COMMUTING OPERATOR OF OFFSET LINEAR CANONICAL TRANSFORM AND ITS APPLICATIONS

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ABSTRACT

Linear canonical transform (LCT) is an attractive and useful tool in optics and signal processing. In previous work, a generalization of the LCT with two extra parameters, called offset LCT (OLCT), has been developed. In this paper, a different definition of OLCT is proposed, which has a more concise form of inverse transform. We find a linear operator that commutes with the proposed OLCT. We prove that the commuting operator and the proposed OLCT have the same set of eigenfunctions with different eigenvalues. We also derive the discrete version of the commuting operator to develop a discrete OLCT. The proposed discrete OLCT has perfect reversibility property.

***Index Terms*—** ABCD transform, fractional Fourier transform, linear canonical transform, offset linear canonical transform, quadratic-phase integrals

1. INTRODUCTION

Linear canonical transform (LCT) is a parameterized general linear integral transform. Fourier transform, Fresnel transform and fractional Fourier transform are all its special cases [1, 2]. It has four parameters and thus more important and useful in optics [3, 4, 5] and many signal processing applications including filter design, radar system analysis, signal synthesis, phase reconstruction, time-frequency analysis, pattern recognition, encryption and modulation [6, 7, 8, 9, 10, 11].

In [12], the LCT is generalized by introducing two extra parameters, corresponding to time shift and frequency modulation. This generalized LCT is called offset LCT (OLCT). The eigenfunctions of the OLCT for all cases have been discussed in [12] in detail. In [13], the lossless uniform sampling theorem of the OLCT is developed, and the effect of OLCT in time-frequency plane is also discussed. In [14], the authors develop the convolution and correlation theorems for the OLCT, and investigate the sampling theorem of the OLCT by the convolution theorem. In [15], spectral of uniformly and nonuniformly sampled signals in the OLCT domain is analyzed. Many applications of the LCT can be extended to the OLCT. Some applications have been presented in [12, 16, 17].

In [18], a linear operator commuting with the LCT operator is proposed. The commuting operator and the LCT operator have the same set of eigenfunctions with different eigenvalues. In this paper, we focus on the development of the commuting operator of the OLCT. First, a different definition of OLCT is proposed. The inverse transform of proposed OLCT is more concise. And it is easier to find a linear operator that commutes with the proposed OLCT. The eigenfunctions and eigenvalues of the commuting operator are discussed. We also derive the discrete version of the commuting operator, the eigenvectors of which can be used to develop a discrete OLCT with perfect reversibility property.

2. PRELIMINARY: COMMUTING OPERATOR OF LINEAR CANONICAL TRANSFORM

Given an input signal $x(t)$, the output $X_{\mathbf{M}}(u)$ of the linear canonical transform (LCT) [5, 2] is given by

$$X_{\mathbf{M}}(u) = \mathcal{L}_{\mathbf{M}}\{x(t)\} = \begin{cases} \sqrt{\frac{1}{j2\pi b}} \int_{-\infty}^{\infty} e^{j\frac{d}{2b}u^2 - j\frac{1}{b}ut + j\frac{a}{2b}t^2} x(t)\, dt, & b \neq 0 \\ \sqrt{d}\, e^{j\frac{cd}{2}u^2} x(du), & b = 0 \end{cases}, \quad (1)$$

where $\mathcal{L}_{\mathbf{M}}$ denotes the LCT operator, and $\mathbf{M} = [a, b; c, d]$ is the 2×2 LCT parameter matrix satisfying $\det(\mathbf{M}) = ad - bc = 1$. The inverse LCT is equivalent to the forward LCT with parameter matrix $\mathbf{M}^{-1}$:

$$[\mathcal{L}_{\mathbf{M}}]^{-1} = \mathcal{L}_{\mathbf{M}^{-1}}. \quad (2)$$

Let ω and ν denote the frequency coordinates with respect to t and u, respectively. [7] shows that the LCT can produce the following affine transformation in time-frequency plane:

$$\begin{bmatrix} u \\ \nu \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ \omega \end{bmatrix}. \quad (3)$$

The eigenfunctions of the LCT for all cases have been discussed in [2]. In [18], the authors find a linear operator $\mathcal{C}_{\mathbf{M}}$ commuting with the LCT operator, i.e. $\mathcal{C}_{\mathbf{M}}\mathcal{L}_{\mathbf{M}} = \mathcal{L}_{\mathbf{M}}\mathcal{C}_{\mathbf{M}}$. The commuting operator is given by

$$\mathcal{C}_{\mathbf{M}} = b\mathcal{D}_t^2 + j\frac{a-d}{2}(t\mathcal{D}_t + \mathcal{D}_t t) + ct^2, \quad (4)$$

where $\mathcal{D}_t = \frac{d}{dt}$. It implies that $x(t)$ through $\mathcal{C}_{\mathbf{M}}$ becomes

$$b\frac{d^2}{dt^2}x(t) + j\frac{a-d}{2}\left(t\frac{d}{dt}x(t) + \frac{d}{dt}\left[tx(t)\right]\right) + ct^2x(t). \quad (5)$$

It is proved that $\mathcal{C}_{\mathbf{M}}$ and $\mathcal{L}_{\mathbf{M}}$ have the same set of eigenfunctions with different eigenvalues.

3. COMMUTING OPERATOR OF OFFSET LINEAR CANONICAL TRANSFORM

In this section, a new definition of offset LCT (OLCT) is proposed. The inverse transform of the proposed OLCT is more concise, and it is easier to find a linear operator that commutes with the proposed OLCT.

3.1. Offset Linear Canonical Transform

The conventional definition of OLCT [12, 13] is given by

$$\widetilde{X}_{\mathbf{M}}^{\tau,\eta}(u) = \widetilde{\mathcal{L}}_{\mathbf{M}}^{\tau,\eta}\left\{x(t)\right\}$$
$$= \begin{cases} \sqrt{\frac{1}{j2\pi b}}e^{j\eta u}\int_{-\infty}^{\infty} e^{j\frac{d}{2b}(u-\tau)^2 - j\frac{1}{b}(u-\tau)t + j\frac{a}{2b}t^2}x(t)dt, & b \neq 0 \\ \sqrt{d}\, e^{j\eta u}e^{j\frac{cd}{2}(u-\tau)^2}x\left(du - d\tau\right), & b = 0 \end{cases}, \quad (6)$$

where the two extra parameters τ and η correspond to time shift and frequency modulation, respectively, and its inverse transform is given by

$$\left[\widetilde{\mathcal{L}}_{\mathbf{M}}^{\tau,\eta}\right]^{-1} = e^{j\frac{cd}{2}\tau^2 - jad\tau\eta + j\frac{ab}{2}\eta^2}\,\widetilde{\mathcal{L}}_{\mathbf{M}^{-1}}^{-d\tau+b\eta,\, c\tau-a\eta}. \quad (7)$$

In (7), there is a constant phase, and the parameters of time shift and frequency modulation change. In the following, a new definition of OLCT with more concise inverse transform is proposed. It is easier to find a commuting operator from the proposed OLCT than from the conventional one.

In time-frequency plane, conventional OLCT results in

$$\begin{bmatrix} u \\ \nu \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x \\ \omega \end{bmatrix} + \begin{bmatrix} \tau \\ \eta \end{bmatrix}. \quad (8)$$

Consider another kind of time-frequency transformation:

$$\begin{bmatrix} u \\ \nu \end{bmatrix} - \begin{bmatrix} \beta \\ \gamma \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\left(\begin{bmatrix} x \\ \omega \end{bmatrix} - \begin{bmatrix} \beta \\ \gamma \end{bmatrix}\right). \quad (9)$$

Let $\mathcal{L}_{\mathbf{M}}^{\beta,\gamma}$ denote the OLCT operator corresponding to (9), and assume $X_{\mathbf{M}}^{\beta,\gamma}(u) = \mathcal{L}_{\mathbf{M}}^{\beta,\gamma}\{x(t)\}$. Then, (9) implies

$$e^{-j\gamma u}X_{\mathbf{M}}^{\beta,\gamma}(u+\beta) = \mathcal{L}_{\mathbf{M}}\left\{e^{-j\gamma t}x(t+\beta)\right\}. \quad (10)$$

Based on (10), the OLCT operator $\mathcal{L}_{\mathbf{M}}^{\beta,\gamma}$ can be derived from the LCT operator $\mathcal{L}_{\mathbf{M}}$ defined in (1):

$$X_{\mathbf{M}}^{\beta,\gamma}(u) = \mathcal{L}_{\mathbf{M}}^{\beta,\gamma}\left\{x(t)\right\}$$
$$= \begin{cases} \sqrt{\frac{1}{j2\pi b}}\int_{-\infty}^{\infty} e^{j\gamma(u-\beta) + j\frac{d}{2b}(u-\beta)^2 - j\frac{1}{b}(u-\beta)(t-\beta)} \\ \qquad {}^{+j\frac{a}{2b}(t-\beta)^2 - j\gamma(t-\beta)}x(t)dt, \; b \neq 0 \quad (11) \\ \sqrt{d}e^{j\gamma(u-\beta) + j\frac{cd}{2}(u-\beta)^2 - j\gamma d(u-\beta)} \\ \qquad \cdot x\left(d(u-\beta)+\beta\right), \; b = 0 \end{cases}$$

From (8) and (9), we can find out that

$$\tau = (1-a)\beta - b\gamma \quad \text{and} \quad \eta = (1-d)\gamma - c\beta. \quad (12)$$

These two relations show that the proposed OLCT in (11) is equivalent to the conventional OLCT in (6) with some constant phase difference. However, from (1), (2) and (10), the inverse transform of the proposed OLCT is given by

$$\left[\mathcal{L}_{\mathbf{M}}^{\beta,\gamma}\right]^{-1} = \mathcal{L}_{\mathbf{M}^{-1}}^{\beta,\gamma}, \quad (13)$$

which is more concise than the inverse transform of the conventional OLCT in (7). **The OLCT mentioned in the rest of this paper refers to the proposed OLCT defined in (11).**

3.2. Commuting Operator of the Proposed OLCT

Assume $b \neq 0$. The first derivative of the OLCT is given by

$$\frac{d}{du}X_{\mathbf{M}}^{\beta,\gamma}(u) = \sqrt{\frac{1}{j2\pi b}}\int_{-\infty}^{\infty} e^{j\gamma(u-\beta) + j\frac{d}{2b}(u-\beta)^2 - j\frac{1}{b}(u-\beta)(t-\beta)}$$
$$^{+j\frac{a}{2b}(t-\beta)^2 - j\gamma(t-\beta)}j\left[\gamma + \frac{d}{b}(u-\beta) - \frac{1}{b}(t-\beta)\right]x(t)dt$$
$$= j\left[\gamma + \frac{d}{b}(u-\beta)\right]X_{\mathbf{M}}^{\beta,\gamma}(u) - j\frac{1}{b}\mathcal{L}_{\mathbf{M}}^{\beta,\gamma}\{(t-\beta)x(t)\}. \quad (14)$$

For simplicity, the equation above is rewritten as the following operator form:

$$\mathcal{D}_t\mathcal{L}_{\mathbf{M}}^{\beta,\gamma} = j\left[\gamma + \frac{d}{b}(t-\beta)\right]\mathcal{L}_{\mathbf{M}}^{\beta,\gamma} - j\frac{1}{b}\mathcal{L}_{\mathbf{M}}^{\beta,\gamma}(t-\beta), \quad (15)$$

where $\mathcal{D}_t = \frac{d}{dt}$. From (15), we have the Property 1:

$$\text{(P1)}\quad \mathcal{L}_{\mathbf{M}}^{\beta,\gamma}(t-\beta) = \left[jb\left(\mathcal{D}_t - j\gamma\right) + d(t-\beta)\right]\mathcal{L}_{\mathbf{M}}^{\beta,\gamma}. \quad (16)$$

Since $[\mathcal{L}_{\mathbf{M}}^{\beta,\gamma}]^{-1} = \mathcal{L}_{\mathbf{M}^{-1}}^{\beta,\gamma}$, replacing $\mathbf{M} = [a,b;c,d]$ by $\mathbf{M}^{-1} = [d,-b;-c,a]$ in (16) leads to

$$\left[\mathcal{L}_{\mathbf{M}}^{\beta,\gamma}\right]^{-1}(t-\beta) = \left[-jb\left(\mathcal{D}_t - j\gamma\right) + a(t-\beta)\right]\left[\mathcal{L}_{\mathbf{M}}^{\beta,\gamma}\right]^{-1}, \quad (17)$$

and it follows that

$$(t-\beta)\mathcal{L}_{\mathbf{M}}^{\beta,\gamma} = -jb\mathcal{L}_{\mathbf{M}}^{\beta,\gamma}\left(\mathcal{D}_t - j\gamma\right) + a\mathcal{L}_{\mathbf{M}}^{\beta,\gamma}(t-\beta). \quad (18)$$

Since $ad - bc = 1$, substituting the Property 1 in (16) to the above equation leads to the Property 2:

$$\text{(P2)}\quad \mathcal{L}_{\mathbf{M}}^{\beta,\gamma}(\mathcal{D}_t - j\gamma) = \left[a\left(\mathcal{D}_t - j\gamma\right) - jc(t-\beta)\right]\mathcal{L}_{\mathbf{M}}^{\beta,\gamma}. \quad (19)$$

From Properties 1 and 2 in (16) and (19), the commuting operator of the OLCT is given by

$$\mathcal{C}_{\mathbf{M}}^{\beta,\gamma} = b\left(\mathcal{D}_t - j\gamma\right)^2 + j\frac{a-d}{2}\left[(t-\beta)\left(\mathcal{D}_t - j\gamma\right) + \left(\mathcal{D}_t - j\gamma\right)(t-\beta)\right] + c(t-\beta)^2. \quad (20)$$

One can also prove that $\mathcal{C}_{\mathbf{M}}^{\beta,\gamma}$ is also the commuting operator for $b = 0$ case. When $\beta = \gamma = 0$ (i.e. no offset), $\mathcal{C}_{\mathbf{M}}^{\beta,\gamma}$ reduces to the commuting operator of the LCT, i.e. $\mathcal{C}_{\mathbf{M}}$, in (4).

4. APPLICATIONS

4.1. Eigenfunctions of Commuting Operator of OLCT

Since $\mathcal{L}_{\mathbf{M}}$ and $\mathcal{C}_{\mathbf{M}}$ commutes, they have the same set of eigenfunctions with different eigenvalues. Assume

$$\mathcal{L}_{\mathbf{M}} E_m(t) = \lambda_m E_m(t), \qquad \mathcal{C}_{\mathbf{M}} E_m(t) = \mu_m E_m(t), \quad (21)$$

where the eigenfunctions $E_m(t)$ and eigenvalues λ_m, μ_m have been discussed in [2, 18] for all cases of $\mathbf{M}$. From the relation between the LCT and the OLCT in (10), $e^{j\gamma(t-\beta)}E_m(t-\beta)$ are the eigenfunctions of the OLCT with the same eigenvalues λ_m:

$$\mathcal{L}_{\mathbf{M}}^{\beta,\gamma} e^{j(t-\beta)} E_m(t-\beta) = \lambda_m e^{j(t-\beta)} E_m(t-\beta). \quad (22)$$

Next, we will prove that $\mathcal{L}_{\mathbf{M}}^{\beta,\gamma}$ and $\mathcal{C}_{\mathbf{M}}^{\beta,\gamma}$ share the same set of eigenfunctions with different eigenvalues. From the definition of $\mathcal{C}_{\mathbf{M}}$ in (4) and $\mathcal{C}_{\mathbf{M}}^{\beta,\gamma}$ in (20), we have

$$\begin{aligned}\mathcal{C}_{\mathbf{M}}^{\beta,\gamma} e^{j(t-\beta)} E_m(t-\beta) = & e^{j(t-\beta)} \Big\{ b\mathcal{D}_t^2 + j\frac{a-d}{2}\left[(t-\beta)\mathcal{D}_t\right. \\ & \left.+\mathcal{D}_t(t-\beta)\right] + c(t-\beta)^2 \Big\} E_m(t-\beta) \\ = & e^{j(t-\beta)} \left[\mathcal{C}_{\mathbf{M}} E_m(t)\right]_{t\to t-\beta}. \end{aligned} \quad (23)$$

From (21), it follows that

$$\mathcal{C}_{\mathbf{M}}^{\beta,\gamma} e^{j(t-\beta)} E_m(t-\beta) = \mu_m e^{j(t-\beta)} E_m(t-\beta). \quad (24)$$

Therefore, $e^{j\gamma(t-\beta)}E_m(t-\beta)$ are also the eigenfunctions of $\mathcal{C}_{\mathbf{M}}^{\beta,\gamma}$ with the same eigenvalues as $\mathcal{C}_{\mathbf{M}}$.

It has been shown in [18] that the eigenvalues λ_m are complex and $|\lambda_m| = 1$, while the eigenvalues μ_m are all real and distinct. Therefore, the set of eigenfunctions obtained from the commuting operator is unique and orthogonal. For example, when $|a+d| < 2$, the eigenfunctions of the LCT and its commuting operator are given by

$$h_{\mathbf{M},m}(t) \triangleq \sqrt{\frac{1}{\sqrt{\pi}2^m m!}} H_m\left(\frac{t}{\sigma}\right) e^{-\frac{1+j\rho}{2\sigma^2}t^2}, \quad (25)$$

$$\text{where } \sigma^2 = \frac{2|b|}{\sqrt{4-(a+d)^2}}, \quad \rho = \frac{\operatorname{sgn}(b)(a-d)}{\sqrt{4-(a+d)^2}}. \quad (26)$$

$H_m(t)$ is the Hermite polynomial of order m. The eigenvalues λ_m and μ_m are given by

$$\lambda_m = e^{-j\left(m+\frac{1}{2}\right)\cos^{-1}\left(\frac{a+d}{2}\right)}, \quad (27)$$

$$\mu_m = -\operatorname{sgn}(b)\sqrt{4-(a+d)^2}\left(m+\frac{1}{2}\right). \quad (28)$$

According to (22) and (24), the eigenfunctions of the OLCT and its commuting operator are given by

$$h_{\mathbf{M},m}^{\beta,\gamma}(t) \triangleq \sqrt{\frac{1}{\sqrt{\pi}2^m m!}} H_m\left(\frac{t-\beta}{\sigma}\right) e^{-\frac{1+j\rho}{2\sigma^2}(t-\beta)^2 + j\gamma(t-\beta)}, \quad (29)$$

where σ and ρ are defined in (26). For two different eigenfunctions, say $h_{\mathbf{M},m_1}^{\beta,\gamma}(t)$ and $h_{\mathbf{M},m_2}^{\beta,\gamma}(t)$ where $m_1 \neq m_2$, the values of μ_{m_1} and μ_{m_2} are different, but the values of λ_{m_1} and λ_{m_1} will be the same if $(m_1 - m_2)\cos^{-1}\left(\frac{a+d}{2}\right) = 2k\pi$. If so, all the linear combinations $c_1 h_{\mathbf{M},m_1}^{\beta,\gamma}(t) + c_2 h_{\mathbf{M},m_2}^{\beta,\gamma}(t)$ with arbitrary c_1 and c_2 are also the eigenfunctions of OLCT. There will be many orthogonal sets of eigenfunctions for the OLCT, but there's only one set for the commuting operator.

4.2. Self-Imaging Phenomena

The eigenfunctions of the OLCT are useful for analyzing the self-imaging phenomena and the resonance phenomena of spherical mirror pair systems in optics [12]. For example, Fig. 1 shows an optical system consisting of a lens with focal length f shifted upward with distance x_0, a free space with distance z_0, and a prism with refractive index n and ratio of bottom width to height being ρ. According to [12], the light propagation through this system can be modeled by the conventional OLCT with $\mathbf{M} = [1 - z_0/f, z_0/k; -k/f, 1]$, $\tau = z_0x_0/f$ and $\eta = kx_0/f - k(n-1)\rho$, where $k = 2\pi/\lambda$ and λ is the wavelength. And from (12), it can also be modeled by the proposed OLCT with the same $\mathbf{M}$, $\beta = x_0 - (n-1)\rho f$ and $\gamma = -k(n-1)\rho$. When $|a+d| < 2$, all the inputs resulting in self-imaging phenomena can be expressed by the eigenfunctions in (29).

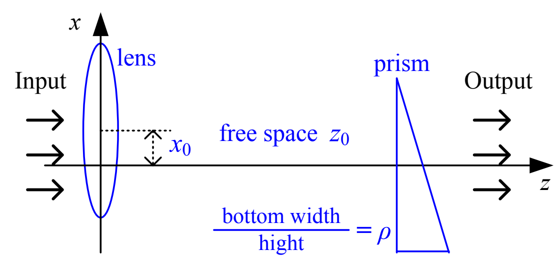

Fig. 1. An optical system with a shifted lens, a free space and a prism.

4.3. Discrete OLCT

The eigenfunctions of the OLCT can form an orthonormal set. Therefore, the eigenfunctions can be used to implement the OLCT. For example, when $|a+d| < 2$, express the input signal $x(t)$ in terms of $h_{\mathbf{M},m}^{\beta,\gamma}(t)$ as follows:

$$x(t) = \sum_{m=0}^{\infty} a_m h_{\mathbf{M},m}^{\beta,\gamma}(t), \quad a_m = \int_{-\infty}^{\infty} x(t)\overline{h_{\mathbf{M},m}^{\beta,\gamma}(t)}dt. \quad (30)$$

Since $h_{\mathbf{M},m}^{\beta,\gamma}(t)$ are the eigenfunctions of the OLCT with eigenvalues λ_m given in (27) when $|a+d| < 2$, the OLCT of $x(t)$ can be obtained from

$$X_{\mathbf{M}}^{\beta,\gamma}(u) = \sum_{m=0}^{\infty} a_m \mathcal{L}_{\mathbf{M}}^{\beta,\gamma}\left\{h_{\mathbf{M},m}^{\beta,\gamma}(t)\right\} = \sum_{m=0}^{\infty} \lambda_m a_m h_{\mathbf{M},m}^{\beta,\gamma}(u). \quad (31)$$

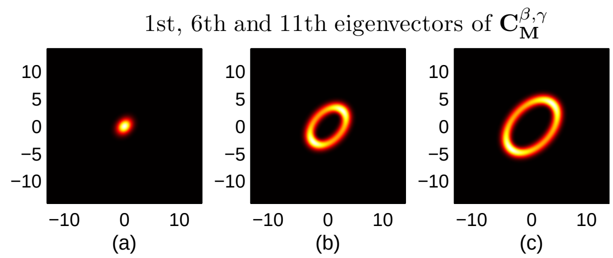

Fig. 2. Time-frequency distributions: (a) 1st, (b) 6th and (c) 11th eigenvectors of $\mathbf{C}_{\mathbf{M}}^{\beta,\gamma}$ used to approximate the sampled OLCT eigenfunctions $h_{\mathbf{M},m}^{\beta,\gamma}(n\Delta)$ with $m = 0, 5, 10$, where $\mathbf{M} = [0.53, 0.63; -0.67, 1.09]$, $\beta = -2$, $\gamma = 3$, $N = 127$ and $\Delta = \sqrt{2\pi/N}$.

However, for the discrete case, we cannot digitally implement the OLCT by sampling (30) and (31) with some sampling period Δ directly. The sampled eigenfunctions, $h_{\mathbf{M},m}^{\beta,\gamma}(n\Delta)$, are no longer orthogonal to each other. Therefore, we need to find an orthogonal set of discrete functions that can approximate $h_{\mathbf{M},m}^{\beta,\gamma}(n\Delta)$. A simple solution is to develop the discrete version of $\mathcal{C}_{\mathbf{M}}^{\beta,\gamma}$, denoted by $\mathbf{C}_{\mathbf{M}}^{\beta,\gamma}$, and then find an orthonormal set of discrete eigenfunctions (i.e. eigenvectors) from $\mathbf{C}_{\mathbf{M}}^{\beta,\gamma}$.

To obtain $\mathbf{C}_{\mathbf{M}}^{\beta,\gamma}$, a method similar to that in [19] is used. Assume the sampling period is $\Delta = \sqrt{2\pi/N}$, and the discrete input is given by $x[n] = x\left((n - (N-1)/2)\Delta\right)$ where $0 \le n \le N-1$. In (20), $\mathcal{C}_{\mathbf{M}}^{\beta,\gamma}$ is composed of two kinds of operators, $t - \beta$ and $\mathcal{D}_t - j\gamma$. It is straightforward to define the discrete version of $t - \beta$ as an $N \times N$ diagonal matrix $\mathbf{T} - \beta\mathbf{I}$, where $\mathbf{I}$ is the identity matrix and

$$[\mathbf{T}]_{kn} = \begin{cases} \left(n - \frac{N-1}{2}\right)\sqrt{\frac{2\pi}{N}}, & n = k \\ 0, & n \neq k \end{cases}, \qquad (32)$$

where $0 \le n, k \le N-1$. Let $\mathcal{F}$ denote the Fourier transform. It is well known that $\mathcal{F}\mathcal{D}_t = jt\mathcal{F}$. Therefore, $\mathcal{D}_t - j\gamma = j\mathcal{F}^{-1}t\mathcal{F} - j\gamma$ and its discrete version can be designed as $j\mathbf{F}^H\mathbf{T}\mathbf{F} - j\gamma\mathbf{I}$, where $\mathbf{F}$ is an $N \times N$ centered DFT matrix:

$$[\mathbf{F}]_{kn} = e^{-j\frac{2\pi}{N}\left(k-\frac{N-1}{2}\right)\left(n-\frac{N-1}{2}\right)}, \quad 0 \le n, k \le N-1. \quad (33)$$

Accordingly, the discrete version of $\mathcal{C}_{\mathbf{M}}^{\beta,\gamma}$ is given by

$$\mathbf{C}_{\mathbf{M}}^{\beta,\gamma} = b\big(j\mathbf{F}^H\mathbf{T}\mathbf{F} - j\gamma\mathbf{I}\big)^2 + j\frac{a-d}{2}\big[(\mathbf{T} - \beta\mathbf{I})\big(j\mathbf{F}^H\mathbf{T}\mathbf{F} - j\gamma\mathbf{I}\big) + \big(j\mathbf{F}^H\mathbf{T}\mathbf{F} - j\gamma\mathbf{I}\big)(\mathbf{T} - \beta\mathbf{I})\big] + c(\mathbf{T} - \beta\mathbf{I})^2. \quad (34)$$

Performing eigendecomposition on $\mathbf{C}_{\mathbf{M}}^{\beta,\gamma}$, the N orthonormal eigenvectors, denoted by $h_{\mathbf{M},m}^{\beta,\gamma}[n]$, can approximate $h_{\mathbf{M},m}^{\beta,\gamma}(n\Delta)$ with some constant difference. For example, assume $\mathbf{M} = [0.53, 0.63; -0.67, 1.09]$, $\beta = -2$, $\gamma = 3$ and $N = 127$. The sampled OLCT eigenfunctions $h_{\mathbf{M},m}^{\beta,\gamma}(n\Delta)$ for $m = 0, 5, 10$ are approximated by the 1st, 6th and 11th eigenvectors of $\mathbf{C}_{\mathbf{M}}^{\beta,\gamma}$, the time-frequency distributions of which are shown in Fig. 2(a), (b) and (c).

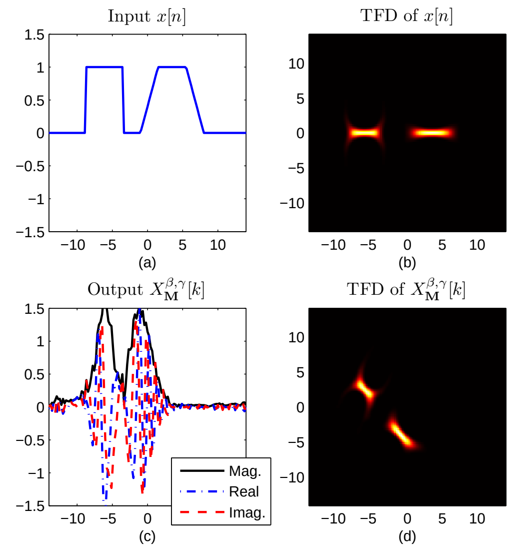

Fig. 3. (a) Discrete input $x[n]$, (b) time-frequency distribution of $x[n]$, (c) output $X_{\mathbf{M}}^{\beta,\gamma}[k]$ of the discrete OLCT and (d) time-frequency distribution of $X_{\mathbf{M}}^{\beta,\gamma}[k]$, where $\mathbf{M} = [0.53, 0.63; -0.67, 1.09]$, $\beta = -2$, $\gamma = 3$ and $N = 127$.

Therefore, a discrete OLCT can be developed as follows

$$X_{\mathbf{M}}^{\beta,\gamma}[k] = \sum_{m=0}^{N-1} \lambda_m a_m h_{\mathbf{M},m}^{\beta,\gamma}[k], \quad a_m = \sum_n x[n]\overline{h_{\mathbf{M},m}^{\beta,\gamma}[n]}.$$

This discrete OLCT features perfect reconstruction because $h_{\mathbf{M},m}^{\beta,\gamma}[n]$'s are orthonormal. In the following, a simulation is given, using the same parameters as in Fig. 2. The discrete input $x[n]$ is real, containing a rectangular function and a trapezoidal function, as shown in Fig. 3(a). The time-frequency distribution of $x[n]$ is depicted in Fig. 3(b). The output $X_{\mathbf{M}}^{\beta,\gamma}[k]$ of the discrete OLCT is shown in Fig. 3(c), and its time-frequency distribution is depicted in Fig. 3(d).

5. CONCLUSION

In this paper, a different definition of OLCT is proposed. The proposed OLCT has a more concise form of inverse transform than the conventional definition of OLCT. A linear operator that commutes with the proposed OLCT is derived. The eigenfunctions and eigenvalues of the commuting operator are analyzed. We prove that the commuting operator and the proposed OLCT share common eigenfunctions with different eigenvalues. We also derive the discrete version of the commuting operator, the eigenvectors of which can be used to develop a discrete OLCT. The proposed discrete OLCT has an important property: perfect reversibility.

6. ACKNOWLEDGMENTS

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