A SPARSE-GRAPH-CODED FILTER BANK APPROACH TO MINIMUM-RATE SPECTRUM-BLIND SAMPLING

Orhan Öçal, Xiao Li and Kannan Ramchandran

Electrical Engineering and Computer Sciences
University of California, Berkeley, USA
{ocal, xiaoli, kannanr}@eecs.berkeley.edu

ABSTRACT

Sampling of bandlimited signals whose frequency support is unknown is called *spectrum-blind* sampling. It has attracted considerable attention due to its potential for sampling much lower than the Nyquist rate. The *minimum rate* for spectrum-blind sampling has been established as twice the measure of the frequency support. We study this sampling problem and propose a novel sampling framework by leveraging tools from *modern coding theory*. Our approach is based on subsampling the outputs of a carefully designed *sparse-graph-coded filter bank*. The key idea is to exploit, rather than avoid, the aliasing artifacts induced by subsampling, which introduces linear mixing of spectral components in the form of parity constraints for *sparse-graph codes*. Under the proposed sampling scheme, signal reconstruction becomes equivalent to the peeling decoding of sparse-graph codes in erasure channels. As a result, we can *simultaneously* approach the *minimum sampling rate*, while also having a *computational cost that is linear in the number of samples*. We support our theoretical findings through numerical experiments.

Index Terms— Spectrum-blind sampling, sparse-graph codes, peeling decoder.

1. INTRODUCTION

Sampling theory fundamentally connects the analog and digital worlds. In particular, the classic Shannon-Nyquist sampling theorem provides the foundation for digital signal processing through analog-to-digital conversion (ADC). The theorem states that any bandlimited signal can be reconstructed from its point-wise samples taken at or above the Nyquist rate of the signal. Advances in signal processing and harmonic analysis have contributed significantly to the field of sampling theory, leading to innovative sampling methods [1, 2, 3].

Recently, *spectrum-blind* sampling has attracted considerable attention, mostly because of possible rate reductions over the Nyquist rate when the signal spectrum is sparse but its frequency support is *not* known *a priori*. This important class of bandlimited signals is commonly referred to as *multiband signals*, which consist of a union of continuous spectral bands spread across a wide spectrum. There have been prolific results on developing theories and algorithms for sampling this class of signals. As shown in the seminal work by Landau in 1960's [4], any bandlimited signal with an *arbitrary but known* frequency support can be recovered from samples taken at a rate equal to the measure of its frequency support, also known as the *Landau rate*. Interestingly, it was later pointed out in [5] that spectrum-blind sampling simply requires a minimum sampling rate at *twice the Landau rate*.

In order to constructively approach this minimum rate, the authors of [6, 7] showed that using multi-coset sampling, the signal can be reconstructed from the discrete time Fourier transform (DTFT) of the samples if the cosets are chosen carefully. Another approach was proposed by the authors of [8, 9] where the problem of spectrum-blind sampling was transformed to a compressed sensing problem through modulation by random periodic functions and low pass filtering. However, the proposed scheme incurs an extra logarithmic factor away from the minimum rate.

In this work, we study this problem from a radically different viewpoint based on *modern coding theory*. This seems at first glance unrelated to the sampling problem. Connections between sampling theory and coding theory have been unexplored in the literature. Our goal is to shed new light on this fundamental connection. Our approach to the sampling problem is based on the use of *sparse-graph codes* which have revolutionized the design of modern communication systems. This allows us to derive a novel sampling framework that reflects interesting connections between the fundamental problem of *minimum-rate spectrum-blind sampling*, and the classic problem of designing *capacity-achieving* sparse-graph codes, such as Low Density Parity Check (LDPC) codes [10, 11].

Our key contribution is the proposal of a new and novel multichannel *sparse-graph-coded* filter bank (cf., Section 2). Our results show that any multiband signal can be sampled asymptotically at the minimum rate in a probabilistic setting, where the probability of reconstruction failure goes to zero. *To the best of our knowledge, this is the first constructive scheme that achieves the minimum rate for spectrum-blind sampling, and also admits a fast reconstruction algorithm.*

1.1. Organization and Notation

The rest of this paper is organized as follows. In Section 2, we state the problem and describe the underlying ideas leading to our results and present the multiband filter bank sampling architecture. In Section 3, we describe our design of sparse-graph coded filter banks that leads to minimum rate sampling of wideband signals, and describe an implementation for realizing such filters. We present empirical validation of our results and conclude in Section 4.

Functions referred to by upper case letters are in Fourier domain, e.g., $X(f) = \int_{-\infty}^{\infty} x(t)\exp(-j2\pi ft)\ \mathrm{d}t$ is the Fourier transform of $x(t)$. Furthermore, $x(t) \overset{\mathcal{F}}{\longleftrightarrow} X(f)$ denotes Fourier transform pairs. Lower case bold letters denote vectors, e.g., $\mathbf{x}$. Upper case bold letters denote matrices, e.g., $\mathbf{A}$, and the ith column of a matrix $\mathbf{A}$ is denoted as $\mathbf{a}_i$. We define the parametrized family of one-sided

This work was supported in part by the grants: NSF CCF EAGER 1439725, AFOSR MURI FA9550-10-1-0567.

boxcar functions $\Pi_B(f) = 1$ if $0 \le f < B$ and 0 otherwise.

2. MAIN IDEA AND FORMULATION

By $\mathrm{BandLimited}(f_{\max}, f_L)$ we denote complex-valued signals $x(t) \in \mathbb{C}$ whose frequency spectrum is zero outside $[0, f_{\max})$, i.e., $X(f) = 0$ if $f < 0$ or $f \ge f_{\max}$, and its frequency support $\mathcal{F}$ has Lebesgue measure $|\mathcal{F}| = f_L$ and is of the form of the union of a finite number of intervals with unknown locations. The choice of the lower band limit 0 is without loss of generality because any given signal having known frequency limits $(f_{\min}, f_{\max})$ can be modulated to such a band. Note that the minimum rate for blind sampling and reconstruction for $\mathrm{BandLimited}(f_{\max}, f_L)$ is $\min(2f_L, f_{\max})$ [5].

Choose a positive integer N and divide the interval $[0, f_{\max})$ into N intervals of equal width $B = f_{\max}/N$. We can write any signal bandlimited to $[0, f_{\max})$ as a sum

$$x(t) = \sum_{k=0}^{N-1} x_k(t) e^{j2\pi kBt}, \tag{1}$$

where each $x_k(t) \in \mathbb{C}$ is bandlimited to $[0, B)$. Because this is a universal representation, $x(t)$ can be reconstructed if each $x_k(t)$ is recovered, and as each $x_k(t)$ is bandlimited to $[0, B)$, it can be recovered from its samples taken at rate B, i.e., $x_k[n] = x_k(n/B)$.

In the case where the signal is from $\mathrm{BandLimited}(f_{\max}, f_L)$ with $f_L \ll f_{\max}$, the number of active components in (1) is significantly less than N as the frequency spectrum is sliced finely. Note that, as the slicing becomes finer, namely, as N grows, we have the ratio of the active components in (1) to N approach the frequency occupancy ratio, that is, $\lim_{N\to\infty} |\mathsf{supp}\,(\mathbf{x}[n])|/N = f_L/f_{\max}$, where $\mathbf{x}[n] \in \mathbb{C}^N$ is the vector having its kth component equal to $x_k[n]$.

2.1. Sampling Framework and Design Objective

Our sampling framework is summarized in Fig. 1 as a conceptual representation, and a more practical implementation is described in Section 3.4. A multiband signal is passed through a filter bank consisting of M filters with frequency response $H_i(f)$ for $i = 1, \cdots, M$. Filter bank outputs are sampled at B Hz to produce M streams of samples $\{y_i[n]\}_{n\in\mathbb{Z}}^{i=1,\cdots,M}$, which are then input to a non-linear reconstruction scheme. Note that with this sampling scheme the aggregate sampling rate is $f_s = MB$.

The goal is to design a blind, lossless sampling and reconstruction scheme that achieves a sampling rate approaching the minimum rate and perfectly recovers the signal (with probability approaching one). Given a reconstructed signal $\widehat{x}(t)$, the performance is characterized by the triplet $(f_s, T, \mathbb{P}_F)$, where f_s is the sampling

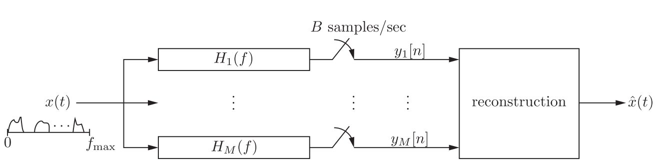


Fig. 1: Proposed spectrum-blind sampling framework consists of a sampling front-end and a reconstruction back-end. The front-end takes sub-Nyquist samples of a carefully designed filter bank, and the back-end performs decoding. Note that this is a conceptual representation (please refer to Section 3.4 for details).

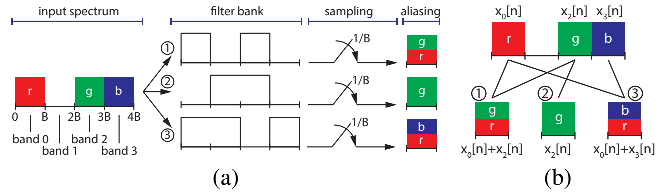


Fig. 2: (a) The aliased spectra of the output samples in different sampling channels and the original spectrum. Each multiband sampling filter passes a controlled subset of bands of the signal, and subsampling introduces aliasing of these components, that is, introduces a linear combination of the signals on the selected bands (cf., Section 2.2). (b) Spectral aliasing induces a *parity check* constraints for each sampling instant. The reconstruction is equivalent to finding the sample values at the bands that satisfy the parity check constraints. Letters r, g and b denote red, green and blue colors respectively.

rate, T is the reconstruction complexity in terms of arithmetic operations per unit of time, and $\mathbb{P}_F$ is the failure probability $\mathbb{P}_F = \mathsf{Pr}\,(\widehat{x}(t) \ne x(t))$. The probability $\mathbb{P}_F$ is evaluated with respect to the randomness associated with the sampling front-end design. In other words, for a given multiband signal $x(t)$, our design generates a random sampling front-end from a specific random ensemble and produces an estimate $\widehat{x}(t)$ with probability $1 - \mathbb{P}_F$ approaching one asymptotically. Some questions of interest are: (1) How to design the filter response $H_i(f)$ and how to choose the sampling rate B? (2) How many filters (i.e., M) are necessary? (3) How are the design parameters B, M related to the minimum rate? (4) How to reconstruct the signal given the filter bank output samples $\{y_i[n]\}_{n\in\mathbb{Z}}^{i=1,\cdots,M}$?

2.2. Design Philosophy Through an Example

Before answering these questions, we first explain our design philosophy through a simple example with reference to Fig. 2a, where we consider a signal from $\mathrm{BandLimited}(4B, 3B)$ with spectrum supported on the three bands each of width B. The support bands 0, 2 and 3 are color-coded in *red, green* and *blue* respectively.

For sampling, we use a filter bank with $M = 3$ channels, where each filter has a frequency response $H_i(f)$ taking a *multiband* structure, where each band is also of width B. The filters have unity gain $H_i(f) = 1$ in some bands and $H_i(f) = 0$ elsewhere. For example, in Fig. 2a, the first filter passes bands 0 and 2, the second filter passes bands 1 and 2, and the third filter passes bands 0, 1 and 3. Conforming with the multiband structure of the sampling filter, the associated filter output is also (sub)-sampled at rate B. Clearly, sampling at rate B results in *aliasing* of the output spectrum because the rate is below the Nyquist rate $4B$. Due to the variations in the multiband structure of sampling filters, each output stream of samples is a different linear combination of signal components in a controlled selection of bands 0 to 3. For example, stream 1 is a linear combination of signals in bands 0 and 2 (red and green), and stream 2 is equal to the signal in band 2 (green) since there is no signal in band 1.

Later in this paper, it will become clear that the design of these filters is guided by coding patterns derived from *sparse-graph codes* that create careful aliasing (linear mixing) of the spectral components. We highlight such mixing, as Fig. 2b, in the form of sparse-graph codes, where each color mixture corresponds to a *parity check* constraint of the codeword for each sampling snapshot n.

This sparse bipartite graph structure allows the decoder to recover the signal in each frequency band through a *fast peeling* operation on the edges of the graph. We describe the peeling process in our running example of Fig. 2b. If the decoder knows that a certain stream of samples is of a single color, it can peel off its contribution from other streams of samples, hopefully uncovering new single color streams. This is similar in spirit to the idea of a "peeling decoder" in packet communication systems. For clarity of presentation, let us first assume that we have an oracle that informs the decoder if an output stream comprises of a single band or not, and if so, which band it comes from (we will later show how to get rid of this oracle). Given this oracle, we perform the following:

- The second stream $y_2[n]$ is composed of a single component coming from band 2 (the aliasing leads to the combination of the signal at bands 1 and 2, but the band 1 has no signal), hence we recover the samples $y_2[n]$ of the spectral component at band 2. Because the component at band 2 contributes to the first stream $y_1[n]$, we peel off the contribution of $y_2[n]$ from the first stream $y_1[n]$;
- The new first stream after the peeling becomes $y_1'[n] = y_1[n] - y_2[n]$, which is from band 0 based on the given mechanism. We then peel $y_1'[n]$ from stream $y_3[n]$, which leads to a single component coming from band 3;
- By peeling stream $y_3[n]$, the entire signal is recovered.

2.3. Generalized Sampling Architecture and Observation Model

To leverage the sparsity of the active components in order to reduce the sampling rate, define M multiband filters, $h_i(t)$, where each filter has frequency response a weighted combination of ideal bandpass filters, i.e., $H_i(f) = \sum_{k=0}^{N-1} c_{i,k}\Pi_B(f - kB)$, where $c_{i,k} \in \mathbb{C}$.

Under this design, M channels, each sampled at rate B samples/second, result in a total sampling rate $f_s = MB$ samples/second. We want to design the filter bank such that as the number of slices N grows, f_s approaches the minimum rate $2f_L$. The resulting M samples output from the filter bank at each sampling instant $n \in \mathbb{Z}$ can be described as

$$y_i[n] = \sum_{k=0}^{N-1} c_{i,k} x_k[n]. \tag{2}$$

The multiple measurements obtained through the use of M different $h_i(t)$ can be represented for each sampling instant n as

$$\mathbf{y}[n] = \mathbf{C}\mathbf{x}[n], \tag{3}$$

where $\mathbf{C} \in \mathbb{C}^{M\times N}$.

The problem then boils down to choosing a "good" $\mathbf{C}$ matrix that enables recovery of $\mathbf{x}[n]$ from $\mathbf{y}[n]$. When the frequency support is sparse, the vector $\mathbf{x}[n]$ is going to be sparse for each n. This sparse recovery problem has been studied extensively in compressed sensing where there is prolific literature with various measurement scaling results for M [3]. A typical construction is to choose $\mathbf{C}$ from some random matrix ensemble. For example, the authors of [8] have proposed using such randomized constructions that require oversampling with a logarithmic factor $\log(N)$ with respect to the number of slices, which does *not* achieve the minimum rate. Another approach is to design $\mathbf{C}$ with a Vandermonde structure, where, by using Prony's method, one can recover the signal using samples twice the number of non-zero entries in $\mathbf{x}[n]$ [12]. However, Prony's method requires solving for the roots of a polynomial. As the number of slices increases, numerical precision of the roots needs to scale with N, increasing the computational complexity and the numerical instability. To approach the minimum sampling rate together with low computational complexity, we use our framework developed in [13] based on sparse-graph codes which circumvents the above issues.

3. SPARSE-GRAPH-CODED FILTER BANK DESIGN

The compressed sensing formulation in (3) captures the filter bank architecture by the $M \times N$ coefficient matrix $\mathbf{C}$, and the goal is to recover the baseband samples $\mathbf{x}[n]$ in each band from the filter bank outputs $\mathbf{y}[n]$. More specifically, each row of the matrix $\mathbf{C}$ characterizes the frequency response of the sampling filter in each channel of the filter bank. In the following, we discuss how to design the frequency response of the filter bank.

3.1. Filter Bank Response

We specify the filter bank frequency response in terms of its *magnitude* and *phase response* over N bands.

In our sampling framework, the *magnitude response* is sparse-graph encoded to introduce peeling-friendly aliasing. We specify R magnitude responses for the sampling filters through an $R \times N$ matrix $\mathbf{H}$ that is equal to the adjacency matrix of a bipartite graph $\mathcal{G}$ consisting of N left nodes and R right nodes. The left nodes correspond to the N frequency slices, and the right nodes correspond to the channels of the filter bank. In Section 3.2, we describe two graph ensembles for choosing $\mathbf{H}$ that results in peeling-friendly aliasing.

Based on each magnitude response (row of $\mathbf{H}$), we design P sampling filters that have the same magnitude response but different *phase responses*. The P phase responses are appropriately chosen in order to determine from its output if it contains the samples of only one frequency slice and further determine which frequency slice it is. We represent P phase responses with a $P \times N$ matrix $\mathbf{S}$ such that there are a total of $M = RP$ sampling channels. The concrete design of $\mathbf{S}$ is given in Section 3.3.

Mathematically, given a *magnitude response matrix* $\mathbf{H}$ and a *phase response matrix* $\mathbf{S}$, the response of the filter bank constructed as above is given by the matrix

$$\mathbf{C} = \mathbf{H} \boxtimes \mathbf{S}, \tag{4}$$

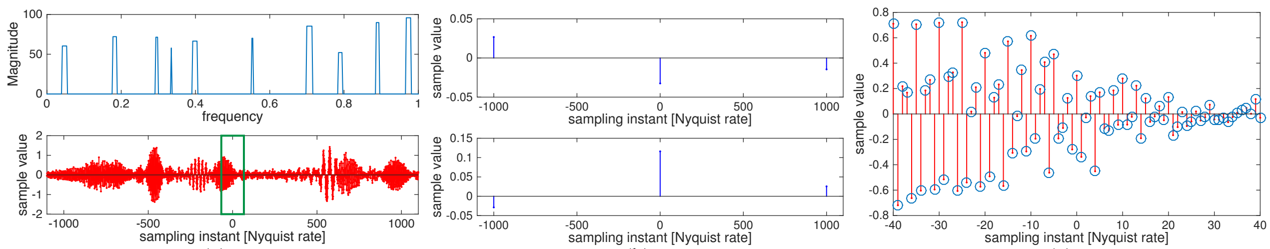
where $\boxtimes$ is the Khatri-Rao product defined as $\mathbf{H} \boxtimes \mathbf{S} = [\mathbf{h}_1 \otimes \mathbf{s}_1 \ \cdots \ \mathbf{h}_N \otimes \mathbf{s}_N]$, and $\otimes$ is the Kronecker product. Next we formally define our filter bank ensemble using this construction.

3.2. Oracle-based Reconstruction with Blind Sampling

We first present the design of the magnitude response matrix $\mathbf{H}$ of our sampling filter bank for blind sampling with the help of a single-ton oracle during reconstruction (as in the example in Section 2.2, the oracle tells the decoder the band location and sample value if an output stream comprises of a single band). We will then discuss in Section 3.3 how to design and use the phase response matrix $\mathbf{S}$ to complete our sampling framework without using the single-ton oracle. We define the *regular graph ensemble* for designing $\mathbf{H}$ [14].

Definition 1 (Regular graph ensemble)**.** *Given N left nodes, R right nodes and an integer $d \geq 2$, regular graph ensemble $\mathcal{G}_{reg}^N(R, d)$ is the set of all graphs where each left node is connected to d right nodes.*

Theorem 1. *Given the filter bank ensemble designed by regular graph ensemble $\mathcal{G}_{reg}^N(R, d)$, with the help of a single-ton oracle in reconstruction, any signal $x(t) \in \mathrm{BandLimited}(f_{\max}, f_L)$ can be sampled at rate $f_s = 1.23 f_L$ and perfectly reconstructed with probability at least $1 - O\left((f_{\max}/f_L)/N\right)$ by performing $O(f_L)$ arithmetic operations per unit time.*



(a) (b) (c)

Fig. 3: (a) Input signal's magnitude spectrum (top) and Nyquist rate samples (bottom). (b) Output samples of two channels of the sparse-graph-coded filter bank, sampled at rate 10^{-3}. (c) Nyquist rate samples corresponding the region in the green box in (a); red lines correspond to the Nyquist rate samples of the input signal, and the blue circles correspond to the reconstruction using our proposed sampling framework.

The proof of this theorem is omitted for length considerations, and it can be found in [15]. The theorem states that, assuming a single-ton oracle in reconstruction, a multiband signal can be sampled a constant 1.23 factor away from the Landau rate. To lower this constant factor, the filter bank can be constructed based on the following graph ensemble.

Definition 2 (Irregular graph ensemble)**.** *Given N left nodes, R right nodes and an integer $D \geq 2$, the edge set in the irregular graph ensemble $\mathcal{G}^N_{\text{irreg}}(R, D)$ is characterized by the degree sequence $\lambda_j = 1/\left(L(D)(j-1)\right)$ for $j = 2, \cdots, D+1$, where λ_j denotes the fraction of edges that connect to a left node with degree j and $L(D) = \sum_{j=1}^{D} 1/j$ is for normalization, i.e., to have $\sum_{j\geq 2} \lambda_j = 1$.*

Filter bank designed based on $\mathcal{G}^N_{\text{irreg}}(R, D)$ allows sampling at rate $f_s = (1+\epsilon)f_L$ for any $\epsilon > 0$, while the probability of perfect reconstruction is at least $1 - O((f_{\max}/f_L)/N)$. Arithmetic operations per unit time is still $O(f_L)$. However, design through irregular graph ensemble requires large N. Hence, we will use the regular graph ensemble for empirical validations.

3.3. Blind Reconstruction with Blind Sampling

We now describe the design and use of the phase response $\mathbf{S}$ of the filter bank to replace the oracle in reconstruction. Choose the $2 \times N$ phase response matrix $\mathbf{S}$ as the first two rows of the N-point inverse DFT matrix as $[\mathbf{S}]_{k,\ell} = \exp(j2\pi k\ell/N)$. Filter bank construction (4) implies that for each filter with a magnitude response specified as a row of $\mathbf{H}$ and a constant phase response, there is an extra filter whose magnitude response is identical but with a piecewise constant phase response over the N bands. This introduces a factor $P = 2$ in the number of sampling channels. The resulting sampling rate f_s is twice that of the oracle-based reconstruction with blind sampling.

Using the sample outputs $\mathbf{y}_r[n] = [y_r[n], y_{c,r}[n]]^\top$ for each pair of the filters with the same magnitude response $r = 1, \cdots, R$, we perform the following tests to reliably identify the single-ton bins and obtain the correct band-sample pair for any single-ton:

- **Zero-ton test**: The bin is a zero-ton if $\|\mathbf{y}_r[n]\| = 0$;
- **Multi-ton test**: The bin is a multi-ton as long as $|y_{c,r}[n]| \neq |y_r[n]|$ and/or $\angle\left(y_{c,r}[n]/y_r[n]\right) \neq 0 \mod 2\pi/N$. The multi-ton test fails when the relative phase is a multiple of $2\pi/N$;
- **Single-ton test**: After the zero-ton and multi-ton tests, if $|y_{c,r}[n]| = |y_r[n]|$ and $y_{c,r}[n]/y_r[n] = \exp(j2\pi\ell/N)$, for some $\ell \in [N]$, the measurement bin is detected as a single-ton with the band-sample pair $\widehat{k}_r[n] = \frac{N}{2\pi}\angle\left(y_{c,r}[n]/y_r[n]\right)$, $\widehat{x}[\widehat{k}_r[n]] = y_r[n]$, which is then used for peeling.

By performing these tests on all the outputs of the sampling channels, the baseband samples from each band can be reconstructed via peeling in the same manner as the oracle-based reconstruction.

3.4. Implementation

The operations described above can be implemented similar to the *modulated wideband converter* architecture of [8].

Let $h(t)$ be an ideal one-sided lowpass filter, with frequency response $H(f) = \Pi_B(f)$. Letting $p_i(t) \in \mathbb{R}$, $i = 1, \cdots, M$ denote periodic signals with period $T_p = N/f_{\max} = 1/B$, that is, $p_i(t) = p_i(t - T_p)$, we define modulated and then filtered signals $y_i(t)$, $i = 1, \cdots, M$, as $y_i(t) \stackrel{\text{def}}{=} (x(t) \times p_i(t)) * h(t)$. Note that $y_i(t)$ is bandlimited to $[0, B)$ and can be recovered from its samples taken at rate B, i.e., $y_i[n] = y_i(n/B) = y_i(nT_p)$.

Because $p_i(t)$ is periodic with $T_p = 1/B$, it can be written as a Fourier series $p_i(t) = \sum_{k=-\infty}^{\infty} c_{i,k} \exp(-j2\pi Bkt)$, where $c_{i,k}$ are Fourier series coefficients of the signal $p_i(t)$, that is, $c_{i,k} = (1/T_p)\int_{t=0}^{T_p} p_i(t)\exp(j2\pi Bkt)\,\mathrm{d}t$. Using the Fourier series expansion for $p_i(t)$, we get $x(t) \times p_i(t) \stackrel{\mathcal{F}}{\longleftrightarrow} \sum_{k=-\infty}^{\infty} c_{i,k}X(f + kB)$. Hence, the spectrum of $y_i(t)$ satisfies $Y_i(f) = \sum_{k=-\infty}^{\infty} c_{i,k}X(f + kB)\Pi_B(f)$. Furthermore, because $X(f)$ is zero outside $[0, f_{\max})$, we can have contribution only for $k = 0, \cdots, N-1$. Hence, we have $k \in \{0, \cdots, N-1\}$ in the above expression. To simplify the notation, define $X_k(f) = X(f + kB)\Pi_B(f)$ hence we can write $Y_i(f) = \sum_{k=0}^{N-1} c_{i,k}X_k(f)$ which yields the same relation as (2).

4. EMPIRICAL VALIDATION AND CLOSING REMARKS

In this section we present a numerical experiment validating the performance of our *spectrum-blind sampling and reconstruction* framework based on sparse-graph-coded filter banks. We generate a random signal from $\mathrm{BandLimited}(1, 0.1)$ having a piecewise constant spectrum on 10 disjoint frequency bands occupying a total of $1/10$ of the interval $[0, 1)$. The frequency spectrum and the Nyquist rate samples of the input signal is shown in Fig. 3a. We choose number disjoint frequency slices $N = 1000$ and use $M = 284$ channels designed through the regular graph ensemble. The sampling rate at the output of each filter is $1/N$. Samples from a choice of two channels of the filter bank are shown in Fig. 3b. The aggregate sampling rate from this setting is equal to $f_s = f_{\max}M/N = 0.284$.

Fig. 3c shows the Nyquist samples from the original signal together with the reconstruction from samples obtained by the sparse-graph-coded filter bank. As can be seen, Nyquist rate samples, hence the whole time domain signal, can be recovered *spectrum-blind* from samples taken at rate 0.284. Our result in Theorem 1 implies a rate of 0.246 for this signal model. The difference is due to finite slicing of the spectrum; as we increase N the empirical aggregate sampling rate f_s approaches the theoretical value.

As a concluding remark, the algorithm can be made robust to noise and non-ideal filters by constructing the phase response $\mathbf{S}$ through robust single-ton bin detection designs described in [13].

5. REFERENCES

[1] M. Unser, “Sampling–50 years after Shannon,” *Proc. IEEE*, vol. 88, no. 4, pp. 569–587, Apr. 2000.

[2] M. Vetterli, P. Marziliano, and T. Blu, “Sampling signals with finite rate of innovation,” *IEEE Trans. Inf. Theory*, vol. 50, no. 6, pp. 1417–1428, Jun. 2002.

[3] D. L. Donoho, “Compressed sensing,” *IEEE Trans. Inf. Theory*, vol. 52, no. 4, pp. 1289–1306, Apr. 2006.

[4] H. J. Landau, “Necessary density conditions for sampling and interpolation of certain entire functions,” *Acta Math.*, vol. 117, pp. 37–52, Jul. 1967.

[5] Y. M. Lu and M. N. Do, “A theory for sampling signals from a union of subspaces,” *IEEE Trans. Signal Process.*, vol. 56, no. 6, pp. 2334–2345, Jun. 2008.

[6] P. Feng and Y. Bresler, “Spectrum-blind minimum-rate sampling and reconstruction of multiband signals,” in *IEEE Int. Conf. on Acoust., Speech and Signal Process.*, vol. 3. Atlanta: IEEE, May 1996, pp. 1688–1691 vol. 3.

[7] Y. Bresler, “Spectrum-blind sampling and compressive sensing for continuous-index signals,” *Inf. Theory and Appl. Workshop, 2008*, pp. 547–554, 2008.

[8] M. Mishali and Y. C. Eldar, “From theory to practice: Sub-Nyquist sampling of sparse wideband analog signals,” *IEEE J. Sel. Topics Signal Process.*, vol. 4, no. 2, pp. 375–391, Apr. 2010.

[9] M. Mishali, Y. C. Eldar, O. Dounaevsky, and E. Shoshan, “Xampling: Analog to digital at sub-Nyquist rates,” *IET circuits*, vol. 5, pp. 8–20, Jan. 2011.

[10] R. G. Gallager, “Low-density parity-check codes,” *IRE Trans. Inf. Theory,*, vol. 8, no. 1, pp. 21–28, 1962.

[11] T. J. Richardson and R. L. Urbanke, “The capacity of low-density parity-check codes under message-passing decoding,” *IEEE Trans. Inf. Theory*, vol. 47, no. 2, pp. 599–618, 2001.

[12] A. Hormati and M. Vetterli, “Annihilating filter-based decoding in the compressed sensing framework,” in *Wavelets XII*, ser. Proc. SPIE, Aug. 2007, pp. 670 121–670 121.

[13] X. Li, S. Pawar, and K. Ramchandran, “Sub-linear time support recovery for compressed sensing using sparse-graph codes,” *arXiv*, Dec. 2014.

[14] T. Richardson and R. Urbanke, *Modern Coding Theory*. New York, NY, USA: Cambridge University Press, 2008.

[15] O. Öçal, X. Li, and K. Ramchandran, “Minimum-rate spectrum blind sampling based on sparse-graph codes,” Sep. 2015. [Online]. Available: http://www.eecs.berkeley.edu/~ocal/sampling-sparse-graph-codes.pdf