SPARSE PHASE RETRIEVAL WITH NEAR MINIMAL MEASUREMENTS: A STRUCTURED SAMPLING BASED APPROACH

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ABSTRACT

The problem of sparse phase retrieval is considered where the goal is to recover a sparse complex valued vector (with s non zero elements) from the magnitudes of its linear measurements. Using a modified and partially randomized version of a newly proposed structured sampler, namely the Partial Nested Fourier Sampler (PNFS), it is shown to be possible to recover the unknown signal (up to a global phase ambiguity) from $O(s\log N)$ phaseless measurements where N is the dimension of the vector. The reconstruction is based on a novel idea of "decoupling" certain quadratic terms in the phaseless measurements acquired by the PNFS, leading to a simple l_1-minimization-based recovery algorithm, without the need for "lifting" the unknown variable to a higher dimensional space. The proposed algorithm is also proved to be stable in presence of bounded noise.

Index Terms— Sparse Phase Retrieval, Nested Sampling, l_1 minimization, Decoupling, Random Partial Fourier Measurements.

1. INTRODUCTION

The problem of reconstructing an unknown signal (up to a global phase) from its phaseless measurements has been studied for decades owing to its wide applications in many areas of imaging science such as X-ray crystallography, diffraction imaging and molecular imaging, and so forth [1, 2, 3]. The problem can be studied under various settings by considering a real or complex signal model, with or without sparsity constraints. A comprehensive review of existing measurement strategies and reconstruction algorithms for phase retrieval is provided in [4].

A central goal in phase retrieval problems is to develop an effective measurement strategy and a recovery algorithm which can provably recover the unknown signal with *minimal* number of measurements. Recent approaches based on the elegant idea of "lifting" can provably recover (non-sparse) signals of dimension N using $O(N)$ or $O(N\log N)$ measurements [3, 5, 6, 7], by solving an appropriate convex problem in the "lifted" variable.

Recovering a sparse signal from its phaseless measurements with near optimal number of measurements (which is $O(s)$ up to a logarithmic factor) is a challenging problem that has received much attention in recent times [8, 9, 10, 11]. In fact, it becomes necessary to impose a sparse prior on the unknown signal to ensure its unique recovery, when Fourier measurements are used. An l_1-minimization-based approach for sparse phase retrieval is proposed in [11], which requires $O(s^2)$ measurements along with an additional Collision-Free-Condition [12] on the autocorrelation of the unknown signal. In [8, 9, 10], the authors use a graph-decoding based approach which requires the sparsity to be at most $O(\sqrt{N})$. Recent iterative approaches using alternating minimization also require the number of measurements to grow quadratically in s [13]. In [14, 15, 16], the problem of sparse phase retrieval is cast as a joint low-rank+sparse matrix recovery problem which minimizes the weighted linear combination of the nuclear norm and l_1 norm of an appropriate lifted variable. However, as pointed out in [17], convex optimization based techniques for such simultaneously structured models (low rank+sparse) will necessarily require the number of measurements to be at least quadratic in s. Very recently, concurrent with our own work on Partial Nested Fourier Sampler (PNFS), a promising approach to overcome this limitation has been suggested in [18], where by using constrained measurement vectors and a two-step recovery algorithm, the authors can guarantee unique solution to the sparse phase retrieval problem using only $O(s\log(N/s))$ measurements.

We recently introduced a new family of structured Fourier-like samplers, namely the Partial Nested Fourier Sampler, which can provably recover a *non-sparse* signal of length N from its phaseless measurements using only $4N-5$ measurements [19]. In this paper, we further develop the theory of PNFS for sparse phase retrieval by proposing a randomized version of the basic PNFS, namely the R-PNFS. By using a certain "decoupling" property of the R-PNFS, along with a new "cancellation" based algorithm (that effectively cancels out certain unwanted quadratic terms in the autocorrelation of the signal), we are able to demonstrate that $O(s\log N)$

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measurements are sufficient to recover the sparse signal with probability 1. We also prove that the proposed algorithm is stable in presence of bounded noise, and present numerical simulations to validate the theoretical claims.

2. PROBLEM SETTING AND REVIEW OF PARTIAL NESTED FOURIER SAMPLERS

2.1. Problem Setting

Let $\mathbf{x} \in \mathbb{C}^N$ be a complex valued vector which may be sparse. Given M measurement vectors $\mathbf{f}_i \in \mathbb{C}^N, i = 1, 2, \cdots, M$, we obtain M noisy magnitude measurements as [5, 20, 15]

$$y_i = |\langle \mathbf{x}, \mathbf{f}_i \rangle|^2 + n_i \quad (1)$$

where n_i denotes the additive noise. The fundamental objective of phase retrieval problem is to recover $\mathbf{x}$ from $y_i, i = 1, 2, \cdots, M$. It is well known that for complex $\mathbf{x}$, we can only recover $\mathbf{x}$ up to a global phase ambiguity [1]. We can equivalently express the measurements as

$$y_i = \left(\mathbf{f_i}^T \otimes \mathbf{f_i}^H\right) \text{Vec}\left(\mathbf{x}\mathbf{x}^H\right) + n_i \quad (2)$$

where $\otimes$ denotes the Kronecker product and Vec$(\cdot)$ is the column-wise vectorized form of a matrix.

2.2. Limitations of Fourier Sampling based phase retrieval

Sparse phase retrieval problem was first studied in the context of Fourier measurement vectors [21, 8, 11, 22]. In the Fourier based phase retrieval problem, we collect measurements $y_i, 1 \leqslant i \leqslant M$ using an (oversampled) Fourier sampling vector [4, 3, 21, 8, 23, 22, 24]

$$\mathbf{f}_i = \alpha \left[1, z_i, z_i^2 \cdots, z_i^{N-1}\right]^T$$

Here $z_i = e^{j2\pi i/M}$ where $M \geqslant N$ is the oversampling factor.

It is readily seen that the Fourier based phase retrieval problem is equivalent to recovering a zero padded (if $M > N$) $\mathbf{x}$ from its autocorrelation sequence. This problem has an inherent ambiguity since two distinct finite length signals $x_1[n]$ and $x_2[n]$ (with same length N) can exhibit identical autocorrelation. This can be seen from the fact that the polynomial $X_1(z)\bar{X}_1(1/\bar{z})$ (denoting the $z-$ transform of the autocorrelation of $x_1[n]$ and $\bar{\cdot}$ is conjugate) can be decomposed into two spectral factors of same length in more than one way. To remove this ambiguity, it is necessary to impose additional priors on the signal $\mathbf{x}$.

Sparsity as a prior: A popular prior knowledge used in recent literature is that $\mathbf{x} \in \mathbb{C}^N$ is sparse with $s < N$ non zero elements. However, even with sparse priors, it is non trivial to ensure unique recovery of $\mathbf{x}$ from its autocorrelation, since the autocorrelation may not be sparse. To remedy this, a "Collision-Free Condition" (CFC) is further imposed in literature[11, 12]. Under this condition, for $s \neq 6$, $\mathbf{x}$ can be uniquely recovered from M Fourier measurements where $M \geqslant s^2 - s + 1$, and M is a prime integer [11, 4].

Drawbacks: A major drawback of CFC is that it imposes an upper bound on the sparsity of $\mathbf{x}$ that we can only recover sufficiently sparse vectors whose sparsity can be at most $s = O(\sqrt{N})$. In practice, the no-collision property may only hold for even smaller values of s as experimentally validated in [19]. Secondly, even with CFC, the l_1-minimization-based recovery algorithm proposed in [11] requires $M = O(s^2)$ measurements, which is larger than the degrees of freedom in a sparse $\mathbf{x}$.

2.3. Partial Nested Fourier Sampler

To overcome the limitations of Fourier based phaseless measurements, we recently proposed a new class of Fourier-like sampler, coined as "Partial Nested Fourier Sampler (PNFS)":

Definition 1. *[19]* **(Partial Nested Fourier Sampler:)** *A Partial Nested Fourier Sampler (PNFS) of dimension N, consists of measurement vectors given by*

$$\mathbf{f}_i^{(N)} = \frac{1}{\sqrt[4]{4N-5}}[z_i^1, z_i^2, \cdots z_i^{N-1}, z_i^{2N-2}]^T, \quad (3)$$

where $z_i = e^{j2\pi n_i/4N-5}, n_i \in [0, 4N-6]$.

In [19], we showed that $M = 4N - 5$ PNFS measurements are sufficient to uniquely recover a *non-sparse* $\mathbf{x}$. We also partially addressed the problem of sparse phase retrieval under certain prior knowledge about sparse $\mathbf{x}$ in the noiseless setting. We now further develop the theory of sparse phase retrieval by removing the need for any prior knowledge and developing stability results in presence of noise.

3. SPARSE PHASE RETRIEVAL USING RANDOMIZED PNFS

We introduce a randomized version of the PNFS for sparse phase retrieval as follows:

Definition 2. *(***Randomized PNFS***) A Randomized PNFS (R-PNFS) consists of measurement vectors*

$$\mathbf{f}_i^{\textit{(R-PNFS)}} = [\mathbf{I}_{N,N} \quad \mathbf{v}]\, \mathbf{f}_i^{(N+1)}$$

where $\mathbf{v} \in \mathbb{C}^N$ is a random vector with independent entries, and $\mathbf{f}_i^{(N+1)}$ is defined in (3) for dimension $N+1$.

Given the unknown signal $\mathbf{x}^\star \in \mathbb{C}^N$, the phaseless measurement obtained using a R-PNFS vector can be expressed as

$$y_i = \left|\left(\mathbf{f}_i^{\text{R-PNFS}}\right)^H \mathbf{x}^\star\right|^2 + n_i = \left|\mathbf{f}_i^{(N+1)H} \begin{bmatrix} \mathbf{x}^\star \\ \mathbf{v}^H\mathbf{x}^\star \end{bmatrix}\right|^2 + n_i \quad (4)$$

The basic idea of R-PNFS is to concatenate an extra element $x_{N+1} = \mathbf{v}^H\mathbf{x}^\star$ to form the vector $\mathbf{x} = [\mathbf{x}^{\star T} \quad x_{N+1}]^T$, and then measure $\mathbf{x}$ using PNFS for dimension $N+1$. Since the elements of $\mathbf{v}$ are independent random variables, it follows that the last entry of $\mathbf{x}$ satisfies $x_{N+1} \neq 0$ with probability 1. This enables us to devise an efficient cancellation based algorithm for sparse phase retrieval as follows.

3.1. A Cancellation Based Algorithm for R-PNFS

We measure a sparse $\mathbf{x}^\star$ (with s non zero elements) using two sets of PNFS samplers, and perform sparse recovery on the difference between the two measurements. This enables us to "cancel" out certain non-zero terms in the autocorrelation of $\mathbf{x}^\star$ and retain only "decoupled terms" (singletons) which have a maximum sparsity of $2s+1$. We begin by introducing a second sampling vector $\tilde{\mathbf{f}}_i^{(N+1)} \in \mathbb{C}^{N+1}$ as

$$\tilde{\mathbf{f}}_i^{(N+1)} = [\mathbf{I}_{N,N} \quad \mathbf{0}]\, \mathbf{f}_i^{(N+1)}$$

This sampler can be thought of as a masked version of the PNFS sampler defined in (3). Following are the main steps of the algorithm:

1. Collect two sets of (noisy) phaseless measurements $\mathbf{y}^{(1)}, \mathbf{y}^{(2)} \in \mathbb{C}^{\tilde{M}}$ as

$$\mathbf{y}_i^{(1)} = \left|\left(\mathbf{f}_i^{\text{R-PNFS}}\right)^H \mathbf{x}^\star\right|^2 + \mathbf{n}_i^{(1)}$$
$$\mathbf{y}_i^{(2)} = \left|\left(\tilde{\mathbf{f}}_i^{(N+1)}\right)^H \mathbf{x}^\star\right|^2 + \mathbf{n}_i^{(2)} \tag{5}$$

   We assume the noise is bounded, i.e. $|\mathbf{n}_i^{(k)}| \leqslant \eta, k = 1,2$. Notice that we collect a total of $M = 2\tilde{M}$ measurements.

2. Compute the difference measurement $\mathbf{\Delta y} = \mathbf{y}^{(1)} - \mathbf{y}^{(2)}$. The *key step is to notice that*

$$\mathbf{\Delta y} = \mathbf{Z}\hat{\mathbf{x}} + \mathbf{\Delta n} \tag{6}$$

   where the unknown vector $\hat{\mathbf{x}} \in \mathbf{C}^{4N-1}$ consists only of "decoupled" quadratic terms (singletons of the form $\bar{x}_{N+1}x_i, i = 1,2,\cdots,N$) given by

$$[\hat{\mathbf{x}}]_m = \begin{cases} |x_{N+1}|^2 & m = 0 \\ 0 & m = 1,2,\cdots,N-1 \\ x_{2N-m}\bar{x}_{N+1} & m = N,\cdots,2N-1 \\ \overline{[\hat{\mathbf{x}}]}_{-m} & m < 0 \end{cases}$$

   Since $x_{N+1} = \mathbf{v}^H\mathbf{x}^\star$ where $\mathbf{v}$ is a random vector with independent entries, it holds that $x_{N+1} \neq 0$ with probability 1. Hence $\hat{\mathbf{x}}$ has exactly $2s+1$ non zero elements. We also have $\mathbf{\Delta n} = \mathbf{n}^{(1)} - \mathbf{n}^{(2)}$, and the matrix $\mathbf{Z} \in \mathbb{C}^{\tilde{M},4N-1}$ is a partial DFT matrix with $[\mathbf{Z}]_{i,k} = \frac{1}{\sqrt{4N-1}} e^{j2\pi\frac{n_i k}{4N-1}}$.

3. Obtain an estimate of $\hat{\mathbf{x}}$ as the solution to the following l_1-minimization problem:

$$\min_\theta \|\theta\|_1 \quad \text{subject to } \|\mathbf{\Delta y} - \mathbf{Z}\theta\|_2 \leqslant \eta\sqrt{\tilde{M}} \quad (\mathbf{P1})$$

4. Given the solution $\hat{\mathbf{x}}^\#$ to (**P1**), the estimate for each entry of $\mathbf{x}^\star$ is given by $x_q^\# = [\hat{\mathbf{x}}^\#]_{2N-q}/\sqrt{[\hat{\mathbf{x}}^\#]_0}$ for $1 \leqslant q \leqslant N$ and $x_{N+1}^\# = \sqrt{[\hat{\mathbf{x}}^\#]_0}$.

3.2. Stability of Noisy Phase Retrieval with R-PNFS

To analyze the performance of the proposed algorithm, we use the following lemma from [25] which is tailored for the form (**P1**):

Lemma 1. *[25] Consider a sparse $\hat{\mathbf{x}} \in \mathbb{C}^{4N-1}$ with $2s+1$ non zero elements and $\mathbf{Z} \in \mathbb{C}^{\tilde{M},4N-1}$ be the DFT matrix with M rows whose indices are chosen uniformly at random from $[0, 4N-2]$. If $\tilde{M} \geqslant c_0(2s+1)\log(4N-1)\log(\varepsilon^{-1})$, then with probability at least $1-\varepsilon$, the solution $\hat{\mathbf{x}}^\#$ of (**P1**) satisfies*

$$\|\hat{\mathbf{x}} - \hat{\mathbf{x}}^\#\|_2 \leqslant c_1\sqrt{2s+1}\,\eta \tag{7}$$

where c_0, c_1 are universal constants.

Theorem 1. *Given a sparse $\mathbf{x}^\star \in \mathbb{C}^N$ (with s non zeros), and the measurement vector $\mathbf{v} \in \mathbb{C}^N$, consider the measurement model (5) where the indices n_i of $\mathbf{f}_i^{(N+1)}, i = 1,2\cdots, M$ are chosen uniformly at random from $[0, 4N-2]$. If $\tilde{M} \geqslant c_0(2s+1)\log(4N-1)\log(\varepsilon^{-1})$ and $|x_{N+1}|^2 > c_1\sqrt{2s+1}\eta$, with probability at least $1-\varepsilon$, the estimates $x_q^\#$ of $x_q^\star, 1 \leqslant q \leqslant N$, satisfy*

$$\sum_{q=1}^N |x_q^\star - e^{j\phi_0}x_q^\#| \leqslant \frac{c_1\sqrt{(2s+1)(4N-1)}}{\sqrt{|x_{N+1}|^2 - c_1\sqrt{2s+1}\eta}}\eta + \|\mathbf{x}^\star\|_1\left(\frac{1}{\sqrt{1 - c_1\frac{\sqrt{2s+1}\eta}{|x_{N+1}|^2}}} - 1\right) \tag{8}$$

where $x_{N+1} = \mathbf{v}^H\mathbf{x}^\star$, $\phi_0 = arg_{\phi\in[0,2\pi)} x_{N+1}/|x_{N+1}|$, and c_0, c_1 are universal constants given in Lemma 1.

Proof. According to the proposed algorithm, the estimate for each entry of $\mathbf{x}^\star$ is given by $x_q^\# = [\hat{\mathbf{x}}^\#]_{2N-q}/\sqrt{[\hat{\mathbf{x}}^\#]_0}$ for $1 \leqslant q \leqslant N$. Then, we have

$$|x_q^\star - e^{j\phi_0}x_q^\#| = |\frac{x_{N+1}}{|x_{N+1}|}\left(\frac{[\hat{\mathbf{x}}]_{2N-q}}{|x_{N+1}|} - \frac{[\hat{\mathbf{x}}^\#]_{2N-q}}{|x_{N+1}^\#|}\right)| \leqslant \beta\frac{\epsilon_{2N-q}}{|x_{N+1}|} + |1-\beta||x_q^\star| \tag{9}$$

where $\epsilon_{2N-q} \triangleq |[\hat{\mathbf{x}}]_{2N-q} - [\hat{\mathbf{x}}^\#]_{2N-q}|$ and $\beta = \frac{|x_{N+1}|}{|x_{N+1}^\#|}$. It follows that

$$\sum_{q=1}^N |x_q^\star - e^{j\phi_0}x_q^\#| \leqslant \beta\frac{\sum_{q=1}^N \epsilon_{2N-q}}{|x_{N+1}|} + |1-\beta|\sum_{q=1}^N |x_q^\star| \leqslant \beta\frac{\|\hat{\mathbf{x}} - \hat{\mathbf{x}}^\#\|_1}{|x_{N+1}|} + |1-\beta|\|\mathbf{x}^\star\|_1 \tag{10}$$

Since $\|\hat{\mathbf{x}} - \hat{\mathbf{x}}^{\#}\|_1 \leqslant \sqrt{4N-1}\|\hat{\mathbf{x}} - \hat{\mathbf{x}}^{\#}\|_2$, Lemma 1 gives us

$$\sum_{q=1}^{N} |x_q^\star - e^{j\phi_0} x_q^{\#}| \leqslant c_1\beta \frac{\sqrt{(2s+1)(4N-1)}\eta}{|x_{N+1}|} + |1-\beta|\|\mathbf{x}^\star\|_1 \quad (11)$$

Since $[\hat{\mathbf{x}}]_0 = |x_{N+1}|^2$, it follows from Lemma 1 that $|1 - \frac{1}{\beta^2}| \leqslant c_1 \frac{\sqrt{2s+1}\eta}{|x_{N+1}|^2}$. If $|x_{N+1}|^2 > c_1\sqrt{2s+1}\eta$, we have

$$1 - c_1 \frac{\sqrt{2s+1}\eta}{|x_{N+1}|^2} \leqslant \frac{1}{\beta^2} \leqslant 1 + c_1 \frac{\sqrt{2s+1}\eta}{|x_{N+1}|^2} \quad (12)$$

The proof completes by plugging (12) in (11). □

Remark 1. In absence of noise, setting $\eta = 0$ in (8) implies exact recovery of $\mathbf{x}^\star$ with a global phase ambiguity ϕ_0 which is explicitly given. This is achieved using a total of $M = 2\tilde{M}$ measurements, where $\tilde{M} = O(s \log N)$. Hence, our algorithm recovers $\mathbf{x}^\star$ with an order-wise minimal (up to a logarithmic factor) number of measurements.

Remark 2. Unlike "lifting" based approaches [18, 3], our method is based on l_1-minimization with only $O(N)$ variables. This implies significant computational saving and allows faster implementation.

4. NUMERICAL RESULTS

We consider a complex valued signal $\mathbf{x}^\star \in \mathbb{C}^N$ with s non zero elements, and $\|\mathbf{x}^\star\|_2 = 1$. Both the nonzero indices and amplitudes are generated randomly.

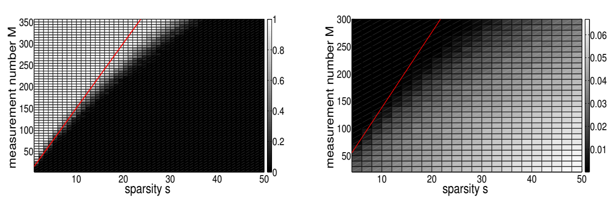

Fig. 1. (Left) Phase transition for noiseless case, averaged over 100 runs with $N = 150$. White and black boxes denote success rates of 1 and 0 respectively. (Right) Phase transition for noisy case averaged over 50 runs with $N = 100$, and entry-wise noise bounded by 0.01. Each box denotes $\frac{1}{N}\sum_{q=1}^{N} |x_q^\star - e^{j\phi_0} x_q^{\#}|$. The red line represents $M = 3s \log N$ for both.

The phase transition plots of the proposed method for both noiseless and noisy signal models is depicted in Fig.1. Here $N = 100$. In the noiseless setting, for each M and s, we declare success if $\max_q |x_q^\star - e^{j\phi_0} x_q^{\#}| < 10^{-6}$. For the noisy model, we assume the entry wise noise to be upper bounded by $\epsilon = 0.01$ and plot the reconstruction error $\frac{1}{N}\sum_{q=1}^{N} |x_q^\star - e^{j\phi_0} x_q^{\#}|$. We also superpose the line corresponding to $M = 3s \log N$ to demonstrate that the proposed approach recovers the true $\mathbf{x}^\star$ with $M = O(s \log N)$ measurements.

In Fig. 2, we show an example of sparse phase retrieval using the proposed R-PNFS sampler and cancellation based algorithm. Here $N = 350, s = 6, M = 100$. It can be seen that the proposed technique recovers the true $\mathbf{x}^\star$ faithfully up to a global phase ambiguity, the value of which is easily obtained from the complex plane representation.

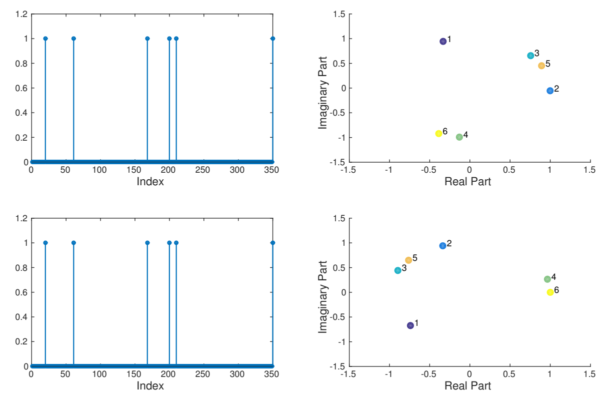

Fig. 2. (Top left) Amplitudes of the original data. (Top right) The complex plane representation of the nonzero part of the original data.(Bottom left) Amplitudes of the recovered data. (Bottom right) The complex plane representation of the recovered data. Here, N=350, s=6 and M = 100.

5. CONCLUSION

We proposed a new structured sampling scheme, namely the Randomized Partial Nested Fourier Sampler (R-PNFS), along with a novel cancellation based algorithm which can provably recover sparse complex valued signals from their amplitude measurements. The proposed technique requires only $M = O(s \log N)$ measurements which is near-optimal compared to the underlying degree of freedom of the sparse signal. We also showed that under mild conditions, the approach is stable to bounded measurement noise.

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