# TARGET DETECTON AND TRACKING VIA STRUCTURED CONVEX OPTIMIZATION

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## ABSTRACT

Moving target detection and tracking in reverberation environment is an important yet challenging problem in many applications such as speech, sonar, radar and seismic signal processing. Extending the early work of online subspace and sparse filtering [1], this paper presents an approach based on structured convex optimization. Exploiting potentially coherent structure of reverberation background, we represent the beamspace image data as the sum of a low-rank and a sparse matrix, where reverberation assumes low-rank structure and moving target signal is modeled as sparse. Detection and tracking is then formulated as a structured convex optimization problem, and solved via an accelerated proximal gradient (APG) algorithm. The performance of proposed algorithm is demonstrated using experimental results.

***Index Terms***— target detection and tracking, reverberation, low-rank, sparse, structured convex optimization, accelerated proximal gradient,

## 1. INTRODUCTION

Target detection and tracking is important in a number of applications such as computer vision, speech, sonar, radar and seismic signal processing. However, it remains a challenging task especially in environments where severe reverberations or background clutters are mixed with highly dynamic target signals. For instance, in speech or active sonar applications, signal quality may be degraded by strong reverberations associated with reflection or diffraction of transmitted signals from boundaries or propagation medium. Reverberation can in addition become strongly correlated with the transmitted signals, overlap or even overwhelm target signal of interest at the receivers, thus rendering correlation-based approaches such as matched filtering or adaptive filtering [9] less effective. Although increasing array size may lower the power of the interfering reverberations within reduced resolution cell, it has the adverse effect of causing reverberations within the cell significantly more heavy-tailed than Rayleigh distribution at the matched filter envelope, potentially leads to an increased false positive rate. Subspace based techniques [2]–[7] can be effectively used to capture reverberation components in the data by assuming and exploiting its potentially coherent structure. This is achieved through the assumption that coherent reverberation components are of reduced-dimension and hence can be extracted from the data via orthonormal decomposition techniques such as singular value decomposition. Measurement data are decomposed into a signal and a noise subspace which need to be updated from received data in slowly time-varying cases. For instance, principal component inverse [13] and Dominant mode rejection (DMR) [14] both operate in a reduced-dimension space spanned by the eigenvectors of the cross-spectral density matrix (CSDM) associated with its largest eigenvalues. These algorithms essentially attempt to separate target signal from reverberation according to assumed orthogonality that is relatively stable over the time duration of interest. However, for data with a complex dynamic structure this signal and noise subspace decomposition is not always an adequate representation. For instance, in moving target tracking, the low-rank subspace could be dominated by structured background reverberation that may also be slowly time-varying. The rapidly moving target signal, of real interest in this case, does not fit cleanly into either subspace to be filtered accurately. Its estimate is often degraded by noise and reverberation interference.

Recently a joint online subspace and sparse filtering algorithm was proposed in [1], which alternate between tracking a low-rank subspace representation of reverberation background and estimating the instantaneously sparse components associated with moving targets, and recursively update both as new data arrives. The algorithm consists of a random projection based subspace updating scheme and a soft-thresholding based sparse estimation. The reported results show significantly improved tracking performance over traditional subspace tracking algorithms.

In this paper, we extend the work in [1] to develop a structured convex optimization formulation of the problem of moving target detection and tracking, and propose to solve it using an accelerated proximal gradient (APG) algorithm. We use an example data set consisting of moving targets embedded in severe background reverberations, obtained in the form of a sequence of two-dimensional range-bearing beamspace snapshots sampled over a certain period of observation time. A global matrix is constructed out of all the data by stacking together each data frame after column vectorization. We then represent the global data matrix as the sum of a low-rank and sparse matrices. The moving targets, covering a small set of data points in each snapshot, is represented by the sparse matrix that may change the support regions over time. The background reverberations, assumed stable and correlated over the considered sequence, contribute to the low-rank matrix. As a result, the problem of moving target detection and tracking in reverberation environments is casted as one of recovering the sparse matrix, time-tracking its variations over columns after being separated from the low-rank component, which we propose to solve iteratively via the APG algorithm.

The rest of this paper is organized as follows. Section II presents the problem formulation. The APG algorithm for low-rank and sparse matrix decomposition is described in Section III. Section IV present experimental data processing and results. Finally, conclusions and discussions are provided in Section V.

## 2. PROBLEM FORMULATION

We start by considering a sequence of 2D data frames, denoted as $Z_i \in \mathbb{R}^{N_R \times N_C}$, for $i = 1, \dots N$. Each represents the snapshot image in the range-bearing domain obtained at the *i*th sampling time. $N_R$ and $N_C$ are the dimensions in the range and bearing directions, respectively. After vectorizing each frame, *i.e.* by column stacking and converting each $Z_i$ into a vector $z_i$ of length $N_R N_C$, we collect them all as columns of a global data matrix $\boldsymbol{Z} \in \mathbb{R}^{M \times N}$, where $M = N_R N_C$, and $N$, the number of columns, is equal to the number of time snapshots:

$$\boldsymbol{Z} = [z_1 \, z_2 \dots z_N] \, . \tag{1}$$

In the context of moving target detection and tracking under reverberation conditions, the data sequence in $\boldsymbol{Z}$ can be realistically modeled as the sum of target signals of interest, background reverberations or clutters, and noises. As mentioned in the introduction, we make the assumption throughout the paper that during the time span of the considered data sequence, the background reverberations are strongly correlated and slowly time-varying from frame to frame, while the target signals have sparse support in each snapshot frame and can change rapidly over frames. As a result, a suitable decomposition of the data matrix $\boldsymbol{Z}$ that reflects this type of structure is the following

$$\boldsymbol{Z} = \boldsymbol{L} + \boldsymbol{S} + \boldsymbol{G} \, , \tag{2}$$

where $\boldsymbol{L}$, $\boldsymbol{S}$ and $\boldsymbol{G}$ are the low-rank background reverberation, the sparse target signal and the noise components of the data, respectively. In the absence of $\boldsymbol{S}$, recovering $\boldsymbol{L}$ at a given rank r from $\boldsymbol{Z}$ involves the familiar low-rank matrix approximation problem, *i.e.*:

$$\min \|\boldsymbol{Z} - \boldsymbol{L}\|_F, \quad \text{such that } \operatorname{rank}(\boldsymbol{L}) = r, \tag{3}$$

which can be solved via singular value decomposition according to the Eckart-Young Theorem [8]. Here $\| \;\; \|_F$ denotes the matrix Frobenius norm.

Based on the data representation specified in (2), the problem of detecting and tracking moving target signal involves extracting the sparse matrix $\boldsymbol{S}$ from the data matrix $\boldsymbol{Z}$, and then tracking the signal variations across the columns of $\boldsymbol{S}$. To recover the matrices $\boldsymbol{L}$ and $\boldsymbol{S}$ from $\boldsymbol{Z}$, we adopt the following relaxed optimization formulation ([1],[31], [27]-[28])

$$\min_{\boldsymbol{L},\boldsymbol{S}} \frac{1}{2} \|\boldsymbol{Z} - \boldsymbol{L} - \boldsymbol{S}\|_F^2 + \eta \|\boldsymbol{L}\|_* + \eta\alpha \|\boldsymbol{S}\|_1, \tag{4}$$

where $\alpha$ and $\eta$ are positive scalars as the weighting coefficients for the two structure penalty terms, *i.e.* low-rank and sparsity. $\| \;\; \|_*$ and $\| \;\; \|_1$ denote the matrix nuclear norm and the matrix $l_1$norm, as the convex surrogates for matrix rank and $l_0$norm, respectively. Given a matrix $\boldsymbol{A}$ with element entries $a_{i,j}$ and singular values $\lambda_i$,

$$\|\boldsymbol{A}\|_* \triangleq \sum_i \lambda_i \tag{5}$$

$$\|\boldsymbol{A}\|_1 \triangleq \sum_{ij} |a_{ij}| \tag{6}$$

$$\|\boldsymbol{A}\|_0 \triangleq \sum_{ij} I(a_{ij}), \tag{7}$$

where $I(a_{ij}) = 0$ if $a_{ij} = 0$ and 1 otherwise.

The decomposition in (2) is generally not unique. Recovery guarantee conditions, mostly for noiseless cases, involve the so-called rank-sparsity incoherence [10], or in terms of the matrix row/column support and the coordinate alignment of its low-rank singular vectors [11], which is beyond the scope of this paper.

## 3. STRUCTURED CONVEX OPTIMIZATION VIA THE APG ALGORITHM

The optimization framework (4) for low-rank and sparse matrix decomposition has been part of an intensive research field in recent years, with many applications such as machine learning, computer vision, and compressive sensing [31]-[33]. In the current application context of moving target detection and tracking, we may view low-rank and sparse matrix decomposition as an extension of subspace-based processing, capable of coping with more complex data structure that does not necessarily conform to a pure low-rank structure. A large number of algorithms have been developed for the same or similar purposes ([19]-[30]), with theoretical aspects such as recovery guarantee, convergence and complexity discussed in [32][33]. A review of some of these algorithms can be found, for instance, in [31]. In this paper we choose the accelerated proximal gradient (APG) algorithm to solve the nonlinear convex problem (4). In the context of proximal mapping [34], a general form for problems such as (4) can be written as follows:

$$\min_{\boldsymbol{X}} F(\boldsymbol{X}) \triangleq f(\boldsymbol{X}) + c(\boldsymbol{X}), \quad (8)$$

where $f(\boldsymbol{X})$ denotes a convex objective term that has an inexpensive proximal operator [34][35] and $c(\boldsymbol{X})$ is differentiable and associated with constraints. Specifically, in the case of (4), we have

$$f(\boldsymbol{X}) = \eta\|\boldsymbol{L}\|_* + \eta\alpha\|\boldsymbol{S}\|_1 \quad (9a)$$

$$c(\boldsymbol{X}) = \frac{1}{2}\|\boldsymbol{Z} - \boldsymbol{L} - \boldsymbol{S}\|_F^2 \quad (9b)$$

with $\boldsymbol{X} \triangleq (\boldsymbol{L}, \boldsymbol{S})$**.** Since $c(\boldsymbol{X})$ is quadratic hence smooth and convex, we have the following Lipschitz condition [34]

$$\|\nabla c(\boldsymbol{X}) - \nabla c(\boldsymbol{Y})\|_F \leq L_c\|\boldsymbol{X} - \boldsymbol{Y}\|_F, \quad (10)$$

with the Lipschitz constant $L_c = 2$ in this case. $\nabla c(\boldsymbol{X})$ here denotes the Fréchet derivative of $c(\boldsymbol{X})$.

The proximal gradient algorithms ([22][34]) consists of the following minimization at each iteration

$$\widehat{\boldsymbol{X}}_{k+1} = prox_{\lambda_k f}(\boldsymbol{Y}_k - \lambda_k \nabla c(\boldsymbol{Y}_k))$$
$$= \operatorname*{argmin}_{\boldsymbol{X}} f(\boldsymbol{X}) + \frac{1}{2\lambda_k}\left\|\boldsymbol{X} - \left(\boldsymbol{Y}_k - \lambda_k \nabla c(\boldsymbol{Y}_k)\right)\right\|_2^2 \quad (11)$$
$$= \operatorname*{argmin}_{\boldsymbol{X}} f(\boldsymbol{X}) + c(\boldsymbol{Y}_k) + \langle \nabla c(\boldsymbol{Y}_k), \boldsymbol{X} - \boldsymbol{Y}_k\rangle + \frac{1}{2\lambda_k}\|\boldsymbol{X} - \boldsymbol{Y}_k\|_2^2$$

where $prox_{\lambda_k f}$ denotes the proximal mapping of a convex function $\lambda_k f(\ \ )$ [34], and $\lambda_k \in (0, 1/L_c]$is the step size. $\boldsymbol{Y}_k$ is updated from $\boldsymbol{X}_k$**,** the estimate at the previous iteration, and is equal to $\boldsymbol{X}_k$ in the basic version of the proximal gradient algorithm. In the accelerated proximal gradient (APG) algorithm [34][35][38],

$$\boldsymbol{Y}_k = \widehat{\boldsymbol{X}}_k + \frac{t_{k-1}-1}{t_k}(\widehat{\boldsymbol{X}}_k - \widehat{\boldsymbol{X}}_{k-1}) \quad (13)$$

which is one step extrapolation based on $\boldsymbol{X}_k$ and $\boldsymbol{X}_{k-1}$. The extrapolation parameter is determined by the sequence $t_k$ which is required to satisfy $t_k^2 - t_k \leq t_{k-1}^2$. Often it suffices to take the equality hence $t_k = \left(1 + \sqrt{4t_{k-1}^2 + 1}\right)/2$.

Let $\boldsymbol{G}_k \triangleq \boldsymbol{Y}_k - \lambda_k \nabla c(\boldsymbol{Y}_k)$, we have

$$\widehat{\boldsymbol{X}}_{k+1} = \operatorname*{argmin}_{\boldsymbol{X}} f(\boldsymbol{X}) + \frac{1}{2\lambda_k}\|\boldsymbol{X} - \boldsymbol{G}_k\|_2^2. \quad (14)$$

Furthermore, denote $\boldsymbol{G}_k \triangleq (\boldsymbol{G}_k^L, \boldsymbol{G}_k^S)$, where $\boldsymbol{G}_k^L$ and $\boldsymbol{G}_k^S$ correspond to the low-rank and sparse components in $\boldsymbol{G}_k$, respectively, similarly with $\boldsymbol{Y}_k \triangleq (\boldsymbol{Y}_k^L, \boldsymbol{Y}_k^S)$,. As a result, $\boldsymbol{G}_k^L$ and $\boldsymbol{G}_k^S$ can be iteratively update as follows,

$$\boldsymbol{G}_k^L = \boldsymbol{Y}_k^L - \lambda_k(\boldsymbol{Y}_k^L + \boldsymbol{Y}_k^S - \boldsymbol{Z}), \quad (15a)$$

$$\boldsymbol{G}_k^S = \boldsymbol{Y}_k^S - \lambda_k(\boldsymbol{Y}_k^L + \boldsymbol{Y}_k^S - \boldsymbol{Z}). \quad (15b)$$

Given the particular form of $f(\boldsymbol{X})$ in (9a), the proximal mapping in (11) involves two relatively simple soft-thresholding operations for estimating the sparse component $\widehat{\boldsymbol{S}}_{k+1}$ and the low-rank component $\widehat{\boldsymbol{L}}_{k+1}$, as explained below.

The sparse component estimate $\widehat{\boldsymbol{S}}_{k+1}$ in $\widehat{\boldsymbol{X}}_{k+1}$ can thus be obtained via direct soft-thresholding $\boldsymbol{G}_k^S$:

$$\widehat{\boldsymbol{S}}_{k+1} = ST_{\eta\alpha\lambda_k}(\boldsymbol{G}_k^S) = \begin{cases} \boldsymbol{G}_k^S - \eta\alpha\lambda_k, \text{if } \boldsymbol{G}_k^S > \eta\alpha\lambda_k \\ \boldsymbol{G}_k^S + \eta\alpha\lambda_k, \text{if } \boldsymbol{G}_k^S < -\eta\alpha\lambda_k \\ 0,\ otherwise \end{cases} \quad (16)$$

where $ST_{\eta\alpha\lambda_k}()$ denotes the soft-thresholding operator for the elements of $\boldsymbol{G}_k^S$. The low-rank component estimate $\widehat{\boldsymbol{L}}_{k+1}$ in $\widehat{\boldsymbol{X}}_{k+1}$ involves soft-thresholding the singular values of $\boldsymbol{G}_k^L$. Let the singular value decomposition(SVD) of $\boldsymbol{G}_k^L = \boldsymbol{U}_k\boldsymbol{\Sigma}_k\boldsymbol{V}_k^T$, then

$$\widehat{\boldsymbol{L}}_{k+1} = \boldsymbol{U}_k ST_{\eta\lambda_k}(\boldsymbol{\Sigma}_k)\boldsymbol{V}_k^T \quad (17)$$

where $ST_{\eta\lambda_k}()$ denotes the soft-thresholding operator for the singular values $\boldsymbol{\Sigma}_k$. The SVD in (17) needed at each iteration can be computationally costly especially when the matrix size is large. This may be replaced by partial SVD. Or in applications like the one we have, the data matrix $\boldsymbol{Z} \in \mathbb{R}^{M\times N}$, with $M = N_R N_C \gg N$, then an eigenvalue decomposition of the $N\times N$ correlation matrix becomes a significantly more efficient alternative, and the soft thresholding only needs to be applied the square root of the first $N$ eigenvalues. The resulting APG algorithm is summarized in Table 1.

**Table 1**. APG algorithm for target detection and tracking

Inputs: global data matrix $\boldsymbol{Z}$ specified in (1)
Outputs: estimates of $\boldsymbol{L}$, $\boldsymbol{S}$ as in (2)
Parameters: weights $\alpha, \eta, \lambda_k, t_k, \varepsilon$, *maxiter*,
Initialization: $\widehat{\boldsymbol{L}}_1 = \widehat{\boldsymbol{L}}_0 = \widehat{\boldsymbol{S}}_1 = \widehat{\boldsymbol{S}}_0 = \boldsymbol{0}$
**For** $k = 1$:*maxiter* and $\left\|\boldsymbol{Z} - \left(\widehat{\boldsymbol{L}}_k + \widehat{\boldsymbol{S}}_k\right)\right\|_F^2 < \varepsilon$
$\quad \boldsymbol{Y}_k^L = \widehat{\boldsymbol{L}}_k + \frac{t_{k-1}-1}{t_k}(\widehat{\boldsymbol{L}}_k - \widehat{\boldsymbol{L}}_{k-1})$
$\quad \boldsymbol{Y}_k^S = \widehat{\boldsymbol{S}}_k + \frac{t_{k-1}-1}{t_k}(\widehat{\boldsymbol{S}}_k - \widehat{\boldsymbol{S}}_{k-1})$
compute $\boldsymbol{G}_k^L, \boldsymbol{G}_k^S$ according to (15)
$(\boldsymbol{V}_k, \boldsymbol{\Lambda}_k) = \boldsymbol{eig}\left[\left(\boldsymbol{G}_k^L\right)^T\boldsymbol{G}_k^L\right], and\ \boldsymbol{\Sigma}_k = (\boldsymbol{\Lambda}_k)^{1/2}$
$\boldsymbol{U}_k = \boldsymbol{\Sigma}_k^{-1}\boldsymbol{G}_k^L\boldsymbol{V}_k$
compute $\widehat{\boldsymbol{S}}_{k+1}$ and $\widehat{\boldsymbol{L}}_{k+1}$from (16) and (17), respectively
update $t_k = \left(1 + \sqrt{4t_{k-1}^2 + 1}\right)/2$
**End**

Where $\lambda_k$can be obtained via line search within $(0, 1/L_c]$ $\varepsilon$ is the tolerate level for the reconstruction error at convergence. The choices of how to choose the values for $\alpha$ and $\eta$ can be found in [31], for instance, $\alpha = 1/\sqrt{\max(M, N)}$.

## 4. EXPERIMENTAL RESULTS

We formulated and tested the APG algorithms described in sections 2 and 3 for moving target detection and tracking on an acoustic field data sets. The data set was chosen for its strong reverberation that appears fairly coherent across observed time duration, and the fact that the target signal has relatively sparse support in each time snapshot. The data set consisted of 60 frames of 2D range-bearing data generated from an array imaging algorithm. Each frame is a matrix of dimension 793×90, with 793 samples along the range axis and 90 samples the radial angle axis. One such data frame is plotted in Figure. 1(a). As shown the moving target signal (circled in white) is overwhelmed by the background reverberation associated with acoustic backscattering, both in terms of magnitude and support area. Across the 60 frames, the background reverberation remains relatively stationary with random scintillation while the target follows a nearly linear motion trajectory.

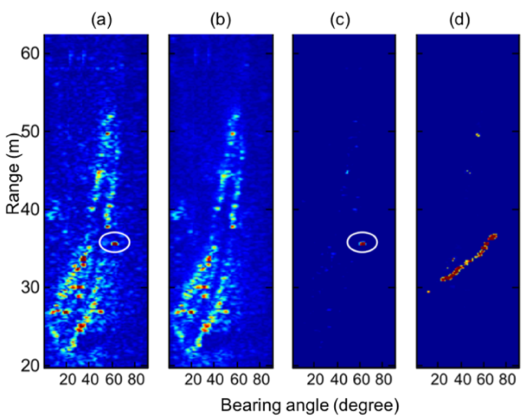


Figure 1. Acoustic moving targets detection and tracking. (a) an example data frame ($\boldsymbol{z_i}$); (b) estimated reverberation ($\boldsymbol{l_i}$), (c) move target signals ($\boldsymbol{s_i}$)) (circled in white), and (d) the target tracking results.

A global data matrix $\boldsymbol{Z}$ of size 71370×60 is assembled from the 60 data frames according to (1). We then apply the APG algorithm to extract the reverberation and target signal as the low-rank component $\boldsymbol{L}$ and the sparse component $\boldsymbol{S}$, respectively. The parameter values used were $\alpha = 0.013$, $\eta = 0.001$ and *maxiter*=200. The matrices $\boldsymbol{Z}$, $\boldsymbol{L}$, $\boldsymbol{S}$, and residual $\boldsymbol{G}$ are plotted in Figure 2, after truncating to exclude the noise-only areas for better visualization. The low-rank structure reflecting cross-frame reverberation coherence is evident in Figure 2(b). The rank and sparse structures are also illustrated in Figure 3. In Figure 3(a), the leading 30 singular values of both $\boldsymbol{L}$ and $\boldsymbol{Z}$ are plotted together. Figure 3(b) plots 500 out of 71370 elements in $\boldsymbol{S}$ that have the largest magnitude. These results confirm the degree of sparsity in the singular value domain (hence low-rank) and element domain (hence sparse), respectively.

The estimated low-rank and sparse matrices, $\boldsymbol{L}$, and $\boldsymbol{S}$, can be dissembled with each of their columns converted back into one data frame, becoming the reverberation and the target signal at the corresponding data frame, respectively. For the data frame shown in Figure 1(a), the associated low-rank and sparse components have been plotted in Figures 1(b) and 1(c), respectively. The target tracking result is plotted in Figure 1(d). As pointed out in [1], in traditional subspace tracking, the background reverberation residual adds up constructively over tracking integral time, and overwhelms the target signal.

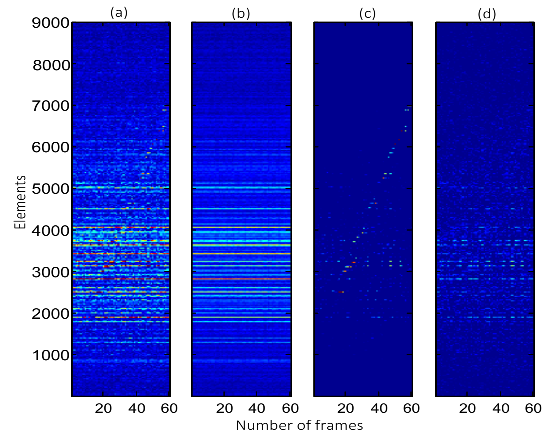


Figure 2. Low-rank and spare matrix decomposition; (a) the global data matrix $\boldsymbol{Z}$; (b) the low-rank component $\boldsymbol{L}$; (c) the sparse matrix $\boldsymbol{S}$, and (d) the residual noise matrix $\boldsymbol{G}$, respectively. Noise only portions have been excluded for better visualization.

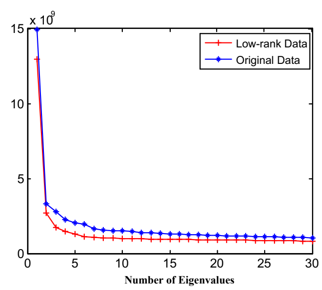


Fig. 3(a) singular values of $\boldsymbol{L}$, $\boldsymbol{Z}$.

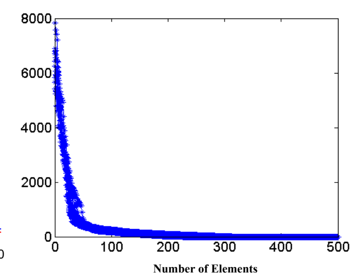


Fig 3(b) Cardinality of $\boldsymbol{S}$

## 5. CONCLUSION

In this paper, we formulate the problem of moving target detection and tracking in reverberation environments as a structured convex optimization problem for low-rank and sparse matrix decomposition, as an extension to early work in [1]. An accelerated proximal gradient (APG) algorithm has been applied to recover the low-rank and sparse components of the data, corresponding to coherent background reverberation and the moving target signal. Acoustic field data results have demonstrated the effectiveness of the approach.

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