PERSISTENT HOMOLOGY OF TOROIDAL SLIDING WINDOW EMBEDDINGS

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ABSTRACT

In this paper we study the persistent homology of sliding window embeddings of quasi-periodic signals. That is, functions which are sums of harmonics with different periods, and in particular, those having incommensurate frequencies. The resulting sliding window point-clouds, in this case, are (dense in) high-dimensional tori. We prove theorems which guide the choice of window size and embedding dimension, and describe the associated persistent homology.

Index Terms— Sliding window, delay embedding, incommensurate frequencies, persistent homology

1. INTRODUCTION

Time series are ubiquitous, so naturally their analysis is a fundamental object of study. To this end, many methods have been developed: Spectral representations, such as the Discrete Fourier and Wavelet transforms, have been successfully applied for many years to several problems. There are, however, instances where the spectrum is only part of the story and it is paramount to characterise the underlying dynamics.

Sliding window (or time-delay) embeddings yield – in generic conditions – a diffeomorphic reconstruction of the underlying state space [1]. The shape of these reconstructions often carries vital information regarding the structure of attractors (e.g. points vs. cycles) and their dimension, as well as that of connecting regions. The main point of this paper and other recent work, is that the shape of sliding window embeddings can be measured and leveraged for applications.

Indeed, given a collection of points (i.e. a point-cloud) in Euclidean space, Persistent Homology (a fundamental tool in topological data analysis) measures the number and nature of the holes of a geometric object approximating the cloud. Recently, the combination of sliding window embeddings and persistent homology has seen several applications. These include analysis of recurrent systems [2], identification of copy number aberrations in breast cancer [3], periodicity quantification in gene expression time series data [4], stability quantification in turning systems [5] and wheeze detection [6].

Understanding what it is that persistent homology of sliding window embeddings captures was a key element in the success of these approaches. Of particular relevance was the theoretical analysis for periodic functions undertaken in [7]. There, it was shown that periodicity was best captured in the sliding window embedding when: the embedding dimension is at least the number of dominant harmonics, and the window size approximates the function's period. In addition, it was shown that the salient features in the 1-dim persistence diagram are controlled by the function's Fourier coefficients.

Here we extend the results of [7] to functions which are sums of harmonics with incommensurate frequencies. These signals will not be periodic, but quasi-periodic, in that their sliding window embeddings are dense in high-dimensional tori. Quasi-periodicity appears naturally, for instance, in biphonation phenomena in mammals [8], as well as in the transition to chaos in rotating fluids [9]. We will prove theorems which guide the choice of time-delay and window size (Theorems 2.1 and 2.2), and also provide explicit bounds on the persistence homology for all dimensions (Theorem 2.8). While defining persistent homology (for the Rips filtration and coefficients on a field $\mathbb{F}$) is out of the scope of this paper, we refer the reader to [10] for a terse introduction.

2. MAIN RESULTS

We are interested in functions $f : \mathbb{R} \longrightarrow \mathbb{C}$ of the form

$$f(t) = \sum_{n=0}^{N} c_n e^{i\omega_n t}$$

where $N \in \mathbb{N}$, the c_n are non-zero complex numbers and the ω_n are **incommensurate** positive real numbers. This means that $1, \omega_0, \ldots, \omega_N$ are linearly independent over $\mathbb{Q}$. If $'$ denotes transpose, $M \in \mathbb{N}$ and $\tau > 0$, then

$$SW_{M,\tau} f(t) = \Big(f(t), f(t+\tau), \ldots, f(t+M\tau) \Big)' \in \mathbb{C}^{M+1}$$

is the **sliding window** embedding of f at t, with embedding dimension $M + 1$, delay τ and window size $M\tau$.

Let $x_f(t) = (c_0 e^{i\omega_0 t}, \ldots, c_N e^{i\omega_N t})'$, and for $c \in \mathbb{C}$ let $S_c^1 = \{z \in \mathbb{C} \; : \; |z| = |c|\}$. It follows from Kronecker's theorem [11] that the set $\mathbb{X}_f = \{x_f(k) : k \in \mathbb{Z}\}$ is dense in the $(N+1)$-torus $\mathbb{T}^{N+1} = S_{c_0}^1 \times \cdots \times S_{c_N}^1$. Moreover, if we let Ω_f be the $(M+1)$-by-$(N+1)$ matrix

$$\Omega_f[m+1, n+1] = e^{i\omega_n m\tau}, \quad 0 \leqslant m \leqslant M, \; 0 \leqslant n \leqslant N$$

then the equality $SW_{M,\tau} f(t) = \Omega_f \cdot x_f(t)$ ($\cdot$ denotes matrix multiplication) implies that $\{SW_{M,\tau} f(k) \; : \; k \in \mathbb{Z}\}$ will be dense in an $(N+1)$-torus whenever $rank(\Omega_f) = N+1$. The following theorem clarifies exactly when this happens.

Theorem 2.1. *If* $0 < \tau < \frac{2\pi}{\max\limits_n \{\omega_n\}}$ *then* Ω_f *is full-rank. If in addition* $M \geqslant N$*, then the **sliding window point cloud***

$$\mathbb{SW}_{M,\tau} f = \{SW_{M,\tau} f(k) \; : \; k \in \mathbb{Z}\}$$

is dense in a space homeomorphic to $\mathbb{T}^{N+1}$*.*

Proof. Let $L = \min\{M, N\}$ and let A be the upper-left $(L+1)$-by-$(L+1)$ block of Ω_f. Assume, by way of contradiction, that $\det(A) = 0$. Since in this case the rows of A are linearly dependent, there are $\rho_0, \ldots, \rho_L \in \mathbb{C}$ not all zero for which

$$\sum_{\ell=0}^{L} \rho_\ell e^{i\omega_n \ell \tau} = 0 \quad, \quad n = 0, \ldots, L.$$

In other words, each $\zeta_n = e^{i\omega_n \tau}$ is a root of the non-zero polynomial $p(z) = \rho_0 + \rho_1 z + \cdots + \rho_L z^L$. Moreover, given that the $\omega_n \tau$ are distinct and satisfy $0 < \omega_n \tau < 2\pi$, then $\zeta_0, \ldots, \zeta_L$ are distinct. It follows from the fundamental theorem of algebra that $L + 1 \leqslant \deg(p(z)) \leqslant L$, which is a contradiction. Thus A is invertible and $rank(\Omega_f) = L + 1$. $\square$

In the transition from $\mathbb{X}_f$ to $\mathbb{SW}_{M,\tau} f$ the toroidal geometry of $\mathbb{X}_f$ will be minimally perturbed when the columns of Ω_f are mutually perpendicular. If † denotes conjugate transpose, then the relevant inner products are the off-diagonal entries of $\Omega_f^\dagger \cdot \Omega_f$. Explicitly, $\Omega_f^\dagger \cdot \Omega_f$ is the $(N+1)$-by-$(N+1)$ matrix with diagonal terms equal to $M + 1$, and off-diagonal entries

$$\left(\Omega_f^\dagger \cdot \Omega_f\right)[k+1, l+1] = \frac{1 - e^{i(M+1)\tau(\omega_l - \omega_k)}}{1 - e^{i\tau(\omega_l - \omega_k)}}$$

One way of maximizing mutual perpendicularity is making each term $(M+1)\tau(\omega_l - \omega_k)$ as close as possible to an integer multiple of 2π. It is important to note that this can be done simultaneously and with arbitrary accuracy, by the Dirichlet approximation theorem [12], but requires loosing control of the embedding dimension in order to maintain $\tau < \frac{2\pi}{\max\limits_n \omega_n}$.

Instead, given $M \in \mathbb{N}$, we will choose $0 < \tau < \frac{2\pi}{\max\limits_n \{\omega_n\}}$ and integers $n_{k,l}$ which minimize

$$\sum_{0 \leqslant k < l \leqslant N} \left[(M+1)\tau(\omega_k - \omega_l) - 2\pi n_{k,l}\right]^2 \quad (1)$$

Indeed, let ω and η be the column vectors of dimension $N(N+1)/2$ whose coordinates are the differences $(\omega_k - \omega_l)$ and the integers $n_{k,l}$, respectively, for $0 \leqslant k < l \leqslant N$. A calculus argument shows that for each $\eta \in \mathbb{Z}^{N(N+1)/2}$ and $M \in \mathbb{N}$, the choice of τ which minimizes equation (1) is

$$\tau = \frac{2\pi \sum\limits_{k<l} (\omega_k - \omega_l) n_{k,l}}{(M+1) \sum\limits_{k<l} (\omega_k - \omega_l)^2} = \frac{2\pi\langle\eta, \omega\rangle}{(M+1)\|\omega\|^2} \quad (2)$$

As far as guaranteing $0 < \tau < \frac{2\pi}{\max\limits_n \{\omega_n\}}$, the Cauchy-Schwartz inequality implies that one should require

$$0 < \|\eta\| < \frac{(M+1)\|\omega\|}{\max\limits_n \{\omega_n\}}$$

Plugging in the value for τ from (2) into (1), and rewriting (1) in terms of ω and η, it follows that (1) is equivalent to

$$4\pi^2 \left\| \eta - \frac{\langle\eta, \omega\rangle}{\|\omega\|^2} \omega \right\|^2 \quad (3)$$

which is nothing but $(4\pi^2$ times) the square distance from η to the line spanned by ω. This analysis proves the following:

Theorem 2.2. *Let* $M \in \mathbb{N}$*, and let* ω *be the vector of differences* $(\omega_k - \omega_l)$*,* $0 \leqslant k < l \leqslant N$*. If* η *is the element in*

$$\left\{ \mathbf{n} \in \mathbb{Z}^{N(N+1)/2} \; : \; 0 < \|\mathbf{n}\| < \frac{(M+1)\|\omega\|}{\max\limits_n \{\omega_n\}} \right\}$$

which is closest to the line spanned by ω*, then* $\tau = \frac{2\pi\langle\eta, \omega\rangle}{(M+1)\|\omega\|^2}$ *satisfies* $0 < \tau < \frac{2\pi}{\max\limits_n \{\omega_n\}}$ *and* (τ, η) *minimizes (1).*

Now that we have settled on how to choose the parameters for the sliding window point-cloud, let us determine explicit bounds for its persistent homology. We begin with a technical result motivating the types of comparisons we will make. For $\epsilon > 0$ let $\mathbf{rk}_\epsilon(\mathsf{dgm})$ be the number of elements in the multiset $\{(\mu, \nu) \in \mathsf{dgm} \; : \; \mu = 0 \text{ and } \nu > \epsilon\}$.

Lemma 2.3. *Let* $\mathbf{K} = \{K_\epsilon\}_{\epsilon>0}$ *and* $\mathbf{K}' = \{K_\epsilon'\}_{\epsilon>0}$ *be filtered simplicial complexes, and let* $V_\epsilon = H_d(K_\epsilon; \mathbb{F})$ *and* $V_\epsilon' = H_d(K_\epsilon'; \mathbb{F})$ *be their* d*-dimensional homology with coefficients in a field* $\mathbb{F}$*. Let* $\gamma, \theta > 0$ *be so that* $\gamma\theta \geqslant 1$*. If there exist linear maps* $\phi : V_\epsilon \longrightarrow V_{\theta\epsilon}'$ *and* $\varphi : V_\epsilon' \longrightarrow V_{\gamma\epsilon}$ *so that*

$$\begin{array}{ccc} V_{\epsilon/\theta} & \longrightarrow & V_{\gamma\epsilon} \\ {\scriptstyle\phi}\downarrow & {\scriptstyle\varphi}\nearrow & \downarrow{\scriptstyle\phi} \\ V_\epsilon' & \longrightarrow & V_{\gamma\theta\epsilon}' \end{array}$$

is a commutative diagram for every $\epsilon > 0$*, then*

$$\mathbf{rk}_{\epsilon/\theta}\left(\mathsf{dgm}_d(\mathbf{K})\right) \geqslant \mathbf{rk}_\epsilon\left(\mathsf{dgm}_d(\mathbf{K}')\right) \geqslant \mathbf{rk}_{\gamma\epsilon}\left(\mathsf{dgm}_d(\mathbf{K})\right)$$

Proof. Since it is enough to check one inequality let us see that $\mathbf{rk}_{\epsilon/\theta}\big(\mathsf{dgm}_d(\mathbf{K})\big) \geqslant \mathbf{rk}_\epsilon\big(\mathsf{dgm}_d(\mathbf{K}')\big)$, and to that end let $k = \mathbf{rk}_\epsilon\big(\mathsf{dgm}_d(\mathbf{K}')\big)$. Then there is $\delta > \epsilon$ and exactly k independent classes $z_1, \ldots, z_k \in V'_\delta$ with zero birth-time and death-time greater than δ. It follows that there are classes $\widetilde{z}_1, \ldots, \widetilde{z}_k \in V'_{\delta/(\gamma\theta)}$ with zero birth-time, so that $\widetilde{z}_j$ is mapped to z_j through the homomorphism induced by $K'_{\delta/(\gamma\theta)} \hookrightarrow K'_\delta$. By commutativity of the diagram and since $z_1, \ldots, z_k$ represent distinct bars in $\mathsf{dgm}_d(\mathbf{K}')$, then $\varphi(\widetilde{z}_1), \ldots, \varphi(\widetilde{z}_k) \in V_{\delta/\theta}$ represent k distinct bars in $\mathsf{dgm}_d(\mathbf{K})$ with zero birth-time and death-time greater than $\delta/\theta > \epsilon/\theta$. $\qquad\square$

We now set the stage to apply Lemma 2.3. Since $\Omega_f^\dagger \cdot \Omega_f$ is a Hermitian positive-semi-definite matrix, it follows that its eigenvalues are real and non-negative. Moreover:

Proposition 2.4. *Let $\lambda_{\max}$ and $\lambda_{\min}$ be the maximum and minimum eigenvalues of $\Omega_f^\dagger \cdot \Omega_f$, respectively. Then*

$$\sqrt{\lambda_{\min}}\|\mathbf{z}\| \leqslant \|\Omega_f \cdot \mathbf{z}\| \leqslant \sqrt{\lambda_{\max}}\|\mathbf{z}\|, \quad \mathbf{z} \in \mathbb{C}^{N+1}$$

For $\epsilon > 0$ and $X \subset \mathbb{C}^{N+1}$ let $R_\epsilon(X)$ be the simplicial complex whose simplices are the non-empty finite subsets of X with diameter less than ϵ. $R_\epsilon(X)$ is called **the Rips complex** at scale ϵ, and it satisfies $R_\epsilon(X) \subseteq R_{\epsilon'}(X)$ whenever $\epsilon \leqslant \epsilon'$. That is, $\{R_\epsilon(X)\}_{\epsilon>0}$ is a filtered simplicial complex. The bound from Proposition 2.4 can now be used to relate the persistent homology of the Rips filtration $\{R_\epsilon(\mathbb{X}_f)\}_{\epsilon>0}$ to that of $\{R_\epsilon(\mathbb{SW}_{M,\tau}f)\}_{\epsilon>0}$:

Proposition 2.5. *Let $M \geqslant N$ and let $0 < \tau < \frac{2\pi}{\max_n\{\omega_n\}}$. Then Ω_f^{-1} (as a linear transformation) exists on $\mathbb{SW}_{M,\tau}f$, and Ω_f and Ω_f^{-1} induce simplicial maps making the diagram*

$$
\begin{array}{ccc}
R_{\epsilon/\sqrt{\lambda_{\max}}}(\mathbb{X}_f) & \hooklongrightarrow & R_{\epsilon/\sqrt{\lambda_{\min}}}(\mathbb{X}_f) \\
\Omega_f \downarrow & \nearrow^{\Omega_f^{-1}} & \downarrow \Omega_f \\
R_\epsilon(\mathbb{SW}_{M,\tau}f) & \hooklongrightarrow & R_{\epsilon\sqrt{\frac{\lambda_{\max}}{\lambda_{\min}}}}(\mathbb{SW}_{M,\tau}f)
\end{array}
$$

commute for every $\epsilon > 0$.

Proof. Theorem 2.1 implies that Ω_f^{-1} exists on $\mathbb{SW}_{M,\tau}f$. For $\epsilon > 0$ the right hand side inequality in Proposition 2.4 implies that if $[\mathbf{x}_0, \ldots, \mathbf{x}_d]$ is a d-simplex in $R_{\epsilon/\sqrt{\lambda_{\max}}}(\mathbb{X}_f)$, then $[\Omega_f\mathbf{x}_0, \ldots, \Omega_f\mathbf{x}_d]$ is a d-simplex in $R_\epsilon(\mathbb{SW}_{M,\tau})$. A similar argument applies to Ω_f^{-1} and we obtain the result. $\quad\square$

We will now study the persistence of $\{R_\epsilon(\mathbb{X}_f)\}_{\epsilon>0}$, which will be combined with Proposition 2.5 to describe the persistent homology of $\{R_\epsilon(\mathbb{SW}_{M,\tau}f)\}_{\epsilon>0}$. We begin by letting $n = 0, \ldots, N$ and considering the projection map

$$
p_n : \begin{array}{ccc} \mathbb{X}_f & \longrightarrow & \mathbb{S}^1_{c_n} \\ x_f(k) & \mapsto & c_n e^{i\omega_n k} \end{array}
$$

where $\mathbb{S}^1_{c_n} = \{c_n e^{i\omega_n k} : k \in \mathbb{Z}\}$. Since p_n is distance non-increasing, it follows that it extends to a simplicial map

$$p_n : R_\epsilon(\mathbb{X}_f) \longrightarrow R_\epsilon(\mathbb{S}^1_{c_n})$$

Using cartesian products of ordered simplicial complexes [13] the p_n's can be combined into $p : R_\epsilon(\mathbb{X}_f) \longrightarrow \overset{N}{\underset{n=0}{\times}} R_\epsilon(\mathbb{S}^1_{c_n})$. The partial order $\prec$ on the vertices of $R_\epsilon(\mathbb{S}^1_{c_n})$ is defined by the rule $c_n e^{i\omega_n k} \prec c_n e^{i\omega_n k'}$ if and only if $k \leqslant k'$, for $k, k' \in \mathbb{Z}$. As we vary ϵ, it follows that p yields a map of filtered simplicial complexes. Hence the persistent homology of $\{R_\epsilon(\mathbb{X}_f)\}_{\epsilon>0}$ can be bounded by that of $\left\{\overset{N}{\underset{n=0}{\times}} R_\epsilon(\mathbb{S}^1_n)\right\}_{\epsilon>0}$:

Proposition 2.6. *For each $\epsilon > 0$ there is a homomorphism*

$$\varphi_\epsilon : H_d\left(\overset{N}{\underset{n=0}{\times}} R_\epsilon(\mathbb{S}^1_{c_n}); \mathbb{F}\right) \longrightarrow H_d\big(R_{\epsilon\sqrt{N+1}}(\mathbb{X}_f); \mathbb{F}\big)$$

between d-dimensional homology with coefficients in a field $\mathbb{F}$, which makes the following diagram commute:

$$
\begin{array}{ccc}
H_d\big(R_\epsilon(\mathbb{X}_f); \mathbb{F}\big) & \longrightarrow & H_d\big(R_{\epsilon\sqrt{N+1}}(\mathbb{X}_f); \mathbb{F}\big) \\
p_* \downarrow & \nearrow^{\varphi_\epsilon} & \downarrow p_* \\
H_d\left(\overset{N}{\underset{n=0}{\times}} R_\epsilon(\mathbb{S}^1_{c_n}); \mathbb{F}\right) & \longrightarrow & H_d\left(\overset{N}{\underset{n=0}{\times}} R_{\epsilon\sqrt{N+1}}(\mathbb{S}^1_{c_n}); \mathbb{F}\right)
\end{array}
$$

Proof. Let σ be a d-simplex in $\overset{N}{\underset{n=0}{\times}} R_\epsilon(\mathbb{S}^1_{c_n})$. This means that $\sigma = [\mathbf{x}_0, \ldots, \mathbf{x}_d]$ for some collection of vertices

$$\mathbf{x}_j = (c_0 e^{i\omega_0 k_j^0}, \ldots, c_N e^{i\omega_N k_j^N}), \quad j = 0, \ldots, d$$

where $k_j^n \in \mathbb{Z}$, $n = 0, \ldots, N$, satisfying the following two conditions: First, $k_j^n \leqslant k_\ell^n$ for all $n = 0, \ldots, d$ and every $0 \leqslant j \leqslant \ell \leqslant d$; and second, that for each $n = 0, \ldots, N$ and after deleting repetitions, $[c_n e^{i\omega_n k_0^n}, \ldots, c_n e^{i\omega_n k_d^n}]$ is a d_n-simplex in $R_\epsilon(\mathbb{S}^1_{c_n})$ for some $0 \leqslant d_n \leqslant d$. In particular, this implies that $\|\mathbf{x}_j - \mathbf{x}_\ell\| < \epsilon\sqrt{N+1}$ for $0 \leqslant j, \ell \leqslant d$.

Let $\delta > 0$ be so that

$$\max_{n,j,\ell}\left(|c_n| \cdot \left|e^{i\omega_n k_j^n} - e^{i\omega_n k_\ell^n}\right|\right) + 2\delta < \epsilon$$

Since $\mathbb{X}_f$ is dense in $\mathbb{T}^{N+1}$, there exist integers $k_0 < \cdots < k_d$ so that $k_0 \geqslant \max_{j,n}\{k_j^n\}$ and $\|\mathbf{x}_j - x_f(k_j)\| < \delta/2$, for all $j = 0, \ldots, d$. Then for each $0 \leqslant n \leqslant N$ and $0 \leqslant j, \ell \leqslant d$

$$
\begin{aligned}
|c_n e^{i\omega_n k_j} - c_n e^{i\omega_n k_\ell}| &\leqslant \delta + |c_n e^{i\omega_n k_j^n} - c_n e^{i\omega_n k_\ell^n}| \\
&< \epsilon
\end{aligned}
$$

and we have that $\sigma' = [x_f(k_0), \ldots, x_f(k_d)]$ is a d-simplex in both $\overset{N}{\underset{n=0}{\times}} R_\epsilon(\mathbb{S}^1_{c_n})$ and $R_{\epsilon\sqrt{N+1}}(\mathbb{X}_f)$. Moreover, if we

join each edge $[\mathbf{x}_j, x_f(x_j)]$ it follows that σ and σ' are the base and top face, respectively, of a $(d+1)$-dimensional prism of height $\delta/2$. This prism can be subdivided (as in the proof of 2.1 in [14]) into $(d+1)$-simplices in $\underset{n=0}{\overset{N}{\times}} R_\epsilon(\mathbb{S}^1_{c_n})$, which shows that every class $[z] \in H_d\left(\underset{n=0}{\overset{N}{\times}} R_\epsilon(\mathbb{S}^1_{c_n}); \mathbb{F}\right)$ can be represented by a d-cycle z_0 with simplices of the form $[x_f(k_0), \ldots, x_f(k_d)] \in R_{\epsilon\sqrt{N+1}}(\mathbb{X}_f)$.

A different choice of integers with the same properties, results in a chain z_0' so that $z_0 - z_0'$ is equal, through prisms in $R_{\epsilon\sqrt{N+1}}(\mathbb{X}_f)$ of height δ, to the boundary of a $(d+1)$-chain. Hence they determine the same homology class $\varphi_\epsilon([z]) = [z_0] \in H_d(R_{\epsilon\sqrt{N+1}}(\mathbb{X}_f); \mathbb{F})$. $\qquad\square$

The persistent homology of $\left\{ \underset{n=0}{\overset{N}{\times}} R_\epsilon(\mathbb{S}^1_n) \right\}_{\epsilon>0}$ can be determined explicitly: For each $\epsilon > 0$ the Künneth theorem [14] implies that the (simplicial) cross product $\times$ induces an isomorphism

$$\bigoplus_{\sum_n d_n = d} \left(\bigotimes_{n=0}^{N} H_{d_n}\left(R_\epsilon(\mathbb{S}^1_{c_n}); \mathbb{F}\right) \right) \cong H_d\left(\underset{n=0}{\overset{N}{\times}} R_\epsilon(\mathbb{S}^1_{c_n}); \mathbb{F} \right)$$

which is natural with respect to simplicial maps. That is, the persistence diagrams from the left-hand-side equal those of the right. A theorem of Adams and Adamaszek [15] describes the persistent homology contribution of each term

$$\left\{ H_{d_n}\left(R_\epsilon(\mathbb{S}^1_{c_n}); \mathbb{F}\right) \right\}_{\epsilon>0}, \quad n = 0, \ldots, N.$$

Explicitly, the d_n-dimensional barcode for $\left\{R_\epsilon(\mathbb{S}^1_{c_n})\right\}_{\epsilon>0}$ has a single interval

$$\left(2|c_n| \sin\left(\pi\frac{\ell}{2\ell+1} \right), \, 2|c_n| \sin\left(\pi\frac{\ell+1}{2\ell+3} \right) \right]$$

in each dimension $d_n = 2\ell + 1$, for $\ell > 0$. Moreover, the only intervals starting at zero appear in dimension 1 and are of the form $[0, \sqrt{3}|c_n|]$. Let χ_n be the indicator function for this interval; that is, constant and equal to 1 inside and 0 outside.

Proposition 2.7. *If dgm_d is the d-dimensional persistence diagram of the filtered complex $\left\{ \underset{n=0}{\overset{N}{\times}} R_\epsilon(\mathbb{S}^1_{c_n}) \right\}_{\epsilon>0}$ then*

$$\mathbf{rk}_\epsilon(\mathsf{dgm}_d) = \binom{\sum_{n=0}^{N} \chi_n(\epsilon)}{d}$$

Proof. $\left\{ \bigotimes_{n=0}^{N} H_{d_n}\left(R_\epsilon(\mathbb{S}^1_{c_n}); \mathbb{F}\right) \right\}_{\epsilon>0}$ has an interval starting at zero if and only if each d_n is either 0 or 1. Hence d is the number of 1-dimensional homology groups in the tensor

product. For each $\epsilon > 0$, $\sum_{n=0}^{N} \chi_n(\epsilon)$ is the number of $|c_n|$'s so that $\epsilon < \sqrt{3}|c_n|$. It follows that the binomial coefficient is exactly the number of ways in which an interval starting at zero and with death-time greater than ϵ can be constructed. $\qquad\square$

Everything now comes together in the following theorem:

Theorem 2.8. *Let $M \geqslant N$ and let τ be as in Thm 2.2. Then*

$$\binom{\sum_{n=0}^{n} \chi_n(\epsilon/\sqrt{\lambda_{\min}})}{d} \leqslant \mathbf{rk}_\epsilon\left(\mathsf{dgm}_d(\mathbb{SW}_{M,\tau}f)\right)$$
$$\leqslant \binom{\sum_{n=0}^{n} \chi_n(\epsilon/\sqrt{(N+1)\lambda_{\max}})}{d}$$

Proof. This follows by applying Lemma 2.3 twice. First, with $K_\epsilon = R_\epsilon(\mathbb{X}_f)$, $K'_\epsilon = R_\epsilon(\mathbb{SW}_{M,\tau}f)$, $\phi = \Omega_{f_*}$, $\varphi = \Omega_{f_*}^{-1}$, $\theta = \sqrt{\lambda_{\max}}$ and $\gamma = \frac{1}{\sqrt{\lambda_{\min}}}$, as in Proposition 2.5. Then

$$\mathbf{rk}_{\epsilon/\sqrt{\lambda_{\min}}}\left(\mathsf{dgm}_d(\mathbb{X}_f)\right) \leqslant \mathbf{rk}_\epsilon\left(\mathsf{dgm}_d(\mathbb{SW}_{M,\tau}f)\right)$$
$$\leqslant \mathbf{rk}_{\epsilon/\sqrt{\lambda_{\max}}}\left(\mathsf{dgm}_d(\mathbb{X}_f)\right)$$

The lower and upper bounds for $\mathbb{X}_f$ follow from applying Lemma 2.3 a second time. One now uses $\phi = p_*$, $\theta = 1$, $\varphi = \varphi_\epsilon$ and $\gamma = \sqrt{N+1}$ as in Proposition 2.6. The explicit expressions with binomial coefficients is Proposition 2.7. $\quad\square$

A direct calculation using [16] allows one to bound $\lambda_{\max}$ and $\lambda_{\min}$ in terms of the spectrum of f and the parameters for the sliding window embedding:

Proposition 2.9. *If F_{M+1} is the $(M+1)$-th Fejér kernel and*

$$\Psi = \sqrt{\sum_{k,l=0}^{N} \frac{F_{M+1}\left(\tau(\omega_k - \omega_l)\right)}{(N+1)(M+1)} - 1}$$

then

$$1 - \sqrt{N}\Psi \leqslant \frac{\lambda_{\min}}{M+1} \leqslant 1 - \frac{1}{\sqrt{N}}\Psi \qquad (4)$$

$$1 + \frac{1}{\sqrt{N}}\Psi \leqslant \frac{\lambda_{\max}}{M+1} \leqslant 1 + \sqrt{N}\Psi \qquad (5)$$

Remark 2.10. Replacing $\lambda_{\min}$ (resp. $\lambda_{\max}$) in Theorem 2.8 by the lower (resp. upper) bound from Proposition 2.9, yields explicit bounds for the persistent homology of $\mathbb{SW}_{M,\tau}f$ in terms of M, τ, N the $|c_n|$'s and the ω_n's.

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