A UNIFIED APPROACH TO THE DESIGN OF IIR AND FIR NOTCH FILTERS

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ABSTRACT

This paper presents a unified, optimization-driven solution for designing IIR and FIR notch filters with prescribed, possibly varying notch levels in the given stop-bands, and near unit magnitude frequency response at the pass-bands. Although the original IIR notch filter optimization problem is non-convex, we show that it can be well approximated by a convex problem, by replacing a non-positive semi-definite 2×2 Hermitian matrix with its nearest positive semi-definite counterpart. With this approach, the IIR filter design can be efficiently solved via Newton iteration. The same approach can be directly applied to the FIR filter design since it is a degenerated case of the IIR filter. Moreover, we show that the FIR design problem is convex and therefore can be solved optimally. Numerical examples are presented to verify the effectiveness of the proposed design.

Index Terms— notch filter, IIR, FIR, convex optimization, interior point method

1. INTRODUCTION

Digital notch filters have been widely used in signal processing applications thanks to its capability of removing single-frequency or narrowband interferences while ensuring the wideband signal unchanged. Depending on the frequency characteristics of the narrowband components, one can apply either fixed [1, 2] or adaptive notch filters [3]. It has found applications in many digital systems, such as power line [4], image processing [5], and ultra-wideband communications [6].

There are two types of notch filters: single-notch filters that only eliminate one narrowband interference, and multi-notch filters that simultaneously remove several narrowband components at difference frequencies. A single-notch IIR filter can be directly designed from a second-order transfer function with bilinear transformation [7, 8]. The design of multi-notch IIR filters is more complicated and the past two decades have seen several methods being proposed. A straightforward approach to obtain a multi-notch filter is to cascade several single-notch filters, each corresponding to one notch frequency. This method leverages the known results from single-notch filter design but suffers from shortcomings such as higher bandwidth and higher sensitivity to coefficient distortion [8]. The second approach synthesizes the notch filter $H(z)$ by using an all-pass filter $A(z)$ and their relation $H(z) = 1/2(1 + A(z))$ [1, 2]. This method leverages some symmetric property to obtain a computationally efficient lattice realization with low sensitivity. However, it is not analytically clear how well the resulting filter performs, especially in comparison to the optimal filter under the same constraint.

Majority of the aforementioned design methods are algebraic in nature. With the rapid development of (convex) optimization [9], it becomes not only theoretically possible to design notch filters using the optimization tools, but also practically feasible to implement such designs. A systematic treatment of FIR filter design using convex optimization is provided in [10], where the authors show that a variety of magnitude FIR filter design can be converted to convex semi-definite optimization problems, and hence the optimal solution can be efficiently computed. Applying convex optimization theory to the design of multi-notch IIR filters first appears in [2]. However, the usage of optimization is strictly limited to finding the optimal pole placement using the denominator polynomial of the transfer function. A genetic algorithm (GA) based approach to the IIR notch filter optimization was proposed in [11]. But it is computationally complicated and often suffers from slow convergence.

In this paper, we propose a unified design approach for both IIR and FIR notch filters. We first formulate the IIR notch filter design as such an optimization problem: given that the prescribed notch levels on the stop-band(s) being strictly met, optimize the magnitude of the frequency response on the pass-band(s) to be as close to unit as possible. It is shown that this optimization problem is non-convex and hence difficult to solve. To address this issue, the original problem is analyzed using the interior point method [9]. We show that by modifying a non-positive semi-definite Hessian matrix to be positive semidefinite, we can use the convex

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optimization method, i.e., the Newton iteration to efficiently obtain a near optimum design of IIR notch filters. We also show that the similar design of FIR notch filters results in a convex problem, to which the optimum solution is guaranteed. Numerical examples are provided to validate the proposed design.

The rest of the paper is organized as follows. IIR and FIR filter design problems are treated in Section 2 and 3, respectively. More specifically, the design problem of IIR notch filter is presented in Section 2.1, and the proposed solution is laid out in great detail in Section 2.2. Numerical examples are presented in Section 4, and Section 5 concludes the paper.

2. IIR NOTCH FILTER DESIGN

2.1. Problem Formulation

The z-transform of an IIR filter is

$$H\left(z^{-1}\right)=\frac{P\left(z^{-1}\right)}{Q\left(z^{-1}\right)}=\frac{\sum_{i=0}^{M-1} x_i z^{-i}}{1+\sum_{i=1}^{N} y_i z^{-i}}, \tag{1}$$

which consists of a feedforward filter (FFF) $P\left(z^{-1}\right)$ of order M in the numerator and a feedbackward filter (FBF) $Q\left(z^{-1}\right)$ of order N in the denominator. If $N=0$, the IIR filter degenerates to an FIR filter.

The goal of our work is to design an IIR filter which forms nulls on any given stop-band $\mathcal{S}$, while forcing the magnitude of the frequency response to be as close to unit as possible on the remaining pass-band $\mathcal{P}$. To that end, we formulate the following optimization problem

$$\begin{array}{ll}\underset{\mathbf{x}\in\mathbb{C}^M,\mathbf{y}\in\mathbb{C}^N}{\text{minimize}} & \sum_{i\in\mathcal{P}}\left(\log|\mathbf{f}_i^*\mathbf{x}|^2-\log|1+\mathbf{g}_i^*\mathbf{y}|^2\right)^2 \\ \text{subject to} & \frac{|\mathbf{f}_i^*\mathbf{x}|^2}{|1+\mathbf{g}_i^*\mathbf{y}|^2}\le\beta_i, \forall i\in\mathcal{S}\end{array} \tag{2}$$

where $\mathbf{x}$ and $\mathbf{y}$ are the vectors whose entries are x_i's and y_i's that appear in (1), $\beta_i<1$ represents the frequency mask at the ith frequency point, $\mathbf{f}_i\in\mathbb{C}^M$ consists of the first M entries of the i-th column of the Fast Fourier Transform (FFT) matrix $\mathbf{F}\doteq\left\{\exp\left(j\frac{2kl\pi}{L}\right)\right\}\in\mathbb{C}^{L\times L}$, $\mathbf{g}_i\in\mathbb{C}^N$ consists of the second to the $(N+1)$st entries of the i-th column of the FFT matrix $\mathbf{F}$, and * denotes the conjugate transpose operation. Note that L defines the size of FFT used in the algorithm design, which should be large enough to achieve a fine spectrum granularity but not too large to incur unnecessary computational complexity.

We can show that the optimization problem (2) is *not* convex, and hence classic convex optimization technique cannot be directly applied. However, a convex approximation to (2) can be made by replacing a 2×2 Hermitian matrix whose eigen-values are of different signs by its nearest 2×2 positive semi-definite (p.s.d.) counterpart. Based on this technique, a near optimal solution to (2) can be derived. Details of this approach are provided in the following section.

2.2. Proposed Solution

We apply the interior point method to problem (2), for which we consider the logarithmic barrier function:

$$\begin{aligned}f_t(\mathbf{x},\mathbf{y})= &\sum_{i\in\mathcal{P}}\left(\log|\mathbf{f}_i^*\mathbf{x}|^2-\log|1+\mathbf{g}_i^*\mathbf{y}|^2\right)^2\\ &-\frac{1}{t}\sum_{i\in\mathcal{S}}\log\left(\beta_i-\frac{|\mathbf{f}_i^*\mathbf{x}|^2}{|1+\mathbf{g}_i^*\mathbf{y}|^2}\right).\end{aligned} \tag{3}$$

The first order derivatives of the barrier function (3) can be derived as:

$$\begin{aligned}\nabla_{\mathbf{x}}f_t=&\sum_{i\in\mathcal{P}}(\log|\mathbf{f}_i^*\mathbf{x}|^2-\log|1+\mathbf{g}_i^*\mathbf{y}|^2)^2\frac{4\mathbf{f}_i^*\mathbf{x}}{|\mathbf{f}_i^*\mathbf{x}|^2}\mathbf{f}_i\\ &+\frac{1}{t}\sum_{i\in\mathcal{S}}\frac{2\mathbf{f}_i^*\mathbf{x}}{\beta_i|1+\mathbf{g}_i^*\mathbf{y}|^2-|\mathbf{f}_i^*\mathbf{x}|^2}\mathbf{f}_i,\end{aligned} \tag{4}$$

and

$$\begin{aligned}\nabla_{\mathbf{y}}f_t=&\sum_{i\in\mathcal{P}}(\log|1+\mathbf{g}_i^*\mathbf{y}|^2-\log|\mathbf{f}_i^*\mathbf{x}|^2)^2\frac{4(1+\mathbf{g}_i^*\mathbf{y})}{|1+\mathbf{g}_i^*\mathbf{y}|^2}\mathbf{g}_i\\ &-\frac{1}{t}\sum_{i\in\mathcal{S}}\frac{2(1+\mathbf{g}_i^*\mathbf{y})}{\beta_i|1+\mathbf{g}_i^*\mathbf{y}|^2\mathbf{g}_i-|\mathbf{f}_i^*\mathbf{x}|^2}\cdot\frac{|\mathbf{f}_i^*\mathbf{x}|^2}{|1+\mathbf{g}_i^*\mathbf{y}|^2}\mathbf{g}_i.\end{aligned} \tag{5}$$

We further write (4) and (5) in the following matrix form:

$$\nabla_{\mathbf{x}}f_t = \mathbf{H}\mathbf{q}_x \tag{6}$$

$$\nabla_{\mathbf{y}}f_t = \mathbf{G}\mathbf{q}_y \tag{7}$$

where both $\mathbf{H}$ and $\mathbf{G}$ are sub-matrices of the L-point FFT matrix $\mathbf{F}$, $\mathbf{q}_x$ and $\mathbf{q}_y$ are both length-L vectors with entries

$$\mathbf{q}_{x,i}=\begin{cases}(\log|\mathbf{f}_i^*\mathbf{x}|^2-\log|1+\mathbf{g}_i^*\mathbf{y}|^2)^2\frac{4\mathbf{f}_i^*\mathbf{x}}{|\mathbf{f}_i^*\mathbf{x}|^2} & i\in\mathcal{P}\\ \frac{1}{t}\cdot\frac{2\mathbf{f}_i^*\mathbf{x}}{\beta_i|1+\mathbf{g}_i^*\mathbf{y}|^2-|\mathbf{f}_i^*\mathbf{x}|^2} & i\in\mathcal{S}\\ 0 & \text{otherwise,}\end{cases}$$

and

$$\mathbf{q}_{y,i}=\begin{cases}(\log|1+\mathbf{g}_i^*\mathbf{y}|^2-\log|\mathbf{f}_i^*\mathbf{x}|^2)^2\frac{4(1+\mathbf{g}_i^*\mathbf{y})}{|1+\mathbf{g}_i^*\mathbf{y}|^2} & i\in\mathcal{P}\\ -\frac{1}{t}\cdot\frac{2(1+\mathbf{g}_i^*\mathbf{y})}{\beta_i|1+\mathbf{g}_i^*\mathbf{y}|^2\mathbf{g}_i-|\mathbf{f}_i^*\mathbf{x}|^2}\cdot\frac{|\mathbf{f}_i^*\mathbf{x}|^2}{|1+\mathbf{g}_i^*\mathbf{y}|^2} & i\in\mathcal{S}\\ 0 & \text{otherwise,}\end{cases}$$

respectively. Note that both $\mathbf{f}_i^*\mathbf{x}$ and $1+\mathbf{g}_i^*\mathbf{y}$ can be obtained by applying FFT on $\mathbf{x}$ and $[1,\mathbf{y}^T]^T$, respectively. Hence $\nabla_{\mathbf{x}}$ and $\nabla_{\mathbf{y}}$ can be efficiently computed via FFT and IFFT operations.

The second order derivatives of the barrier function (3) with respect to vector $\mathbf{z}\doteq[\mathbf{x}^T,\mathbf{y}^T]^T$ can be written as

$$\nabla^2 f_t=\left[\begin{array}{c}\mathbf{H}\\ \mathbf{G}\end{array}\right]\left[\begin{array}{cc}\mathcal{D}(\mathbf{a}) & \mathcal{D}(\mathbf{c})\\ \mathcal{D}^*(\mathbf{c}) & \mathcal{D}(\mathbf{b})\end{array}\right]\left[\begin{array}{c}\mathbf{H}\\ \mathbf{G}\end{array}\right]^* \tag{8}$$

where $\mathcal{D}(\mathbf{a})$ is a diagonal matrix whose diagonal elements are vector $\mathbf{a}$. Note that $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are all length-L vector whose entries are defined as

$$a_i=\begin{cases}(2-\log|\mathbf{f}_i^*\mathbf{x}|^2+\log|1+\mathbf{g}_i^*\mathbf{y}|^2)\frac{4}{|\mathbf{f}_i^*\mathbf{x}|^2} & i\in\mathcal{P}\\ \frac{2}{t}\cdot\frac{\beta_i|1+\mathbf{g}_i^*\mathbf{y}|^2+|\mathbf{f}_i^*\mathbf{x}|^2}{(\beta_i|1+\mathbf{g}_i^*\mathbf{y}|^2-|\mathbf{f}_i^*\mathbf{x}|^2)^2} & i\in\mathcal{S}\\ 0 & \text{otherwise}\end{cases}$$

$$b_i = \begin{cases} \left(2+\log|\mathbf{f}_i^*\mathbf{x}|^2 - \log|1+\mathbf{g}_i^*\mathbf{y}|^2\right)\frac{4}{|1+\mathbf{g}_i^*\mathbf{y}|^2} & i\in\mathcal{P} \\ \frac{2}{t}\cdot\frac{3\beta_i|1+\mathbf{g}_i^*\mathbf{y}|^2-|\mathbf{f}_i^*\mathbf{x}|^2}{(\beta_i|1+\mathbf{g}_i^*\mathbf{y}|^2-|\mathbf{f}_i^*\mathbf{x}|^2)^2}\cdot\frac{|\mathbf{f}_i^*\mathbf{x}|^2}{|1+\mathbf{g}_i^*\mathbf{y}|^2} & i\in\mathcal{S} \\ 0 & \text{otherwise} \end{cases}$$

$$c_i = \begin{cases} \frac{-8\mathbf{f}_i^*\mathbf{x}(1+\mathbf{g}_i^*\mathbf{y})^*}{|\mathbf{f}_i^*\mathbf{x}|^2|1+\mathbf{g}_i^*\mathbf{y}|^2} & i\in\mathcal{P} \\ -\frac{4}{t}\cdot\frac{\beta_i\mathbf{f}_i^*\mathbf{x}(1+\mathbf{g}_i^*\mathbf{y})^*}{(\beta_i|1+\mathbf{g}_i^*\mathbf{y}|^2-|\mathbf{f}_i^*\mathbf{x}|^2)^2} & i\in\mathcal{S} \\ 0 & \text{otherwise} \end{cases}$$

Upon further examination, the Hessian matrix $\nabla^2 f_t$ defined in (8) is not positive definite. This is because

$$a_i b_i \le |c_i|^2, \forall i \in \mathcal{P}\cup\mathcal{S}$$

and the equality holds if $|\mathbf{f}_i^*\mathbf{x}| = |1+\mathbf{g}_i^*\mathbf{y}|$ for $i\in\mathcal{P}$ and $|\mathbf{f}_i^*\mathbf{x}|^2 = \beta_i|1+\mathbf{g}_i^*\mathbf{y}|^2$ for $i\in\mathcal{S}$. That is, the eigenvalues of the 2×2 matrix

$$\mathbf{T} \doteq \begin{pmatrix} a_i & c_i \\ c_i^* & b_i \end{pmatrix}, \tag{9}$$

denoted as λ_1, λ_2, are of different signs: $\lambda_1 \ge 0 \ge \lambda_2$. We thereby conclude that problem (2) is not convex.

We propose to approximate the 2×2 matrix (9) by a positive semi-definite matrix of the same dimension:

$$\tilde{\mathbf{T}} = \arg\min_{\mathbb{S}\in\mathbb{C}^{2\times2}:\mathbb{S}\succeq 0} ||\mathbf{S}-\mathbf{T}||_F^2.$$

Some arithmetic (details omitted due to lack of spaces) leads to the optimal solution

$$\tilde{\mathbf{T}} = \mathbf{T} - \frac{\lambda_2}{\mathbf{v}^*\mathbf{v}}\mathbf{v}\mathbf{v}^*, \text{where } \mathbf{v} = \begin{bmatrix} \lambda_1 - a \\ -c^* \end{bmatrix}.$$

We then replace (8) by the p.s.d. matrix and have

$$\tilde{\nabla}^2 f_t = \begin{bmatrix} \mathbf{H} \\ \mathbf{G} \end{bmatrix} \begin{bmatrix} \mathcal{D}(\tilde{\mathbf{a}}) & \mathcal{D}(\tilde{\mathbf{c}}) \\ \mathcal{D}^*(\tilde{\mathbf{c}}) & \mathcal{D}(\tilde{\mathbf{b}}) \end{bmatrix} \begin{bmatrix} \mathbf{H} \\ \mathbf{G} \end{bmatrix}^*. \tag{10}$$

Now that since the Hessian matrix is modified to be positive definite, we can apply Newton iteration to solve the barrier function optimization problem (3), which in turn solves the original problem (2).

To understand the rationale behind approximating $\mathbf{T}$ by $\tilde{\mathbf{T}}$, we note that the iterative minimization of (3) with respect to $\mathbf{z}$ is done by minimizing the local quadratic approximation

$$\mathbf{z}^*\nabla^2 f_t\mathbf{z} + 2\,\mathrm{Re}\{\nabla^* f_t\mathbf{z}\}. \tag{11}$$

Since $\nabla^2 f_t$ is not p.s.d., we replace (11) by

$$\mathbf{z}^*\tilde{\nabla}^2 f_t\mathbf{z} + 2\,\mathrm{Re}\{\nabla^* f_t\mathbf{z}\}, \tag{12}$$

which dominates (11) for any $\mathbf{z}$. Thus, minimizing (12) amounts to minimizing an upper bound, which guarantees monotonous decrease of the original barrier function (3).

We summarize this section with the pseudo-code for the proposed iterative design in Algorithm 1.

Algorithm 1: Proposed algorithm for IIR notch filter design.

Initialize: $\mathbf{x}_0 = [\frac{\sqrt{\min_{i\in\mathcal{S}}\beta_i}}{2}, 0, \cdots, 0]^T, \mathbf{y}_0 = \mathbf{0}$; $t=1, k=1$.
while $t < \frac{|\mathcal{S}|}{\epsilon}$ AND $k < N_{\max}$ **do**
  Compute $\mathbf{d} = [\nabla_{\mathbf{x}} f_t^T, \nabla_{\mathbf{y}} f_t^T]^T$ by (6) and (7);
  Compute the Hessian matrix $\tilde{\nabla}^2 f$ based on (10);
  Solve $\mathbf{u}_{nt} = \tilde{\nabla}^2 f^{-1}\mathbf{d}$ via conjugate gradient;
  if $\mathbf{u}_{nt}^*\mathbf{d} > \epsilon_{nt}$ **then**
    Find the step size s by the backtracking method;
    $\mathbf{x}_{k+1} = \mathbf{x}_k - s\mathbf{u}_{nt}(1:M)$;
    $\mathbf{y}_{k+1} = \mathbf{y}_k - s\mathbf{u}_{nt}(M+1:M+N)$;
    $k = k+1$;
  else
    $t = t\mu$;
  end
end

3. FIR NOTCH FILTER DESIGN

For the case of FIR notch filter, we have $Q\left(z^{-1}\right) = 1$ and the original optimization problem (2) is simplified to

$$\begin{array}{ll} \underset{\mathbf{x}}{\text{minimize}} & \sum_{i\in\mathcal{P}}\left(\log|\mathbf{f}_i^*\mathbf{x}|^2\right)^2 \\ \text{subject to} & |\mathbf{f}_i^*\mathbf{x}|^2 \le \beta_i, \forall i\in\mathcal{S} \end{array} \tag{13}$$

To verify the claim that (13) is a convex optimization problem, we again analyze the logarithmic barrier function

$$f_t(\mathbf{x}) \doteq \sum_{i\in\mathcal{P}}\left(\log|\mathbf{f}_i^*\mathbf{x}|^2\right)^2 - \frac{1}{t}\sum_{i\in\mathcal{S}}\log\left(\beta_i - |\mathbf{f}_i^*\mathbf{x}|^2\right). \tag{14}$$

The first order derivative of (14) is

$$\nabla_{\mathbf{x}} = \sum_{i\in\mathcal{P}}\log|\mathbf{f}_i^*\mathbf{x}|^2\cdot\frac{4\mathbf{f}_i^*\mathbf{x}}{|\mathbf{f}_i^*\mathbf{x}|^2}\mathbf{f}_i + \frac{1}{t}\sum_{i\in\mathcal{S}}\frac{2\mathbf{f}_i^*\mathbf{x}}{\beta_i - |\mathbf{f}_i^*\mathbf{x}|^2}\mathbf{f}_i.$$

It can be written in a matrix form as

$$\nabla_{\mathbf{x}} = \mathbf{G}\mathbf{q}_x,$$

where $\mathbf{q}_x$ is a length-L vector with entries

$$q_i = \begin{cases} \log|\mathbf{f}_i^*\mathbf{x}|^2\cdot\frac{4\mathbf{f}_i^*\mathbf{x}}{|\mathbf{f}_i^*\mathbf{x}|^2} & i\in\mathcal{P} \\ \frac{1}{t}\cdot\frac{2\mathbf{f}_i^*\mathbf{x}}{\beta_i-|\mathbf{f}_i^*\mathbf{x}|^2} & i\in\mathcal{S} \\ 0 & \text{otherwise} \end{cases}$$

The second order derivative is

$$\nabla^2 f_t = \mathbf{G}\mathcal{D}(\mathbf{a})\mathbf{G}^*$$

where $\mathcal{D}(\mathbf{a})$ is an $L \times L$ diagonal matrix whose diagonal elements are

$$a_i = \begin{cases} \left(2 - \log|\mathbf{f}_i^* \mathbf{x}|^2\right) \frac{4}{|\mathbf{f}_i^* \mathbf{x}|^2} & i \in \mathcal{P} \\ \frac{2}{t} \cdot \frac{\beta_i + |\mathbf{f}_i^* \mathbf{x}|^2}{(\beta_i - |\mathbf{f}_i^* \mathbf{x}|^2)^2} & i \in \mathcal{S} \\ 0 & \text{otherwise} \end{cases}$$

It is straightforward to verify that the Hessian matrix $\nabla^2 f_t$ is positive definite. Hence the FIR notch filter design is a convex problem and global optimum can be efficiently found via classic convex optimization tools, e.g., interior point method.

We conclude the discussion of the algorithm by remarking that there is no guarantee that the proposed approach will converge to a minimum phase filter. Hence the conversion to the minimum phase should be conducted as the final step using, e.g., the spectrum factorization method [10].

4. NUMERICAL EXAMPLES

Due to the space limitation, we only present two examples of IIR notch filter design to illustrate the effectiveness and superiority of our proposed design. More comprehensive comparisons, including the computational complexity, are reported in the journal version.

Example 1: We consider a design requirement which has two notch frequencies $\{\omega_1, \omega_2\} = \{0.3\pi, 0.8\pi\}$, with the corresponding 3dB bandwidth $\{dw_1, dw_2\} = \{0.2\pi, 0.1\pi\}$. Since both the cascading method and the genetic algorithm are based on the second-order all-pass filter, all algorithms in the simulation design an IIR filter of order $M = N = 4$. Note that this requirement is purely for the purpose of a fair comparison as cascading and GA must use order 2 for each notch [2, 11]. Fig. 1 shows the magnitude response of all three algorithms. The mask requirement for our new algorithm is also plotted for reference. It is clear that the new algorithm has better roll-off characteristics around the required edge. Furthermore, the magnitude in the pass-band moves around the target value, while the other methods always achieve lower magnitude. The variation of the magnitude in the pass-band is also smaller for our algorithm.

Example 2: We further compare the algorithms for higher-order multi-notch IIR filters with much lower notch levels. Note that GA does not work in this case, as the extension to $M, N > 2$ is non-trivial. The design requirements include two notch frequencies $\{\omega_1, \omega_2\} = \{0.45\pi, 0.85\pi\}$. It is required that the stop-band attenuation be -40dB and -20dB for both notches, respectively, and algorithms design an IIR filter of order $M = N = 16$. Fig. 2 gives the magnitude response of both cascading and the new algorithm. Again, we can observe that the new algorithm offers better performance in terms of the sharp roll-off at the required edge, deep attenuation in the stop-band, and better pass-band response.

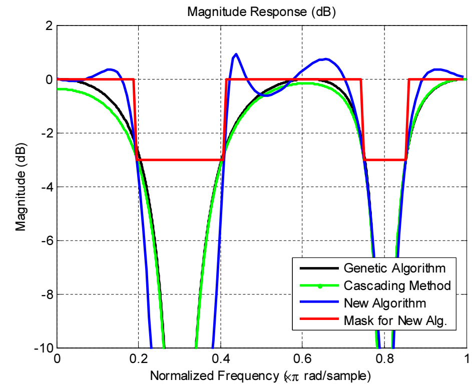

Fig. 1. Multi-notch IIR filter comparison, with M=N=6.

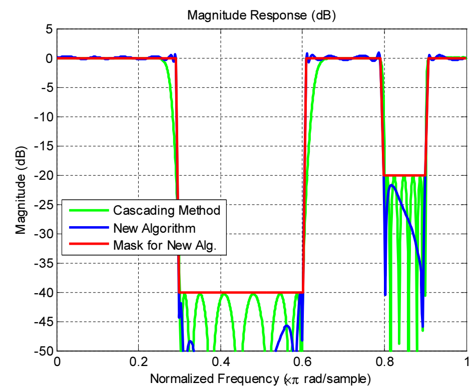

Fig. 2. Multi-notch IIR filter comparison, with M=N=16 and varying notch levels.

5. CONCLUSION

In this paper, we develop a novel optimization-based approach to coherently address the IIR and FIR notch filter design problem. The problem is formulated such that the pre-determined notch level, be it single or multiple notch, can be strictly guaranteed in the given stop-bands, while maintaining near unit magnitude response at the pass-bands. The technical difficulty arises from the non-convexity of the original optimization problem, and an approximation using the nearest 2×2 Hermitian matrix to replace the non-positive semi-definite one is proved to yield near optimal performance with low computational complexity. As a special case, the FIR notch filter design is further shown to be convex and hence renders optimal solution.

6. REFERENCES

[1] Y.V. Joshi and S.C. Dutta Roy, "Design of IIR multiple notch filters based on all-pass filters," *IEEE Trans. Circuits Syst. II*, vol. 46, no. 2, pp. 134–138, 1999.

[2] Chien-Cheng Tseng and Soo-Chang Pei, "Stable IIR notch filter design with optimal pole placement," *IEEE Trans. Signal Processing*, vol. 49, no. 11, pp. 2673–2681, November 2001.

[3] J. Levin, N.O. Perez-Arancibia, and P.A. Ioannou, "Adaptive notch filter using real-time parameter estimation," *IEEE Trans. Control Syst. Technol.*, vol. 19, no. 3, pp. 673–681, May 2011.

[4] D. Olguin, F. Bouchereau, and S. Martinez, "Adaptive notch filter for EEG signals based on the LMS algorithm with variable step-size parameter," in *Conference on Information Sciences and Systems (CISS)*, March 2005.

[5] T. Tholeti, P. Ganesh, and P. Ramanujam, "Frequency domain filtering techniques of halftone images," in *2nd IEEE SPIN*, Feb 2015, pp. 427–430.

[6] K.U. Ahmed and B.S. Virdee, "Ultra-wideband bandpass filter based on composite right/left handed transmission-line unit-cell," *IEEE Trans. Microw. Theory Techn.*, vol. 61, no. 2, pp. 782–788, Feb 2013.

[7] K. Hirano, S. Nishimura, and S.K. Mitra, "Design of digital notch filters," *IEEE Trans. Commun.*, vol. 22, no. 7, pp. 964–970, Jul 1974.

[8] Y.V. Joshi and S.C. Dutta Roy, "Design of IIR digital notch filters," *Circuits, Systems and Signal Processing*, vol. 16, no. 4, pp. 415–427, 1997.

[9] S. Boyd and L. Vandenberghe, *Convex Optimization*, Cambridge University Press, New York, NY, USA, 2004.

[10] S.-P. Wu, S. Boyd, and L. Vandenberghe, "FIR filter design via spectral factorization and convex optimization," in *Applied and Computational Control, Signals and Circuits*, B. Datta, Ed., pp. 215–245. Birkhauser, 1998.

[11] Q. Wang, J. Song, and H. Yuan, "Digital multiple notch filter design based on genetic algorithm," in *IEEE IMCCC*, Sept. 2014, pp. 180–183.