PARAMETER ESTIMATION OF POLYNOMIAL PHASE SIGNAL BASED ON LOW-COMPLEXITY LSU-EKF ALGORITHM IN ENTIRE IDENTIFIABLE REGION

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ABSTRACT

Fast implementation of parameter estimation for polynomial phase signal (PPS) is considered in this paper. A method which combines the least squares unwrapping (LSU) estimator and the extended Kalman filter (EKF) is proposed. A small number of initial samples are used to estimate the PPS's parameters and then these coarse estimates are used to initial the EKF. The proposed LSU-EKF estimator greatly reduces the computation complexity of the LSU estimator and can work in entire identifiable region which inherits from the LSU estimator. Meanwhile, in the EKF stage its output is in point-by-point wise which is useful in real applications.

Index Terms— polynomial phase signal (PPS), parameter estimation, least squares unwrapping (LSU), extended Kalman filter (EKF), identifiable region

1. INTRODUCTION

Polynomial phase signals are common in fields including radar, sonar, geophysics, radio communication and biology [1, 2, 3]. The estimation of the parameters of polynomial phase signals has received considerable interest in recent years and several methods have been proposed to solve the problem. Among these methods, the maximum likelihood estimator (MLE) [4, 5] is an effective approach, being both computationally efficient and statistically accurate. However, this method becomes extremely complex when the order of PPSs is over 1.

In order to estimate the parameters of higher-order polynomial phase signals, researchers have developed many more efficient alternative methods based on 'multilinear transforms' or 'phase unwrapping'. Examples of these multilinear transform techniques include the cubic phase function (CPF) [6], the higher-order ambiguity function (HAF) [7] and its product version (PHAF) [8], which have good performance with acceptable computation and can be adapted to process multiple component signals. Their main limitation is the small identifiable region on which they can operate, owing to the error propagation. One alternative method is phase unwrapping, which utilizes the fact that the phase of a complex sinusoid is a linear function that is wrapped modulo 2π. A classical estimator based on phase unwrapping is the least squares unwrapping estimator [9, 10]. A significant advantage of this approach is that it works for polynomial phase parameters contained anywhere inside the identifiable region. The estimator also appears to perform well after thresholds at low signal-to-noise ratio (SNR) are reached. The major drawback of the LSU estimator is that computing a nearest lattice point is, in general, computationally difficult. Especially, the algorithm becomes prohibitively expensive as the number of samples is more than 1000 [9].

The parametric estimation for polynomial phase signals affected by Gaussian noise by extended Kalman filtering has been investigated in [11]. In this paper, we propose a PPS parameter estimation method which combines the LSU and the EKF. Firstly, the coarse estimates of the PPS's parameters are obtained by the LSU estimator which is performed on a small number of samples. Then, these coarse estimates are used to initial the EKF. The proposed method can work in entire identifiable region which is inherent from the LSU estimator and the estimation is output in point-by-point wise in the EKF stage. The numerical simulations show that the proposed method operating at a low SNR threshold exhibits high performance with small computation, and it is accurate over a far wider range of parameters than many popular existing estimators.

2. SIGNAL MODEL

A received polynomial phase signal can be written as

$$y(n) = Ae^{2\pi j\theta(n)} + w(n) \tag{1}$$

where A is the amplitude of the signal and $w(n)$ is a complex additive white Gaussian noise (AWGN) with zero mean and variance σ^2. $\theta(n)$ is the deterministic polynomial phase of order m, which is expressed by

$$\theta(n) = u_0 + u_1 n + u_2 n^2 + \cdots + u_m n^m \tag{2}$$

where the coefficients u_i $(i = 0, \cdots, m)$ are assumed to be real and unknown, and their identifiable regions are written as

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[12]

$$-\frac{1}{2i!} \leq u_i < \frac{1}{2i!}(i = 0, 1, 2, \cdots m). \tag{3}$$

2.1. The State-Space Modelization

The discrete-time state vector $\boldsymbol{X}_k$ in the case of polynomial phase signals is given by the phase and the derivatives of the phase

$$\boldsymbol{X}_k = \begin{bmatrix} \theta(k) & \theta^{(1)}(k) & \theta^{(2)}(k) & \cdots & \theta^{(m)}(k) \end{bmatrix}^T \tag{4}$$

where $\theta^{(m)}(k)$ denotes m order derivation.

The model of polynomial phase signals can be described by a set of continuous-time differential equations cast in state-space form as

$$d\boldsymbol{X}(t)/dt = \boldsymbol{A}\boldsymbol{X}(t) + \boldsymbol{L}\boldsymbol{v}(t) \tag{5}$$

where the constant matrices $\boldsymbol{A}_{m+1,m+1}$ and $\boldsymbol{L}_{m+1,m}$, which characterize the behavior of the model, are written as

$$\boldsymbol{A} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix}, \boldsymbol{L} = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} \tag{6}$$

and $\boldsymbol{v}(t)$ is a white-noise process. There is a process-noise matrix $\boldsymbol{Q}_c$ that is related to the process-noise vector $\boldsymbol{v}(t)$ according to

$$\boldsymbol{Q}_c = E\big[\boldsymbol{v}(t)\boldsymbol{v}^T(t)\big]. \tag{7}$$

To be able to use the state (4), the model (5) must be discretized somehow [13], so that it can be described with a model of the form

$$\boldsymbol{X}_k = \boldsymbol{F}_k\boldsymbol{X}_{k-1} + \boldsymbol{v}_{k-1} \tag{8}$$

where $\boldsymbol{F}_k$ is the transition matrix of the dynamic model, which is evaluated as

$$\boldsymbol{F}_k = \boldsymbol{\vartheta}(T) = e^{\boldsymbol{A}T} = \boldsymbol{I} + \boldsymbol{A}T + \cdots + \frac{(\boldsymbol{A}T)^n}{n!} + \cdots \tag{9}$$

where T is the step-size of the discretization. On the other hand, $\boldsymbol{v}_{k-1}$ with variance $\boldsymbol{Q}_{k-1}$ is the discretized form of $\boldsymbol{v}(t)$ and $\boldsymbol{Q}_{k-1}$ satisfies

$$\boldsymbol{Q}_{k-1} = \int_0^T \boldsymbol{\vartheta}(\tau)\boldsymbol{L}\boldsymbol{Q}_c\boldsymbol{L}^{\boldsymbol{T}}\boldsymbol{\vartheta}^T(\tau)d(\tau). \tag{10}$$

Therefore, the state-space modelization of polynomial phase signals is linear.

2.2. The Observation Equations

For the signal (1) is complex, it is necessary to express the measured signal as a 2×1 vector in terms of its real and imaginary part

$$\boldsymbol{Y}_n = \begin{bmatrix} \mathrm{Re}(y(n)) & \mathrm{Im}(y(n)) \end{bmatrix}^T. \tag{11}$$

In this sense, the observation equation takes the form

$$\boldsymbol{Y}_k = h(\boldsymbol{X}_k) + \boldsymbol{\omega}_k \tag{12}$$

where the 2×1 nonlinear function $h(\boldsymbol{X}_k)$ is written as

$$h(\boldsymbol{X}_k) = \begin{bmatrix} h_{1k} \\ h_{2k} \end{bmatrix} = \begin{bmatrix} A\cos(2\pi\theta(k)) \\ A\sin(2\pi\theta(k)) \end{bmatrix}. \tag{13}$$

The observation noise vector can be defined as $\boldsymbol{\omega}_k = [w_R(k), w_I(k)]^T$, whose correlation matrix $\boldsymbol{R}_k$ is given by

$$\boldsymbol{R}_k = \begin{bmatrix} \frac{\sigma_\omega^2}{2} & 0 \\ 0 & \frac{\sigma_\omega^2}{2} \end{bmatrix}. \tag{14}$$

In this paper, we assume that the process-noise $\boldsymbol{v}_k$ and the observation noise $\boldsymbol{\omega}_k$ are not correlated, i.e.

$$\mathrm{cov}\big[\boldsymbol{\omega}_k, \boldsymbol{v}_j\big] = E\big[\boldsymbol{\omega}_k\boldsymbol{v}_j^T\big] = 0(k, j = 0, 1, 2, 3, \cdots). \tag{15}$$

Consequently, the observation model of polynomial phase signals is nonlinear.

3. THE EXTEND KALMAN FILTER ALGORITHM

As far as the observation is nonlinear, we use the extended Kalman filter algorithm to estimate the parameters of polynomial phase signals:

EKF Algorithm

Initial Conditions $(k = 0)$:

$$\begin{aligned} &\hat{\boldsymbol{X}}_0 = E(\boldsymbol{X}_0) \\ &\hat{\boldsymbol{P}}_0 = E\left[(\boldsymbol{X}_0 - \hat{\boldsymbol{X}}_0)(\boldsymbol{X}_0 - \hat{\boldsymbol{X}}_0)^T\right] \end{aligned} \tag{16}$$

Predict Equations:

$$\begin{aligned} &\boldsymbol{X}_k' = \boldsymbol{F}_k\hat{\boldsymbol{X}}_{k-1} \\ &\boldsymbol{P}_k' = \boldsymbol{F}_k\hat{\boldsymbol{P}}_{k-1}\boldsymbol{F}_k^T + \boldsymbol{Q}_{k-1} \\ &\boldsymbol{Y}_k' = \boldsymbol{H}_k\boldsymbol{X}_k' \\ &\boldsymbol{H}_k = \left.\frac{\delta h}{\delta \boldsymbol{X}}\right|_{\boldsymbol{X}_k'} \end{aligned} \tag{17}$$

Update Equations:

$$\begin{aligned} &\boldsymbol{K}_k = \boldsymbol{P}_k'\boldsymbol{H}_k^T \cdot \left[\boldsymbol{H}_k\boldsymbol{P}_k'\boldsymbol{H}_k^T + \boldsymbol{R}_k\right]^{-1} \\ &\hat{\boldsymbol{X}}_k = \boldsymbol{X}_k' + \boldsymbol{K}_k\left[\boldsymbol{Y}_k - \boldsymbol{Y}_k'\right] \\ &\hat{\boldsymbol{P}}_k = \left[\boldsymbol{I} - \boldsymbol{K}_k\boldsymbol{H}_k\right]\boldsymbol{P}_k' \end{aligned} \tag{18}$$

The relationship between the parameters $\boldsymbol{u} = [u_0, u_1, \cdots, u_m]^T$ of polynomial phase signals and the initial state vector $\boldsymbol{X}_0$ is described by

$$\boldsymbol{u} = \boldsymbol{C}\boldsymbol{X}_0 \tag{19}$$

where the matrix $\boldsymbol{C}$ is derived as

$$\boldsymbol{C} = diag\left(\begin{bmatrix} 1 & 1 & \frac{1}{2!} & \cdots & \frac{1}{m!} \end{bmatrix}\right). \tag{20}$$

Therefore, we can evaluate the parameters of polynomial phase signals in the form

$$\boldsymbol{u} = \boldsymbol{C} \times \left(\prod_{m=1}^{k} \boldsymbol{F}_m^{-1}\right) \times \hat{\boldsymbol{X}}_k. \tag{21}$$

The extended Kalman filter extends the scope of Kalman filter to nonlinear optimal filtering problems by forming a Gaussian approximation to the joint distribution of state $\boldsymbol{X}$ and measurements $\boldsymbol{Y}$ using a Taylor series based transformation. In this sense, the extended Kalman filter is sensitive to the initial conditions and if the filter is improperly initialized, the filtering may be failed. In this paper, in order to make the filter work effectively, we utilize the estimates of the least squares unwrapping estimator to initialize the EKF. Accounting for the computation load of the LSU estimator, only a few samples are used to obtain the initial estimates.

4. THE LEAST SQUARES UNWRAPPING (LSU) ESTIMATOR

The LSU estimator can estimate the parameters $\boldsymbol{u} = [u_0, u_1, \cdots, u_m]^T$ from the phase of the observations $y(1)$, $y(2)$, ..., $y(n)$. The phase θ_n of polynomial phase signals is written as

$$\theta_n = \angle y(n)/2\pi = \phi_n + \theta(n)(mod 2\pi) \tag{22}$$

where $\angle$ denotes the angle of a complex number, and ϕ_n are random variables representing the phase noise induced by $w(n)$ and can be calculated as

$$\phi_n = \angle\left(1 + \frac{w(n)}{Ae^{2\pi j\theta(n)}}\right) \Big/ 2\pi \tag{23}$$

where ϕ_n is known as the projected normal distribution [14], if $w(n)$ is complex Gaussian noise.

According to the fundamental of the least square method, the parameters u_0, u_1, ..., u_m can be evaluated by

$$SS(\boldsymbol{u}) = \text{argmin}\left(\sum\nolimits_{n=1}^{N}\left(\theta_n - \sum\nolimits_{k=0}^{m} u_k n^k\right)^2\right). \tag{24}$$

Due to the problem of the phase ambiguity in $\angle y(n)$, we may define

$$\left\langle \theta_n - \sum\nolimits_{k=0}^{m} u_k n^k \right\rangle = \theta_n - \sum\nolimits_{k=0}^{m} u_k n^k - \lfloor \varphi_n \rfloor \tag{25}$$

where $\lfloor x \rfloor$ denotes the nearest integer to x. By considering $\lfloor \varphi_n \rfloor = \left\lfloor \theta_n - \sum_{k=0}^{m} u_k n^k \right\rfloor$ as nuisance parameters, (24) can be rewritten as

$$SS(\boldsymbol{u}) = \text{argmin}\left(\sum\nolimits_{n=1}^{N}\left\langle \theta_n - \sum\nolimits_{k=0}^{m} u_k n^k \right\rangle^2\right) \tag{26}$$

where N represents the number of samples.

The LSU estimator is the minimum of $SS(\boldsymbol{u})$ over the identifiable region of PPSs, which can be represented as a nearest lattice point problem [15]. When the number of samples is less than 60, the sphere decoder [16] can exactly computes a nearest lattice point to evaluate the parameters u_0, u_1, ..., u_m. In the other way, if the number of samples is more than 60, we can apply the K-best algorithm [17] having complexity $O(N^3 \log N)$ for approximating the LSU estimator. As a result, the LSU estimator is observed to occur at larger operations as the number of samples, N, grows. Especially, the LSU estimator is not suitable to run for the number of samples over 1000 [9].

The mean square error (MSE) of the LSU estimator can be obtained by

$$MSE = \frac{P_{signal}}{2\left[1 - f\left(-\frac{1}{2}\right)\right]^2 10^{\frac{SNR}{10}}} \times \boldsymbol{D}^{-1}./\boldsymbol{G} \tag{27}$$

where P_{signal} denotes the power of the polynomial phase signal, $f(\cdot)$ is the probability density function of ϕ_n and $\boldsymbol{A}./\boldsymbol{B}$ means element-by-element division between matrix $\boldsymbol{A}$ and matrix $\boldsymbol{B}$. The superscript -1 denotes the inverse operation and $\times$ denotes the product between a scalar and a matrix. Matrix $\boldsymbol{D}$ is a $m+1$ by $m+1$ Hilbert matrix with elements

$$D_{i,j} = 1/(i+j+1)(i,j \in [0,1,2,\cdots,m]) \tag{28}$$

and the matrix $\boldsymbol{G}$ can be defined as

$$\boldsymbol{G} = \begin{bmatrix} N & N^2 & \cdots & N^{m+1} \\ N^2 & N^3 & \cdots & N^{m+2} \\ \vdots & \vdots & \ddots & \vdots \\ N^{m+1} & N^{m+2} & \cdots & N^{2m+1} \end{bmatrix}. \tag{29}$$

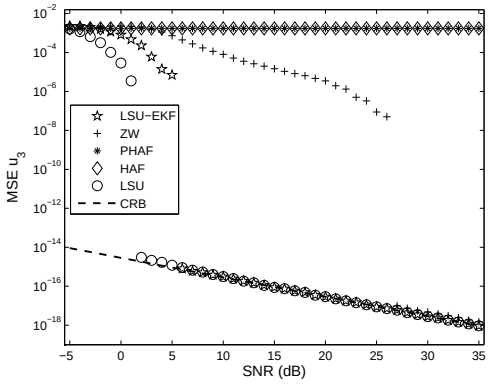
According to (27), to make the EKF work effectively at a low SNR, we should choose more samples for the LSU estimator. However, larger sample number means larger computation load. In this paper, the number of samples used for the LSU estimator is $N_1 = 64$.

5. SIMULATION

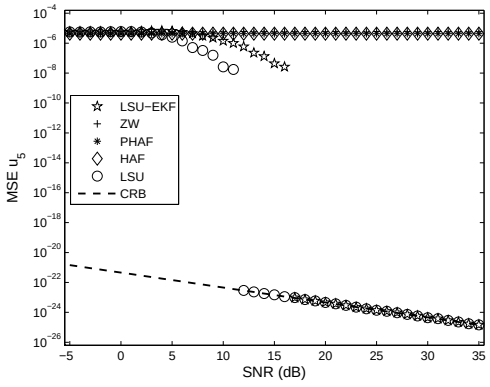
In this section, Monte-Carlo simulations are performed to verify the characteristic of our method. The numerical values are obtained over 2000 simulations. The number of samples used in the simulations is 199 and the number of initial samples used in our method is 64.

Fig.1 shows the MSEs of the LSU estimator, the HAF estimator, the PHAF estimator, and the proposed estimator, where the values of coefficients are $\boldsymbol{u} = \left[\frac{1}{4}, \frac{1}{4}, \frac{1}{8}, \frac{1}{24}, \frac{1}{96}, \frac{1}{480}\right]^T$ for the 5-order PPSs and $\boldsymbol{u} = \left[\frac{1}{4}, \frac{1}{4}, \frac{1}{8}, \frac{1}{24}\right]^T$ for the 3-order PPSs. We can see that the HAF and PHAF estimators fail consistently, as expected. Although the ZW estimator [18] can operate on the entire identifiable region, its performance is poor, including its MSE and the threshold SNR.

The proposed LSU-EKF estimator reaches a threshold at 6 dB for the 3-order PPS and 17 dB for the 5-order PPS. The corresponding threshold SNRs for the LSU estimator are 2 dB for the 3-order PPS and 12 dB for the 5-order PPS respectively. The reason why the threshold SNRs of the LSU-EKF estimator are higher than those of the LSU estimator is that the number of samples used for the LSU estimator is larger than that for the LSU-EKF estimator in the parameter initialization step. Table 1 shows the running time of all the estimators, it is shown that the running time of the LSU estimator is approximately 24 times as much as that of the LSU-EKF estimator for the 3-order PPS and 15 times for the 5-order PPS.



(a)



(b)

Fig. 1: Sample MSEs for the coefficient u_3 for PPSs of order 3 and the coefficient u_5 for PPSs of order 5.

Table 1: Running time in seconds with different methods for PPSs of order 3 and PPSs of order 5.

	HAF	**PHAF**	**ZW**	**LSU**	**LSU-EKF**
$\boldsymbol{m = 3}$	100.4s	180.5s	119.6s	25624.1s	1063.7s
$\boldsymbol{m = 5}$	114.3s	174.1s	177.4s	44883.4s	3113.7s

In order to make the volume of the set of coefficients suit for the HAF and PHAF estimator, we chose the coefficients $u = \left[\frac{1}{4}, \frac{1}{4}, \frac{1}{8N}, \frac{1}{24N^2}, \frac{1}{96N^3}, \frac{1}{480N^4}\right]^T$ for the 5-order PPSs and $u = \left[\frac{1}{4}, \frac{1}{4}, \frac{1}{8N}, \frac{1}{24N^2}\right]^T$ for the 3-order PPSs. In Fig.2, we can see that the HAF estimator reaches a threshold at 6 dB, which is the same as our method for the 3-order PPSs and a threshold at 18 dB approximately a little higher than our method for the 5-order PPS. Meantime, the PHAF estimator has a good performance on the estimation of the coefficients u_3 for the 3-order PPS and u_5 for the 5-order PPS. As expected, the performance of our method remains unchanged for the set of small coefficients.

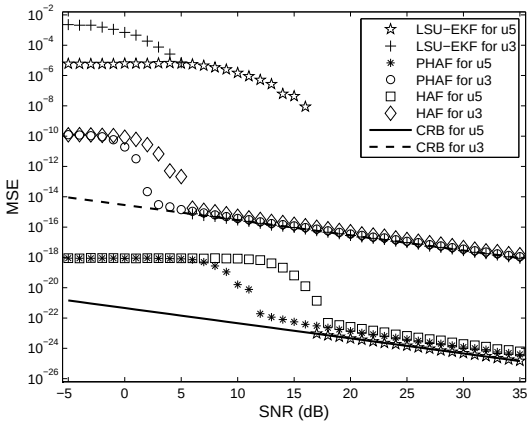


Fig. 2: Sample MSEs for the coefficient u_3 for PPSs of order 3 and the coefficient u_5 for PPSs of order 5.

6. CONCLUSION

The proposed LSU-EKF estimator greatly reduces the computation complexity of the LSU estimator while at the same time it can still work in entire identifiable region. The threshold SNR of our method is a few decibels higher than the LSU estimator because fewer samples are used in the LSU stage of the LSU-EKF estimator compared to the LSU estimator. Since the number of samples to initial the EKF is relative with the threshold SNR and the computation complexity, the size of the initial sample number should be careful chosen according to the tradeoff between these two factors.

7. REFERENCES

[1] Anish Turlapaty and Yuanwei Jin, "Parameter estimation and waveform design for cognitive radar by minimal free-energy principle," in *Acoustics, Speech and Signal Processing (ICASSP), 2013 IEEE International Conference on*. IEEE, 2013, pp. 6244–6248.

[2] Xin Yuan, "Polynomial-phase signal source tracking using an electromagnetic vector-sensor," in *2012 IEEE International Conference on Acoustics, Speech and Signal Processing (ICASSP)*, 2012.

[3] Shuai Yao, Shiliang Fang, Xiaoyan Wang, and Li Wang, "Parameter estimation for hfm signals using combined stft and iteratively reweighted least squares linear fitting," *Signal Processing*, vol. 99, pp. 92–102, 2014.

[4] Marko Simeunovic, Slobodan Djukanovic, and Igor Djurovic, "Quasi-maximum likelihood estimator of multiple polynomial-phase signals," in *Acoustics, Speech and Signal Processing (ICASSP), 2014 IEEE International Conference on*. IEEE, 2014, pp. 4200–4203.

[5] François Vincent, Olivier Besson, and Eric Chaumette, "Approximate maximum likelihood estimation of two closely spaced sources," *Signal Processing*, vol. 97, pp. 83–90, 2014.

[6] Igor Djurović, Pu Wang, and Cornel Ioana, "Parameter estimation of 2-d cubic phase signal using cubic phase function with genetic algorithm," *Signal Processing*, vol. 90, no. 9, pp. 2698–2707, 2010.

[7] Cornel Ioana, Cédric Cornu, and André Quinquis, "Polynomial phase signal processing via warped high-order ambiguity function," in *Signal Processing Conference, 2004 12th European*. IEEE, 2004, pp. 1159–1162.

[8] Sergio Barbarossa, Anna Scaglione, and Georgios B Giannakis, "Product high-order ambiguity function for multicomponent polynomial-phase signal modeling," *Signal Processing, IEEE Transactions on*, vol. 46, no. 3, pp. 691–708, 1998.

[9] Robby G McKilliam, Barry G Quinn, Bill Moran, Badri N Vellambi, et al., "Polynomial phase estimation by least squares phase unwrapping," *Signal Processing, IEEE Transactions on*, vol. 62, no. 8, pp. 1962–1975, 2014.

[10] Robby G McKilliam, Barry G Quinn, Bill Moran, et al., "Frequency estimation by phase unwrapping," *Signal Processing, IEEE Transactions on*, vol. 58, no. 6, pp. 2953–2963, 2010.

[11] Peter J Kootsookos and Joanna Spanjaard, "An extended kalman filter for demodulation of polynomial phase signals," *IEEE Signal Processing Letters*, vol. 5, no. 3, pp. 69–70, 1998.

[12] Robby G McKilliam et al., "Identifiability and aliasing in polynomial-phase signals," *Signal Processing, IEEE Transactions on*, vol. 57, no. 11, pp. 4554–4557, 2009.

[13] Zarchan Paul and Musoff Howard, "Fundamentals of kalman filtering: A practical approach," *Virginia, Published by the American Institute of Aeronautics and Astronautics*, 2005.

[14] Kanti V Mardia and Peter E Jupp, *Directional statistics*, vol. 494, John Wiley & Sons, 2009.

[15] Erik Agrell, Thomas Eriksson, Alexander Vardy, and Kenneth Zeger, "Closest point search in lattices," *Information Theory, IEEE Transactions on*, vol. 48, no. 8, pp. 2201–2214, 2002.

[16] Gianmarco Romano, Domenico Ciuonzo, Pierluigi Salvo Rossi, and Francesco Palmieri, "Low-complexity dominance-based sphere decoder for mimo systems," *Signal Processing*, vol. 93, no. 9, pp. 2500–2509, 2013.

[17] Zhan Guo and Peter Nilsson, "Algorithm and implementation of the k-best sphere decoding for mimo detection," *Selected Areas in Communications, IEEE Journal on*, vol. 24, no. 3, pp. 491–503, 2006.

[18] G Tong Zhou and Yang Wang, "Exploring lag diversity in the high-order ambiguity function for polynomial phase signals," in *Higher-Order Statistics, 1997., Proceedings of the IEEE Signal Processing Workshop on*. IEEE, 1997, pp. 103–106.