PSD constraint, we consider the following modified problem instead which uses PSD constraint:

$$\begin{aligned} \mathbf{r}_{\mathbb{V}^+}^{NUC} &= \arg\min_{\hat{\mathbf{x}}_{\mathbb{V}}} \|\mathcal{T}(\hat{\mathbf{x}}_{\mathbb{V}^+})\|_* \quad (\textbf{NUC-PSD}) \\ s.t. \quad & \|[\mathcal{T}(\hat{\mathbf{x}}_{\mathbb{V}^+})]_{\mathbb{U}^+,1} - \tilde{\mathbf{r}}_{\mathbb{U}^+}\|_2 \leqslant \epsilon \\ & \|[\mathcal{T}(\hat{\mathbf{x}}_{\mathbb{V}^+})]_{\mathbb{D}^+\setminus\mathbb{U}^+,1} - \tilde{\mathbf{r}}_{\mathbb{D}^+\setminus\mathbb{U}^+}\|_2 \leqslant \tilde{\epsilon} \\ & \mathcal{T}(\hat{\mathbf{x}}_{\mathbb{V}^+}) \succeq 0 \end{aligned}$$

where $\epsilon, \tilde{\epsilon}$ are parameters (dependent on estimation error) to ensure that (**NUC-PSD**) feasible (i.e. so that the true solution is contained in the feasible set).

3. A UNIFIED ANALYSIS OF EXTRAPOLATION ERROR

For notational simplicity, let r_n denote the nth entry of $\mathbf{r}_{\mathbb{V}^+}$. Then, using the representation (3), the desired value of r_n is given by

$$r_n = \sum_{i=1}^{D} \sigma_i^2 e^{j2\pi n\theta_i} + \sigma^2\delta(n) \quad n \in \mathbb{V}^+$$

Let $\mathbf{r}_{\mathbb{V}^+}^{\#}$ denote *any estimate* of $\mathbf{r}_{\mathbb{V}^+}$ such that

$$\mathcal{T}(\mathbf{r}_{\mathbb{V}^+}^{\#}) \succeq \mathbf{0} \tag{7}$$

Notice that this automatically implies $\mathcal{T}(\mathbf{r}_{\mathbb{U}^+}^{\#}) \succeq \mathbf{0}$. We now present a fundamental result, upper bounding the extrapolation error $|r_n^{\#} - r_n|$ (for any missing or unobserved lag n outside the range of $\mathbb{D}$) in terms of the estimation error in the correlation values for the lags in $\mathbb{D}$. The proof follows from a closely related lemma in [15, 14]. The stability analysis requires a separation condition on the true directions and we define $\rho(\cdot,\cdot)$ as the distance function in wraparound manner over $\mathbb{T}$ [17].

Theorem 1. *Let $\mathbf{r}_{\mathbb{V}^+}^{\#}$ denote* any estimate *of* $\mathbf{r}_{\mathbb{V}^+}$ *such that (7) holds. If the true DOAs $\{\theta_l\}_{l=1}^D$ in (2) satisfy*

$$\min_{p\neq q} \rho(\theta_p, \theta_q) > 4/|\mathbb{U}^+| \tag{8}$$

and $|\mathbb{U}^+| > 256$, then there exist positive constants $\bar{c}_1, \bar{c}_2, \bar{c}_3, \bar{c}_4$ such that for $|\mathbb{U}^+| \leqslant n < |\mathbb{V}^+|$

$$\begin{aligned} &|r_n - r_n^{\#}| \qquad (9) \\ &\leqslant \left(\bar{c}_1 + \frac{\bar{c}_2\pi n}{|\mathbb{U}^+|} + \frac{\bar{c}_3\pi^2 n^2}{|\mathbb{U}^+|^2}\right)\left(\frac{\bar{c}_4 D\xi}{|\mathbb{U}^+|} + [\mathbf{x}_{\mathbb{U}^+}^{\#}]_0 - [\mathbf{x}_{\mathbb{U}^+}]_0\right) \\ &\leqslant \left(\bar{c}_1 + \frac{\bar{c}_2\pi n}{|\mathbb{U}^+|} + \frac{\bar{c}_3\pi^2 n^2}{|\mathbb{U}^+|^2}\right)\left(\frac{\bar{c}_4 D\xi}{|\mathbb{U}^+|} + \|\mathbf{x}_{\mathbb{U}^+}^{\#} - \mathbf{x}_{\mathbb{U}^+}\|_2\right) \end{aligned}$$

where $\xi \triangleq \sup_{\theta\in\mathbb{T}} |\langle \mathbf{a}_{\mathbb{U}^+}(\theta), \mathbf{x}_{\mathbb{U}^+}^{\#} - \mathbf{x}_{\mathbb{U}^+}\rangle|$

Remark 1. Notice that the above bound on extrapolation error holds *irrespective of any specific algorithm used* as long as the algorithm enforces the PSD constraint (7).

Remark 2. Theorem 1 indicates that the upper bound on extrapolation error bound (on missing lags) is controlled by estimation error of the correlation supported on the observed set $\mathbb{U}^+$. Depending on the algorithm used, the extrapolation error can be magnified by a factor of $O(n^2/|\mathbb{U}^+|^2)$ (with respect to the estimation error on the observed entries in $\mathbb{U}$). A similar quadratic scaling between high frequency reconstruction error and low frequency observation error has also been reported in [16, 19, 17] for super-resolution imaging using TV-norm based reconstruction.

3.1. Analysis of Specific Extrapolation Algorithms

We now apply the result from Theorem 1 to perform a unified analysis of the extrapolation algorithms presented earlier.

3.1.1. Analysis of Maximum Entropy Method

In this case, $\mathbf{r}_{\mathbb{V}^+}^{\#} = \mathbf{r}_{\mathbb{V}^+}^{ME}$ and (ME-2) ensures that $\mathcal{T}(\mathbf{r}_{\mathbb{V}}^{ME}) \succeq \mathbf{0}$. Hence Theorem 1 applies, and we have following result:

Theorem 2. *If the true DOAs $\{\theta_l\}_{l=1}^D$ satisfy*

$$\min_{p\neq q} \rho(\theta_p, \theta_q) > 4/|\mathbb{U}^+|$$

*and $|\mathbb{U}^+| > 256$, then there exist positive constants $\bar{c}_1, \bar{c}_2, \bar{c}_3, \bar{c}_4$ such that for $|\mathbb{U}^+| \leqslant n < |\mathbb{V}^+|$ and $n \notin \mathbb{D}^+$, the solution $\mathbf{r}_{\mathbb{V}^+}^{ME}$ to (*__ME-2__*) satisfies*

$$\begin{aligned} &|r_n - r_n^{ME}| \qquad (10) \\ &\leqslant \left(\bar{c}_1 + \frac{\bar{c}_2\pi n}{|\mathbb{U}^+|} + \frac{\bar{c}_3\pi^2 n^2}{|\mathbb{U}^+|^2}\right)\left(\frac{\bar{c}_4 D}{\sqrt{|\mathbb{U}^+|}} + 1\right)\|\mathbf{r}_{\mathbb{U}^+}^{\#} - \mathbf{r}_{\mathbb{U}^+}\|_2 \\ &\leqslant 2\left(\bar{c}_1 + \frac{\bar{c}_2\pi n}{|\mathbb{U}^+|} + \frac{\bar{c}_3\pi^2 n^2}{|\mathbb{U}^+|^2}\right)\left(\frac{\bar{c}_4 D}{\sqrt{|\mathbb{U}^+|}} + 1\right)\|\mathbf{w}_{\mathbb{U}^+}\|_2 \end{aligned}$$

where $\mathbf{w}_{\mathbb{U}^+}$ denotes the finite-snapshot estimation error (supported on $\mathbb{U}^+$) as given in (4).

Proof. By triangle inequality, we have

$$\xi \leqslant \|\mathbf{a}_{\mathbb{U}^+}(\theta)\|_2\|\mathbf{r}_{\mathbb{U}^+}^{\#} - \mathbf{r}_{\mathbb{U}^+}\|_2 = \sqrt{|\mathbb{U}^+|}\|\mathbf{r}_{\mathbb{U}^+}^{\#} - \mathbf{r}_{\mathbb{U}^+}\|_2 \tag{11}$$

From (**ME-1**), we have

$$\begin{aligned} \|\mathbf{r}_{\mathbb{U}^+}^{\#} - \mathbf{r}_{\mathbb{U}^+}\|_2 &\leqslant \|\mathbf{r}_{\mathbb{U}^+}^{\#} - \tilde{\mathbf{r}}_{\mathbb{U}^+}\|_2 + \|\tilde{\mathbf{r}}_{\mathbb{U}^+} - \mathbf{r}_{\mathbb{U}^+}\|_2 \\ &\leqslant 2\|\tilde{\mathbf{r}}_{\mathbb{U}^+} - \mathbf{r}_{\mathbb{U}^+}\|_2 = 2\|\mathbf{w}_{\mathbb{U}^+}\|_2 \end{aligned} \tag{12}$$

Since $\mathcal{T}(\mathbf{r}_{\mathbb{V}^+}^{ME}) \succeq \mathbf{0}$ and the separation condition is satisfied, we know from Theorem 1 that (9) holds. The proof then follows by substituting (11) and (12) in (9). □

3.1.2. Nuclear Norm Minimization

In this case, $\mathbf{r}_{\mathbb{V}^+}^{\#} = \mathbf{r}_{\mathbb{V}^+}^{NUC}$ and the PSD constraint ensures that $\mathcal{T}(\mathbf{r}_{\mathbb{V}^+}^{NUC}) \succeq \mathbf{0}$. Then Theorem 1 applies, leading to the following results

Theorem 3. *If the true DOAs $\{\theta_l\}_{l=1}^D$ satisfy*

$$\min_{p\neq q} \rho(\theta_p, \theta_q) > 4/|\mathbb{U}^+|$$

and $|\mathbb{U}^+| > 256$, then there exist positive constants $\bar{c}_1, \bar{c}_2, \bar{c}_3, \bar{c}_4$ such that for $|\mathbb{U}^+| \leqslant n < |\mathbb{V}^+|$ and $n \notin \mathbb{D}^+$,

$$\begin{aligned} &|r_n - r_n^{NUC}| \\ &\leqslant \left(\bar{c}_1 + \frac{\bar{c}_2 \pi n}{|\mathbb{U}^+|} + \frac{\bar{c}_3 \pi^2 n^2}{|\mathbb{U}^+|^2}\right) \frac{\bar{c}_4 D}{\sqrt{|\mathbb{U}^+|}} (\epsilon + \|\mathbf{w}_{\mathbb{U}^+}\|_2) \end{aligned} \quad (13)$$

Proof. For any feasible PSD Toeplitz matrix, we have

$$\|\mathcal{T}(\mathbf{x}_{\mathbb{V}^+})\|_* = N[\mathbf{x}_{\mathbb{V}^+}]_0$$

where $[\mathbf{x}_{\mathbb{V}^+}]_0$ denotes the entry corresponding to the zero lag. Therefore, the minimizing solution $\mathbf{r}_{\mathbb{V}^+}^{NUC}$ of (**NUC-PSD**) satisfies $[\mathbf{r}_{\mathbb{V}^+}^{NUC}]_0 \leqslant [\mathbf{r}_{\mathbb{V}^+}]_0$. Also, notice that

$$\begin{aligned} \|\mathbf{r}_{\mathbb{U}^+}^{NUC} - \mathbf{r}_{\mathbb{U}^+}\|_2 &\leqslant \|\mathbf{r}_{\mathbb{U}^+}^{NUC} - \tilde{\mathbf{r}}_{\mathbb{U}^+}\|_2 + \|\tilde{\mathbf{r}}_{\mathbb{U}^+} - \mathbf{r}_{\mathbb{U}^+}\|_2 \\ &\leqslant \epsilon + \|\mathbf{w}_{\mathbb{U}^+}\|_2 \end{aligned}$$

The proof completes by applying the triangle inequality (11) on ξ and using these results in (9). □

Letting $\epsilon = \|\mathbf{w}_{\mathbb{U}^+}\|_2$ ensures a non-empty feasible set in (**NUC-PSD**) and the bound in (13) becomes proportional to $\|\mathbf{w}_{\mathbb{U}^+}\|_2$. Hence, using the universal result presented in Theorem 1, we have derived upper bounds on the extrapolation error corresponding to (**ME-2**) and (**NUC-PSD**), explicitly in terms of the finite snapshot error $\mathbf{w}_{\mathbb{U}^+}$. These results indicate that the extrapolation error for these algorithms goes to zero asymptotically in the number of snapshots (since $\|\mathbf{w}_{\mathbb{U}^+}\|_2$ tends to zero with increasing snapshots), indicating stability of extrapolation. To the best of our knowledge, Theorems 2,3 present the first results on stability of both algorithms in terms of extrapolation error.

4. NUMERICAL RESULTS

We follow the same experimental setting as in [11] which uses a coprime array with sensors located at $\mathbb{S} = [0, 3, 5, 6, 9, 10, 12, 15, 20, 25]$. In this case, $|\mathbb{D}| = 43, |\mathbb{U}| = 35$ and $|\mathbb{V}| = 51$. For a given number (D) of sources, we generate the true DOAs as $\theta_i = -0.4 + 0.8(n-1)/(D-1)$ for $1 \leqslant n \leqslant D$ [11]. We choose both signal and noise powers to be 1 (i.e. SNR of 0 dB). We estimate the correlation matrix at the output of the coprime array by averaging over L snapshots as

$$\tilde{\mathbf{R}}_{\mathbb{S}} = \frac{1}{L}\sum_{i=1}^{L} \mathbf{x}_i \mathbf{x}_i^H$$

Hence, the error $\mathbf{w}_{\mathbb{D}}$ in (4) is due to finite snapshot averaging. We study the interpolation error of (**ME-2**) and (**NUC-PSD**) as a function of L. Let $\mathbf{r}_{\mathbb{V}^+}^{\#}$ be the estimate of $\mathbf{r}_{\mathbb{V}^+}$ obtained from either algorithm. The normalized interpolation error is defined as

$$NMSE_{int} = \frac{1}{\|\mathbf{r}_{\mathbb{V}^+\setminus\mathbb{D}^+}\|_2^2} E\left(\|\mathbf{r}_{\mathbb{V}^+\setminus\mathbb{D}^+}^{\#} - \mathbf{r}_{\mathbb{V}^+\setminus\mathbb{D}^+}\|_2^2\right) \quad (14)$$

In Fig. 1, we plot the $NMSE_{int}$ (averaged over 100 Monte Carlo runs) as a function of L for both algorithms corresponding to different number of sources. The interpolation error decreases monotonically with increasing L, indicating stability of reconstruction. It can also been that (**NUC-PSD**) performs better than (**ME-2**), especially for larger L.

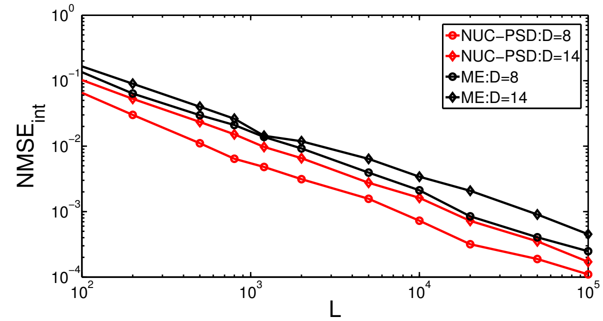

Fig. 1. $NMSE_{int}$ (averaged over 100 runs) as a function of L for (**NUC-PSD**) and (**ME**) algorithms.

In Fig. 2, we compare the MUSIC spectra obtained by applying co-array MUSIC algorithm on $\mathcal{T}(\tilde{\mathbf{r}}_{\mathbb{U}^+})$ (i.e. the correlation matrix corresponding to only the contiguous ULA segment $\mathbb{U}^+$) and on $\mathcal{T}(\mathbf{r}_{\mathbb{V}^+}^{NUC})$ (correlation matrix interpolated using (**NUC-PSD**)). It can be seen that the quality of DOA estimation can be improved by using the full interpolated co-array $\mathbb{V}^+$ instead of using only the contiguous ULA segment $\mathbb{U}^+$.

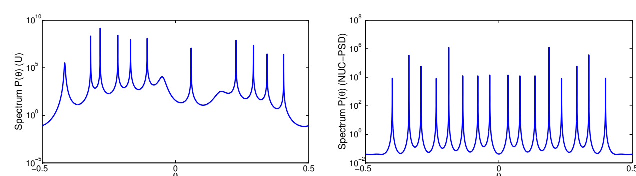

Fig. 2. MUSIC Spectrum obtained by using co-array MUSIC algorithm on (Left) $\mathcal{T}(\tilde{\mathbf{r}}_{\mathbb{U}^+})$, and (Right) $\mathcal{T}(\mathbf{r}_{\mathbb{V}^+}^{NUC})$, interpolated using (**NUC-PSD**) algorithm. Here, D = 16, L = 50.

5. CONCLUSION

In this paper, we analyzed the problem of co-array extrapolation that allows us to estimate correlation values at missing lags (or holes) in the co-array of partially augmentable arrays. We provided a universal upper bound on the extrapolation error for these missing correlation values, in terms of the estimation error corresponding to the contiguous ULA segment of the co-array. This bound is universal in the sense that it is obeyed by any extrapolation algorithm that exploits the PSD constraint on the autocorrelation matrix. Using this unified framework, we analyzed the performance of two extrapolation algorithms and established the stability of extrapolation (with respect to finite-snapshot error). Their performance is further illustrated through numerical experiments.

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