INFINITE-DIMENSIONAL SVD FOR ANALYZING MICROPHONE ARRAY

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ABSTRACT

Nowadays, various types of microphone array are used in many applications. However, it is not easy to compare arrays of different types because each array has been treated by a specific theory depending on the type of an array. Although several criteria have been proposed for microphone arrays for evaluating and/or designing an array, most of them are application-oriented criteria and the best configuration for some criterion may not be a better one in the other criterion. Therefore, an analysis and comparing method for microphone arrays which does not depend on an array configuration and application are necessary. In this paper, infinite-dimensional SVD is proposed for analyzing and comparing properties of arrays. The singular values and functions obtained by proposed method show sampling property of an array and can be unified criterion.

***Index Terms*—** Helmholtz equation, plane wave approximation, Herglotz wave function, singular value decomposition (SVD), analytic calculation.

1. INTRODUCTION

A microphone array is a quite fundamental device for acquiring spatial information of a sound field. A lot of signal processing methods effectively utilizing the spatial information of sound have been investigated for many applications including direction-of-arrival estimation, noise reduction, and blind source separation [1–4]. While the most widely used microphone array is two-channel because it is ubiquitous, an array consists of more than two microphones is getting more and more popular today [5].

Nowadays, various types of microphone array are used in the above applications, such as linear, planar, spiral, spherical, and random configurations. Arguably, some array configuration must be better than the others. However, it is not easy to compare arrays of different types because each array has been treated by a specific theory depending on the type of an array. Therefore, an analysis method for microphone arrays which does not depend on an array configuration is necessary for the comparison.

Several criteria have been proposed for microphone arrays in the context of evaluating and/or designing an array [6–22]. By quantifying effect of array signal processing methods, those criteria enable optimization of microphone positions to achieve better performance. Although they have been proved to be useful in the literatures, there is one limitation: most of them are application-oriented criteria. That is, the best configuration for some criterion may not be a better one in the other criterion. Such application dependent assessment is useful for optimizing performance of a specific processing method. Nevertheless, an application *independent* criterion of characteristics of microphone arrays should also be important for comparing and analyzing properties of arrays.

In this paper, a continuous analogous of the singular value decomposition (SVD), namely infinite-dimensional SVD, is proposed for analyzing microphone arrays. Since microphone array signals of any sound field can be represented by a linear transformation [see Eq. (14)], SVD of the transformation provides information of the sampling property of a microphone array which does not depend on an application. Although it can be regarded as decomposition of an infinite-dimensional matrix, difficulty of calculation is totally avoided by analytic solution.

2. SAMPLING OF SOUND FIELD AND ITS PLANE WAVE REPRESENTATION

In linear acoustics, sound propagation is modeled by the wave equation [23],

$$\left(\triangle - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right) p(\boldsymbol{x}, t) = 0, \tag{1}$$

where $\triangle = \sum_n \partial^2/\partial x_n^2$ is the Laplace operator, t is time, $\boldsymbol{x} = (x_1, x_2, x_3)$ is position, p is sound pressure, and c is the speed of sound. The Fourier transform on time variable $\mathscr{F}_t$ converts Eq. (1) into the Helmholtz equation:

$$\left(\triangle + k^2\right) u(\boldsymbol{x}, \omega) = 0, \tag{2}$$

where $u = \mathscr{F}_t p$, $k = \omega/c$ is the wave number, and ω is the angular frequency. For simplicity, the dependency on ω will be omitted hereafter as $u(\boldsymbol{x})$. Let the sound field u be sampled by M microphones placed at $\{\boldsymbol{x}_m\}_{m=1}^M$ as $\{u(\boldsymbol{x}_m)\}_{m=1}^M$. Then, the aim of this paper is to analyze the sampling property based on the above physical model. For considering every function satisfying Eq. (2), some convenient representation of the solutions is necessary in order to make the problem tractable. Here, plane waves are adopted for characterizing them.

2.1. Plane wave approximation of sound field

It is well known that any solution to the homogeneous Helmholtz equation can be approximated arbitrarily well by linear combination of plane waves [24]:

$$u(\boldsymbol{x}) \approx \sum_{n=1}^{N} \alpha_n \exp(jk\langle \boldsymbol{x}, \boldsymbol{\nu}_n \rangle), \tag{3}$$

where $\alpha_n \in \mathbb{C}$, $j = \sqrt{-1}$, $\langle \cdot, \cdot \rangle$ is the standard inner product, $\{\boldsymbol{\nu}_n\}_{n=1}^{N} \subset \mathbb{S}^2$ is set of unit vectors which corresponds to the direction of propagation of the plane waves. By taking the sampling into account, Eq. (3) becomes

$$u(\boldsymbol{x}_m) \approx \sum_{n=1}^{N} \alpha_n \exp(jk\langle \boldsymbol{x}_m, \boldsymbol{\nu}_n \rangle), \tag{4}$$

which can be rewritten in a matrix form:

$$\boldsymbol{u} = H\boldsymbol{\alpha}, \tag{5}$$

where $\boldsymbol{\alpha}$ and $\boldsymbol{u}$ are respectively N- and M-dimensional column vectors defined as

$$\boldsymbol{\alpha} = [\alpha_1,\, \alpha_2,\, \ldots,\, \alpha_N]^T, \tag{6}$$

$$\boldsymbol{u} = [u(\boldsymbol{x}_1),\, u(\boldsymbol{x}_2),\, \ldots,\, u(\boldsymbol{x}_M)]^T, \tag{7}$$

and H is an $M \times N$ matrix,

$$H = \begin{bmatrix} \exp(jk\langle \boldsymbol{x}_1, \boldsymbol{\nu}_1 \rangle) & \cdots & \exp(jk\langle \boldsymbol{x}_1, \boldsymbol{\nu}_N \rangle) \\ \vdots & \ddots & \vdots \\ \exp(jk\langle \boldsymbol{x}_M, \boldsymbol{\nu}_1 \rangle) & \cdots & \exp(jk\langle \boldsymbol{x}_M, \boldsymbol{\nu}_N \rangle) \end{bmatrix}. \tag{8}$$

This equation shows that any sampled sound field $\boldsymbol{u}$ is related to some $\boldsymbol{\alpha}$ through H. Hence, analyzing the matrix H gives information about the spaces where $\boldsymbol{u}$ and $\boldsymbol{\alpha}$ are lying.

3. INFINITE-DIMENSIONAL SVD OF SOUND FIELD

As in the previous section, a sound field can be characterized by the matrix H consisting of plane waves. Therefore, the sampling property of a microphone array can be analyzed by SVD of H. However, discrete sampling of $\boldsymbol{\nu} \in \mathbb{S}^2$ may cause significant error in the analysis. In this section, an SVD-based analysis method which does not involve such discrete approximation is proposed.

3.1. Singular value decomposition (SVD)

For analyzing a matrix, one of the most popular methodologies is SVD which decomposes an $M \times N$ matrix A into three matrices as

$$A = U\Sigma V^*, \tag{9}$$

where U is an $M \times M$ unitary matrix, Σ is an $M \times M$ diagonal matrix, V is an $N \times M$ column orthonormal matrix, and V^* is conjugate transpose of V. This decomposition reveals mapping properties of the matrix, and thus the sampling property of a microphone array can be analyzed by decomposing the corresponding matrix H.

However, Eq. (3) is just an approximation and not exact when the summation is taken up to a finite integer. This implies that accurate decomposition is impossible in practice even though one may increase N as much as the computational resource allows [1]. Indeed, we observed that results of the decomposition can be notably different depending on N and choice of $\{\boldsymbol{\nu}_n\}_{n=1}^{N}$. Therefore, it is necessary to overcome the approximation error in order to achieve reliable analysis which does not depend on the construction process of the matrix.

3.2. Herglotz wave function and its operator as infinite dimensional matrix

When the set $\{\boldsymbol{\nu}_n\}_{n=1}^{N}$ is chosen properly, Eq. (3) can be regarded as direct discretization of the Herglotz wave function [25]:

$$u(\boldsymbol{x}) = \int_{\mathbb{S}^2} \alpha(\boldsymbol{\nu}) \exp(jk\langle \boldsymbol{x}, \boldsymbol{\nu} \rangle)\, dS(\boldsymbol{\nu}), \tag{10}$$

where α is a square-integrable function defined on the unit sphere $\mathbb{S}^2$. By defining an integral operator $\mathscr{H}$ as

$$(\mathscr{H}\alpha)(\boldsymbol{x}) = \int_{\mathbb{S}^2} \alpha(\boldsymbol{\nu}) \exp(jk\langle \boldsymbol{x}, \boldsymbol{\nu} \rangle)\, dS(\boldsymbol{\nu}), \tag{11}$$

Eq. (10) can be written as

$$u = \mathscr{H}\alpha. \tag{12}$$

Since every solution u can be represented by the Herglotz wave function [26], α can be considered as a representative of the sound field.

In a similar manner to Eq. (4), sampled version of the Herglotz wave function can be defined as

$$u(\boldsymbol{x}_m) = \int_{\mathbb{S}^2} \alpha(\boldsymbol{\nu}) \exp(jk\langle \boldsymbol{x}_m, \boldsymbol{\nu} \rangle)\, dS(\boldsymbol{\nu}), \tag{13}$$

which is also denoted shortly as

$$\boldsymbol{u} = \mathscr{H}_M\alpha, \tag{14}$$

where $\mathscr{H}_M$ is the corresponding integral operator. This operator can be considered as the continuous analogous of the matrix H as taking the limit $N \to \infty$, i.e., $\mathscr{H}_M$ can be regarded as an $M \times \infty$ matrix [27]. Therefore, the approximation error is overcome by decomposing the continuous version $\mathscr{H}_M$ instead of its approximation H. Then, the only issue for this approach is a decomposition algorithm: how to decompose a matrix whose dimension is not finite.

[1] Since $\{\boldsymbol{\nu}_n\}_{n=1}^{N}$ is defined on the unit sphere, the perfectly uniform set cannot be obtained except the five Platonic bodies ($N \in \{4, 6, 8, 12, 20\}$). Therefore, one should suffer from error not only on the approximation in Eq. (3) but also on the asymmetry of $\{\boldsymbol{\nu}_n\}_{n=1}^{N}$ if N is finite.

3.3. Infinite-dimensional SVD of sampled sound field

There are several routes for computing SVD. Since

$$AA^* = U\Sigma V^* V\Sigma U^* = U\Sigma^2 U^*, \tag{15}$$

U and Σ can be obtained through eigendecomposition of AA^*. Then, V is calculated from U and Σ as

$$\Sigma^{-1}U^*A = \Sigma^{-1}U^*U\Sigma V^* = \Sigma^{-1}\Sigma V^* = V^*. \tag{16}$$

Although $\mathscr{H}_M$ is infinite-dimensional, $\mathscr{H}_M\mathscr{H}_M^*$ becomes a finite dimensional matrix which allows the ordinary matrix decomposition. Therefore, SVD of $\mathscr{H}_M$ can be calculated by decomposing the $M\times M$ matrix $\mathscr{H}_M\mathscr{H}_M^*$ and using Eq. (16). Then, one has to evaluate $\mathscr{H}_M\mathscr{H}_M^*$ whose (m,n)-th entry is

$$\begin{aligned}(\mathscr{H}_M\mathscr{H}_M^*)_{mn} &= \int_{\mathbb{S}^2} \exp(jk\langle \boldsymbol{x}_m, \boldsymbol{\nu}\rangle)\overline{\exp(jk\langle \boldsymbol{x}_n, \boldsymbol{\nu}\rangle)}\, dS(\boldsymbol{\nu}) \\ &= \int_{\mathbb{S}^2} \exp(jk\langle \boldsymbol{x}_m - \boldsymbol{x}_n, \boldsymbol{\nu}\rangle)\, dS(\boldsymbol{\nu}),\end{aligned} \tag{17}$$

where $\overline{z}$ is complex conjugate of z. In the above equation, the inner product can be rewritten as

$$\langle \boldsymbol{x}_m - \boldsymbol{x}_n, \boldsymbol{\nu}\rangle = \|\boldsymbol{x}_m - \boldsymbol{x}_n\|\cos(\theta), \tag{18}$$

where $\|\cdot\| = \sqrt{\langle\cdot,\cdot\rangle}$ is the Euclidean norm, $\|\boldsymbol{\nu}\| = 1$ by definition, and θ is the angle between the vectors. For simplicity, let $\boldsymbol{x}_m - \boldsymbol{x}_n$ be lying on the one-dimensional subspace $(0,0,x_3)$. Then, the angle θ coincides with that of the polar coordinate $(\sin(\theta)\cos(\varphi), \sin(\theta)\sin(\varphi), \cos(\theta))$. This observation further simplifies Eq. (17) as

$$\begin{aligned}(\mathscr{H}_M\mathscr{H}_M^*)_{mn} &= \int_{\mathbb{S}^2} \exp(jk\langle \boldsymbol{x}_m - \boldsymbol{x}_n, \boldsymbol{\nu}\rangle)\, dS(\boldsymbol{\nu}) \\ &= \int_{-\pi}^{\pi}\int_0^{\pi} \exp(jkd\cos(\theta))\sin(\theta)\, d\theta d\varphi \\ &= 2\pi\int_0^{\pi} \exp(jkd\cos(\theta))\sin(\theta)\, d\theta,\end{aligned} \tag{19}$$

where $d = \|\boldsymbol{x}_m - \boldsymbol{x}_n\|$. This integration can be solved analytically:

$$\begin{aligned}(\mathscr{H}_M\mathscr{H}_M^*)_{mn} &= 2\pi\int_0^{\pi} \exp(jkd\cos(\theta))\sin(\theta)\, d\theta \\ &= 2\pi\Big[\frac{j}{kd}\exp(jkd\cos(\theta))\Big]_0^{\pi} \\ &= \frac{2\pi}{kd}\big[-\sin(-kd) + \sin(kd) \\ &\qquad + j\cos(-kd) - j\cos(kd)\big] \\ &= 4\pi\frac{\sin(kd)}{kd}.\end{aligned} \tag{20}$$

Until here, the vector $\boldsymbol{x}_m - \boldsymbol{x}_n$ has been assumed to be on the one-dimensional subspace $(0,0,x_3)$. Nevertheless, the above result also holds for any vector because spherical integration is independent of rotation. That is, one may rotate the coordinate first to set $\boldsymbol{x}_m - \boldsymbol{x}_n$ on that subspace, and then the same derivation can be applied to obtain the analytical solution. Therefore, (m,n)-th entry of the matrix $\mathscr{H}_M\mathscr{H}_M^*$ is

$$(\mathscr{H}_M\mathscr{H}_M^*)_{mn} = 4\pi\,\mathrm{sinc}(k\|\boldsymbol{x}_m - \boldsymbol{x}_n\|), \tag{21}$$

where $\mathrm{sinc}(x) = \sin(x)/x$ if $x \neq 0$, and $\mathrm{sinc}(0) = 1$.

Since the matrix is real and symmetric, it can be diagonalized by an orthonormal matrix U as

$$\mathscr{H}_M\mathscr{H}_M^* = U\Sigma^2 U^*. \tag{22}$$

Then, m-th column of V, which is a continuous function on the unit sphere, is obtained by Eq. (16) as

$$v_m(\boldsymbol{\nu}) = \frac{1}{\sigma_m}\sum_{\ell=1}^{M} U_{\ell m}\exp(jk\langle \boldsymbol{x}_\ell, \boldsymbol{\nu}\rangle), \tag{23}$$

where $V = [v_1,\, v_2,\, \ldots,\, v_M]$, and $\Sigma = \mathrm{diag}(\sigma_1, \sigma_2, \cdots, \sigma_M)$. Note that Eq. (21) does not involve any approximation, and therefore the obtained decomposition should be accurate up to the machine precision.

4. NUMERICAL EXPERIMENT

In this section, the proposed method was applied to several microphone arrays in order to show its property. Table 1 and Fig. 1 show the microphone array configurations used here. The sound speed was assumed to be 340 m/s.

Figure 2 shows normalized singular values $\{\sigma_m/\sigma_1\}_{m=1}^M$ of the sampling operator $\mathscr{H}_M$. Although all three cases contain twelve singular values which coincide with the number of microphones, some of them are overlapped and invisible. For example, only three singular values are visible for low-frequency part of the spherical microphone array because their multiplicities are three, five, and three from top to bottom. Note that the first normalized singular value always overlaps with the ceiling as $\sigma_1/\sigma_1 = 1$.

Some specific frequencies related to the types of arrays, the Nyquist frequency or Dirichlet eigenfrequencies, are also depicted in Fig. 2 as vertical lines. A Dirichlet eigenfrequency is a frequency which may be problematic for array

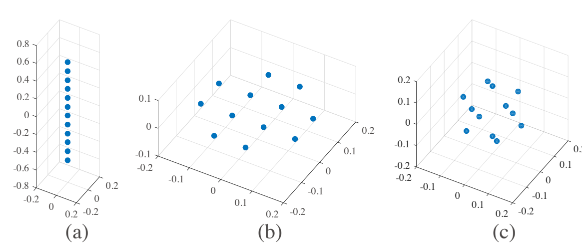

Fig. 1. The microphone arrays used in the numerical experiment. Their details are listed in Table 1.

Table 1. Configurations of the microphone arrays used in the numerical experiment. (a)–(c) correspond to that of Fig. 1.

array	type	configuration
(a)	linear	Line array consisting of microphones placed in a alignment of interval 0.1 m whose length is 1.1 m.
(b)	planar	Flat square array consisting of microphones placed at the square lattice of interval 0.1 m.
(c)	sphere	Spherical array consisting of microphones placed at quasi-uniform sampling points [28] on the surface of a sphere whose radius is 0.1175 m.

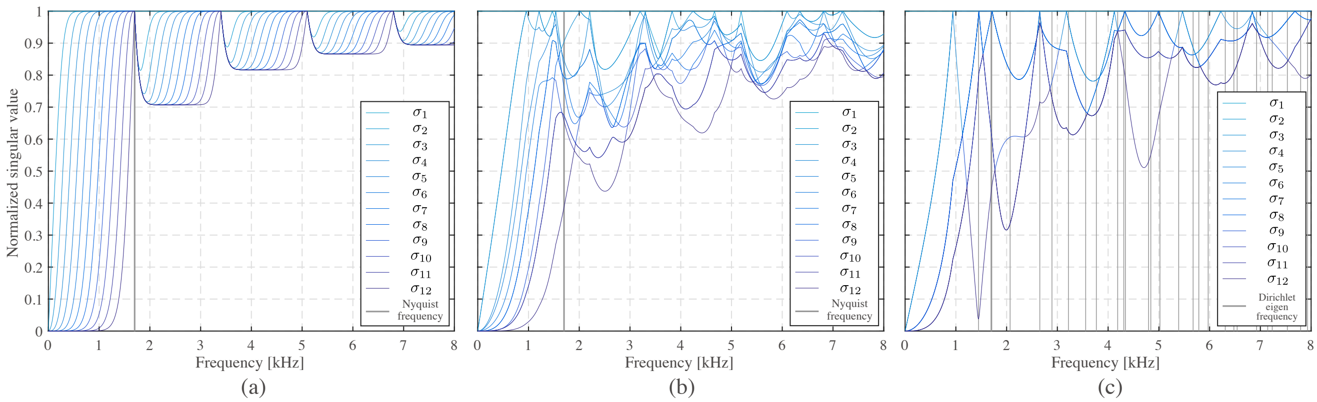

Fig. 2. Normalized singular values of the microphone arrays (a)–(c) in Fig. 1 and Table 1. The vertical lines of (a) and (b) show the spatial Nyquist frequency, and those of (c) represent the Dirichlet eigenfrequencies of the sphere on which microphones lie.

signal processing using an open spherical array [21]. Interestingly, some of those vertical lines agree with frequencies where some singular values have extrema. This agreement suggests that the singular values possibly indicate frequencies where an array is not suitable.

Figure 3 illustrates each right singular function $v_m(\boldsymbol{\nu})$ in Eq. (23) weighted by corresponding singular value σ_m for 1.5 kHz. Since the right singular function is a continuous function on the sphere, its magnitude is represented by the radius. These functions are basis elements of the infinite-dimensional space where α lies. The shape of them seems to indicate some symmetric properties of the arrays. For instance, as in Fig. 3 (a), rotational symmetry of the linear array can be seen from the right singular functions. The symmetry of the front and back side of the planar array is also apparent in Fig. 3 (b). Therefore, right singular functions should provide an alternative method for qualitative assessment of a microphone array.

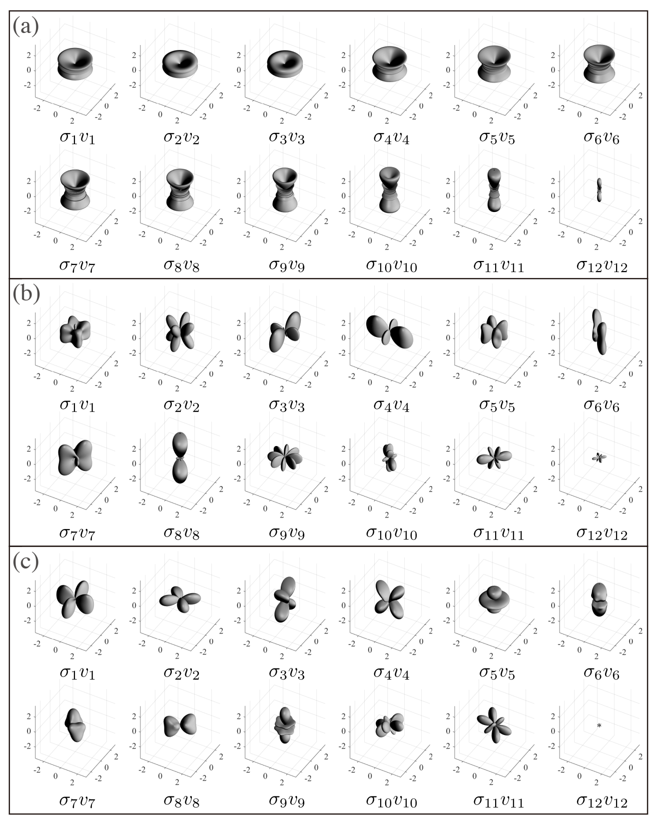

Fig. 3. Visual examples of the weighted right singular functions $\{\sigma_m v_m(\boldsymbol{\nu})\}$ of microphone arrays (a)–(c) for 1.5 kHz. Magnitude of each function is represented by the radius.

5. CONCLUSIONS

In this paper, we proposed the infinite-dimensional SVD for analyzing microphone arrays. Although such infinite-dimensional decomposition is not possible in general, the proposed method enables a simple and easy algorithm to compute it through analytic calculation of the integral. Some interesting properties of the proposed decomposition can be seen from the numerical examples. As is well known, SVD is a quite fundamental and important tool for both theory and application. Therefore, applying infinite-dimensional SVD to many practical problems of a microphone array, and other recent measurement techniques [29–31], is definitely the next step which should be investigated in the future.

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