DYNAMIC GRAPH FOURIER TRANSFORM ON TEMPORAL FUNCTIONAL CONNECTIVITY NETWORKS

Marisel Villafañe-Delgado and Selin Aviyente

Department of Electrical and Computer Engineering, Michigan State University, East Lansing, MI, 48824, USA. aviyente@egr.msu.edu

ABSTRACT

Graph signal processing extends the notion of frequency from signals in the time domain to signals defined on graphs. Graph signals arise in many applications including brain signals defined on functional connectivity networks. Most of the current work on graph signal processing focuses on static graphs. However, functional connectivity networks are dynamic and the signals on these networks change with time. In this paper, we introduce a new transform for dynamic networks named as Dynamic Graph Fourier Transform (DGFT). The proposed approach extends the notion of graph Laplacian from the static case to the dynamic case through the network Laplacian tensor. The basis functions for the transform are obtained through the Tucker decomposition of this Laplacian tensor. The proposed method detects nonstationary activity in the network structure and allows us to obtain information about the regions in the brain that contribute to different frequency contents in a cognitive control experiment.

Index Terms— Dynamic Graph Fourier Transform, Multilinear Analysis, Functional Connectivity Networks, Graph Signal Processing.

1. INTRODUCTION

The recent field of signal processing over graphs has provided the tools for processing signals defined on irregular domains such as graphs [1]. In many applications, such as social networks, sensor networks, energy networks, and brain networks, among others, signals lie on the set of vertices of the network. Recently, it has been shown [2] how signal processing methods adapted to signals on graphs such as filtering and Fourier transform defined in the context of signal processing over graphs provide insights about learning processes in the brain.

Various transforms from signal processing have been adapted to the graph domain to analyze the spectral content of signals over graphs. The first one of these is the graph Fourier transform (GFT), which aims to compute the Fourier transform of a signal defined on the vertices of a graph by employing a basis obtained from the network's adjacency matrix [3] or Laplacian matrix [1]. Another transform defined on graph signals is the windowed graph Fourier transform [4], which considers the nonstationarity of the graph signals and transforms them to the vertex-frequency domain. Recently, the joint time-vertex Fourier transform [5] and the dynamic graph wavelet transform [6] have been proposed for the case of graph signals evolving over time. However, in both of these approaches the graph that the signal is defined on is fixed across time.

In certain applications, such as functional connectivity networks in the brain, the underlying network structure varies over time [7, 8]. This requires the adaptation of the previously mentioned graph signal transforms in order to consider the nonstationary network structure. For example, in the case of the graph Fourier transform, the adjacency matrix or the Laplacian matrix of the network changes for each time instance, and a unique spectral representation is not possible. Therefore, there is no unique definition of frequency across time as the graph evolves. This problem has been previously addressed by defining a common Laplacian across time, where a common subspace was found by means of Grassmann manifolds [9]. However, the accuracy of a common subspace is compromised as the number of time points increases. Another alternative would be averaging of the adjacency matrix or the Laplacian [10]. However, averaging does not necessarily find the optimal subspace across time.

In this paper, two major contributions are presented. First, unlike prior work, we propose to find a common subspace estimate across time for the temporal network by means of tensor decomposition. The temporal network adjacency matrices or Laplacian matrices over time constitute a 3-way tensor. The Tucker decomposition of this tensor results in the orthonormal component matrices which define the basis of the time-varying Laplacian operator. The obtained basis and the corresponding subspace are optimal in the sense of finding the best low-rank approximation to the Laplacians across time. Second, we introduce a windowed transform for non-

This work was supported by the National Science Foundation under Grants No. CCF-1218377 and DGE-1424871.

stationary networks. The proposed method is able to detect anomalies in the networks and, unlike averaging based methods, provides information about relevant vertices in the network contributing to intervals of high energy.

This paper is organized as follows. Section 2 introduces background on graph theory, graph Fourier transform and tensor decomposition. Section 3 presents the proposed tensor based dynamic graph Fourier transform. Section 4 presents results on simulated graph signals and dynamic functional connectivity networks (dFCNs) constructed from electroencephalogram (EEG) data. Section 5 presents the conclusions and future work.

2. BACKGROUND

2.1. Graph Theory

A graph G = (V, E, A) is defined by a set of N vertices, V, and a set of M edges, E, $e_{ij}, i, j \in \{1, ..., N\}$. The adjacency matrix $\mathbf{A} = [a_{ij}]$ represents the relationship between vertices. The Laplacian matrix is defined as $\mathbf{L} = \Delta - \mathbf{A}$, where Δ is the degree matrix, and is defined as a diagonal matrix with δ_i equal to the degree of the i^{th} vertex, $k_i = \sum_{j=1}^{N} a_{ij}$. The Laplacian \mathbf{L} is a positive semidefinite and real matrix and thus has a complete set of orthonormal eigenvectors $\{\mathbf{u}_l\}_{l=0,1,...,N-1}$, and eigenvalues $\{\lambda_l\}_{l=0,1,...,N-1}$, $0 = \lambda_0 \leq \lambda_1 \leq \cdots \leq \lambda_{N-1}$. In this work, we only consider undirected, weighted connected graphs.

2.2. Graph Signal Processing

Let the signal $f: V \to \mathbb{R}$ be defined on the vertices of the graph G. It is represented by a vector $f \in \mathbb{R}^{N \times 1}$, and the i^{th} element of this vector corresponds to the signal at vertex v_i [3]. The graph Fourier transform (GFT) of a signal f defined on the vertices V is given by [1]

$$\hat{f}(\lambda_l) = \langle f, \mathbf{u}_l \rangle = \sum_{i=1}^N f(i) \mathbf{u}_l^*(i), \qquad (1)$$

where $\mathbf{u}_l, l = 0, 1, \dots, N - 1$ correspond to the eigenvectors of the graph Laplacian. The inverse graph Fourier transform (GFT) is obtained by

$$f(i) = \sum_{l=0}^{N-1} \hat{f}(\lambda_l) \mathbf{u}_l(i).$$
(2)

As observed in (1), the underlying structure of the network plays a fundamental role on the graph spectral content [1]. In particular, the Laplacian spectrum provides a sense of frequency. The eigenvectors \mathbf{u}_l are ordered with respect to their corresponding eigenvalues, with \mathbf{u}_0 being a constant equal to $\frac{1}{\sqrt{N}}$ for connected graphs, and increasing their oscillations as l increases.

2.3. Tucker Decompositions

Let $\mathcal{X} \in \mathbb{R}^{I \times J \times K}$ be a 3rd-order tensor. The tensor \mathcal{A} can be decomposed by means of Tucker decomposition as

$$\mathcal{X} = \mathcal{C} \times_1 \mathbf{B}^{(1)} \times_2 \mathbf{B}^{(2)} \times_3 \mathbf{B}^{(3)}, \tag{3}$$

where $C \in \mathbb{R}^{I \times J \times K}$ is the core tensor and the factor matrices $\mathbf{B}^{(1)} \in \mathbb{R}^{I \times I}$, $\mathbf{B}^{(2)} \in \mathbb{R}^{J \times J}$, and $\mathbf{B}^{(3)} \in \mathbb{R}^{K \times K}$ are orthogonal. The matrices $\mathbf{B}^{(1)}$, $\mathbf{B}^{(2)}$, and $\mathbf{B}^{(3)}$ can be obtained as the left singular vectors of $\mathbf{X}_{(1)} \in \mathbb{R}^{I \times JK}$, $\mathbf{X}_{(2)} \in \mathbb{R}^{J \times KI}$, and $\mathbf{X}_{(3)} \in \mathbb{R}^{K \times IJ}$, respectively [11].

3. DYNAMIC GRAPH FOURIER TRANSFORM ON TEMPORAL NETWORKS

3.1. Dynamic Graph Fourier Transform

Consider the dynamic network $G^{(t)} = (V, E^{(t)}, A^{(t)}), t =$ $1, 2, \ldots, T$, to be a time-varying network whose edges vary with time and the vertex set remains constant. The adjacency matrices $A^{(t)}$ over time constitute the 3-way tensor $\mathcal{A} \in \mathbb{R}^{N \times N \times T}$, where N is the total number of vertices, T is the total number of time points, and $A(:,:,t) = A^{(t)}$. Similarly, we define the 3-way tensor $\mathcal{D} \in \mathbb{R}^{N \times N \times T}$ from the degree matrices $D^{(t)}$ over time, where $D(:,:,t) = D^{(t)}$. Since in traditional GFT the eigenvectors of the Laplacian define the basis for the transform, we use the same idea to find the common subspace of the Laplacians, $L^{(t)}$, across time. Some possible approaches to combining multiple Laplacians include averaging, weighted averaging [12, 13] and a more recent optimization framework based on a maximum likelihood criterion [14]. In order to find the common subspace, we define the 3-way tensor from the Laplacians of the time varying graph as $\mathcal{L} \in \mathbb{R}^{N \times N \times T}$, where L(:,:,t) = D(:,:,t) - A(:,:,t), and find the subspace information through Tucker decomposition as

$$\mathcal{L} = \mathcal{C} \times_1 \mathbf{U} \times_2 \mathbf{U} \times_3 \mathbf{V}, \tag{4}$$

where $C \in \mathbb{R}^{N \times N \times T}$ is the core tensor and the factor matrices $\mathbf{U} \in \mathbb{R}^{N \times N}$, and $\mathbf{V} \in \mathbb{R}^{T \times T}$. Due to the symmetry of \mathcal{L} along the first and second modes the corresponding factor matrices are identical.

We propose to consider the left singular vectors of the matrix $\mathbf{L}_{(1)} \in \mathbb{R}^{N \times NT}$, $\mathbf{u}_l, l = 0, 1, \dots, N-1$, as the common basis to be employed in the graph Fourier transform of the time-varying network $G^{(t)}$. This procedure avoids the need of finding a common Laplacian matrix as an intermediate step and uses the orthogonal basis that spans the connectivity mode across all time. Let $f^{(t)}$ be the signal defined on the vertices V at time t. The dynamic graph Fourier transform of $f^{(t)}$ is then given by

$$\hat{f}^{(t)}(\lambda_l) = \langle f^{(t)}, \mathbf{u}_l \rangle = \sum_{i=1}^N f^{(t)}(i) \mathbf{u}_l^*(i),$$
 (5)

where \mathbf{u}_l is the *l*th column of U, in (4). In this work, we compare the results from the proposed method on (5) with that based on the eigenvectors of the Laplacian matrix \mathbf{L}_A obtained from the average of the adjacency matrices over time, $\hat{A} = \frac{1}{T} \sum_{t=1}^{T} A^{(t)}$. We denote this transform as DGFT_A.

3.2. Windowed Dynamic Graph Fourier Transform

In order to compute the DGFT of nonstationary networks, we introduce the windowed DGFT (wDGFT) similar to short-time Fourier transform. For a signal f(i) defined on the vertices of the network and for a rectangular window of length w, wDGFT is defined as:

$$F(m,\lambda_l) = \sum_{i=1}^{N} f^{\left(\left\lceil m + \frac{w}{2} \right\rceil\right)}(i) \mathbf{u}_l^m(i), \tag{6}$$

where w is the window length, and \mathbf{u}_l^m is the *l*th column of the factor matrix U obtained from the tensor defined over the window of length w starting at t = m.

4. RESULTS

In this section, we first compare the dynamic graph Fourier transform obtained from the averaged temporal network and the proposed method. Next, we demonstrate the robustness of the proposed transform to network anomalies. Finally, we assess the dynamic graph Fourier transform on dynamic functional connectivity networks from a cognitive control study from EEG data.

4.1. Simulations

We simulated a weighted ring lattice network with N = 100nodes, with average neighbor connections K = 4 for T = 80. At each time instance, the edge weights were varied uniformly from the interval [0.75, 1] in order to simulate slight variations present in real networks. The signal $f^{(t)} \in \mathbb{R}^{N \times 1}$ is defined as

$$f^{(t)} = \begin{cases} \mathbf{v}_{10}^{(5)}, \ t = 0, \dots, 20, \\ \mathbf{v}_{10}^{(5)} + \mathbf{v}_{40}^{(12)} + \mathbf{v}_{60}^{(50)}, \ t = 21, \dots, 50 \\ \mathbf{v}_{5}^{(75)} + \mathbf{v}_{40}^{(12)} + \mathbf{v}_{80}^{(35)}, \ t = 51, \dots, T, \end{cases}$$
(7)

where $\mathbf{v}_i^{(t)}$ is the i th eigenvector of the network Laplacian at time t.

Fig. 1 shows the results from the proposed DGFT (5). The results from DGFT_A in this simulation are similar, and not shown, since the network is stationary. In order to facilitate the interpretation of the results, the frequency axis is normalized by the largest eigenvalue. As expected, in the interval $0 \le t \le 20$ the frequency content of the network corresponds to $v_{10}^{(5)}$, which extends until t = 50. In the second interval,

 $21 \le t \le 50$, there are in addition the signals corresponding to eigenvectors 40 and 60. Finally, during the last interval there are components corresponding to the eigenvectors 5, 40, and 80.



Fig. 1. DGFT of a ring network with N = 100 nodes and K = 4 over T = 80 seconds. The graph signal is composed of different components over time, which are extracted by the proposed method.

Next, we assess the accuracy of detecting anomalies in dynamic networks. In this simulation, a weighted Small-world network with average degree K = 6, p = 0.05, and N = 60vertices was simulated for t = 1, 2, ..., T, where T = 40. The graph signal $f^{(t)}(n)$ is uniformly distributed on the interval [-1, -0.9] for $1 \le n \le 30$ and between [0.9, 1]for $31 \leq n \leq N$. An anomaly was introduced by altering the network structure at t = 20 and t = 21, when the network structure changed to a random network. The wDGFT was computed with a sliding window of length 5. Fig. 2 (a) and Fig. 2 (b) show the average magnitude wDGFT over 50 realizations based on the eigenvectors from the tensor formulation and the eigenvectors of the average Laplacian, respectively. As expected, the highest energies are concentrated at the low frequencies since neither the signal over the graph nor the network structure are changing considerably over time. However, the DGFT captures the high frequency component corresponding to the anomaly, caused by the random structure of the network, whereas averaging cannot capture this nonstationarity in the network structure.

4.2. Dynamic Functional Connectivity Networks

The proposed method is applied to EEG data obtained from a cognitive control experiment. In the experiment, subjects were required to identify the correct target letter on a fivestring letter in the context of a speeded-reaction flanker task. In this experiment, it is of interest to identify the eventrelated negativity (ERN) potential, which is a brain potential response whose peak occurs between 25-75 ms after the com-



Fig. 2. wDGFT of a dynamic network with an anomaly computed with a window of length 5 (a) wDGFT based on the tensor decomposition, (b) wDGFT based on the average Laplacian.

mission of errors. In particular, it has been shown [15] that there is increased coordination among the lateral prefrontal cortex (IPFC) and the medial prefrontal cortex (mPFC) in the theta band (4-8 Hz). For this experiment, the EEG data was recorded from 62 electrodes according to the 10/20 system on a Neuroscan Synamos2 system (Neuroscan, Inc.). EEG data was preprocessed following standard procedures for artifact removal and volume conduction correction [16]. A total of 19 subjects were considered from this dataset. The adjacency matrices for each subject S at time t, $\mathbf{A}_{S}^{(t)}$, were created by computing the pairwise phase-locking value (PLV) based on a previously proposed method based on time-frequency phase synchrony [17]. For nodes i and j, the PLV measure results in a time-frequency map, $PLV_{i,j}(t,\omega)$. This measure is averaged over the theta band (4-8 Hz) and the subjects to construct the dynamic functional connectivity network. The Laplacian matrix $\mathbf{L}_{\hat{A}}^{(t)}$ is then computed from this average. A 3-way tensor $\mathcal{L} \in \mathbb{R}^{N \times N \times T}$ is constructed, where N corresponds to the number of electrodes, N = 58, and T to the total number of time points, T = 52 corresponding to the interval from 0-75 ms. The tensor \mathcal{L} is decomposed following (4) and the wDGFT with a sliding window of length 25 ms is computed.

As observed in Fig. 3, the spectral energy is high around the ERN time interval, specifically in the low frequencies. The presence of high energy at low frequencies within the ERN interval reflects that during this time the graph signal is smooth with respect to the underlying network structure. In addition, the eigenvectors provide information about network structure. Fig. 4 shows the 8th eigenvector corresponding to t = 0.022 ms. Peaks from this figure identify electrodes FPz, F1, FC1, and Cz as those contributing to the high energy shown at that particular frequency at t = 22 ms. This is consistent with findings from previous works which relate lateral



Fig. 3. wDGFT of the ERN dFCN over the interval 0-75 ms.



Fig. 4. Eigenvector corresponding to the eight eigenvalue at t = 22 ms as a function of electrodes.

and central regions to be relevant during the ERN.

5. CONCLUSIONS

In this paper, a dynamic graph Fourier transform based on the common basis obtained from the Tucker decomposition of the temporal network Laplacian tensor has been introduced to assess nonstationary networks. The proposed method allows us to determine network anomalies across time. These instantaneous anomalies may be missed when the basis is obtained from the Laplacian of the average adjacency matrices across time. Furthermore, the proposed method was applied to EEG data from a cognitive control study to determine the brain regions that are highly involved in the ERN and to better understand the smoothness of the network during ERN. Future work will concentrate on extending the proposed method to detect nonstationarities in both the network structure and the graph signals, simultaneously.

6. ACKNOWLEDGMENT

The authors would like to thank Dr. Jason Moser from the Department of Psychology at Michigan State University for providing the EEG data.

7. REFERENCES

- [1] David I. Shuman, Sunil K. Narang, Pascal Frossard, Antonio Ortega, and Pierre Vandergheynst, "The emerging field of signal processing on graphs: Extending highdimensional data analysis to networks and other irregular domains," *IEEE Signal Processing Magazine*, vol. 30, no. 3, pp. 83–98, 2013.
- [2] Weiyu Huang, Leah Goldsberry, Nicholas F Wymbs, Scott T. Grafton, Danielle S. Bassett, and Alejandro Ribeiro, "Graph frequency analysis of brain signals," arXiv preprint arXiv:1512.00037, 2015.
- [3] Aliaksei Sandryhaila and Jose M.F. Moura, "Discrete signal processing on graphs: Frequency analysis," *IEEE Transactions on Signal Processing*, vol. 62, no. 12, pp. 3042–3054, 2014.
- [4] David I. Shuman, Benjamin Ricaud, and Pierre Vandergheynst, "Vertex-frequency analysis on graphs," *Applied and Computational Harmonic Analysis*, vol. 40, no. 2, pp. 260–291, 2016.
- [5] Andreas Loukas and Damien Foucard, "Frequency analysis of temporal graph signals," *arXiv preprint arXiv:1602.04434*, 2016.
- [6] Francesco Grassi, Nathanael Perraudin, and Benjamin Ricaud, "Tracking time-vertex propagation using dynamic graph wavelets," arXiv preprint arXiv:1606.06653, 2016.
- [7] Selin Aviyente, Edward M. Bernat, Westley S. Evans, and Scott R. Sponheim, "A phase synchrony measure for quantifying dynamic functional integration in the brain," *Human brain mapping*, vol. 32, no. 1, pp. 80–93, 2011.
- [8] R. Matthew Hutchison, Thilo Womelsdorf, Elena A. Allen, Peter A. Bandettini, Vince D. Calhoun, Maurizio Corbetta, Stefania Della Penna, Jeff H. Duyn, Gary H. Glover, Javier Gonzalez-Castillo, et al., "Dynamic functional connectivity: promise, issues, and interpretations," *Neuroimage*, vol. 80, pp. 360–378, 2013.
- [9] Arash Golibagh Mahyari and Selin Aviyente, "Fourier transform for signals on dynamic graphs," in 2014 48th Asilomar Conference on Signals, Systems and Computers. IEEE, 2014, pp. 2001–2004.
- [10] Weiyu Huang, Leah Goldsberry, Nicholas Wymbs, Scott Grafton, Danielle Bassett, and Alejandro. Ribeiro, "Graph frequency analysis of brain signals," *presented at the Graph Signal Processing Workshop, Philadelphia, United States, 2016.*

- [11] Xinhai Liu, Shuiwang Ji, Wolfgang Glänzel, and Bart De Moor, "Multiview partitioning via tensor methods," *IEEE Transactions on Knowledge and Data Engineering*, vol. 25, no. 5, pp. 1056–1069, 2013.
- [12] Wei Tang, Zhengdong Lu, and Inderjit S. Dhillon, "Clustering with multiple graphs," in 2009 Ninth IEEE International Conference on Data Mining. IEEE, 2009, pp. 1016–1021.
- [13] Andreas Argyriou, Mark Herbster, and Massimiliano Pontil, "Combining graph laplacians for semi– supervised learning," in Advances in Neural Information Processing Systems, 2005, pp. 67–74.
- [14] Hilmi E. Egilmez, Antonio Ortega, Onur G. Guleryuz, Jana Ehmann, and Sekoon Yea, "An optimization framework for combining multiple graphs," in 2016 IEEE International Conference on Acoustics, Speech and Signal Processing (ICASSP). IEEE, 2016, pp. 4114–4118.
- [15] James F. Cavanagh, Michael X. Cohen, and John J.B. Allen, "Prelude to and resolution of an error: Eeg phase synchrony reveals cognitive control dynamics during action monitoring," *The Journal of Neuroscience*, vol. 29, no. 1, pp. 98–105, 2009.
- [16] Jürgen Kayser and Craig E. Tenke, "Principal components analysis of laplacian waveforms as a generic method for identifying erp generator patterns: I. evaluation with auditory oddball tasks," *Clinical neurophysiology*, vol. 117, no. 2, pp. 348–368, 2006.
- [17] Selin Aviyente and Ali Yener Mutlu, "A time-frequencybased approach to phase and phase synchrony estimation," *Signal Processing, IEEE Transactions on*, vol. 59, no. 7, pp. 3086–3098, 2011.