LOWER BOUNDS ON THE L_2 -NORMS OF DIGITAL RESAMPLING FILTERS WITH ZERO-VALUED INPUT SAMPLES

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ABSTRACT

This paper derives lower bounds on the L_2 -norms of digital resampling filters with zero-valued input samples. This emanates from uniform-grid sampling but where some of the samples are missing. One application is found in timeinterleaved analog-to-digital converters with missing samples due to calibration at certain time instances. The square of the L_2 -norms correspond to scaling of the round-off noise that in practice is always present at the input of the resampling filter. As will be shown through the derived bounds, the L_2 -norm of the corresponding filter that recovers the missing samples is generally much larger than unity. Consequently, the noise variance is generally much larger for the recovered samples than for the other samples obtained in the sampling process. Based on this observation, the paper also proposes an alternative resampling scheme for which the maximum of all L_2 -norms in the resampling is reduced.

Index Terms— L_2 -norms, missing samples, nonuniform sampling, resampling, time-interleaved ADCs.

1. INTRODUCTION

Digital resampling can be carried out through a digital filter (linear system) with a time-varying impulse response or, equivalently, a set of time-invariant resampling filters (in general an infinite set) [1, 2]. In practice, it is customary to assume a certain amount of oversampling to account for filter transition bands. In other words, the underlying signal to be resampled has a bandwidth that is smaller than the Nyquist band. In this way, one can achieve arbitrarily small approximation errors in the resampling process, with a reasonable filter order.

However, the input of the resampling filter always contains quantization noise in addition to the signal. The quantization is typically not bandlimited but can in many cases be appropriately modeled as a wide-sense stationary (WSS) white-noise process with zero mean and a certain variance. Then, since the resampling filter is time varying, the noise at the output will not be WSS and each output sample will have a noise variance that is scaled by the squared L_2 -norm of the corresponding filter [3]. When the digital input of the resampling filter corresponds to a uniformly sampled version of an underlying analog signal, all L_2 -norms will be slightly smaller than unity, if the filter is properly designed. In the case of a nonuniformly sampled version, on the other hand, some of the resampling filters' L_2 -norms become much larger than unity. The L_2 -norms can be reduced by incorporating a penalty factor on them in the filter design, but it turns out that there are lower bounds that one cannot go below.

This paper derives such lower bounds for a particular class of nonuniform-sampling grids, namely a uniform-sampling grid but with one missing sample at an arbitrary location in a batch of consecutive samples. Mathematically, this can be handled by setting the corresponding sample to zero in a regular uniform-grid-sampling digital signal. In the signal resampling, it corresponds to setting one of the filter taps to zero (details are provided in Section 3). This type of sampling appears, e.g., in time-interleaved analog-to-digital converters (TI-ADCs) with missing samples [4-6], where some sampling instances are reserved for a calibration signal whose samples are then used for estimating the channel mismatches that are always present in TI-ADCs [7]. As will be shown in this paper, based on the derived bounds, the L_2 -norm of the corresponding filter that recovers the missing samples is generally much larger than unity. Consequently, the noise variance is generally much larger for the recovered samples than for the other samples obtained in the sampling process. Based on this observation, the paper also proposes an alternative resampling scheme for which the maximum of all L_2 -norms in the resampling is reduced. Reference [6] also observed the large L_2 -norm of the filter that recovers the missing samples, but it did not consider the more general resampling case that is addressed in this paper. Interpolation in the presence of noise has also been considered, see e.g. [8,9], but to the best of the authors' knowledge, the L2-norm bounds presented here have not been reported before.

Following this introduction, Section 2 provides the necessary prerequisites of digital resampling filters. Section 3 derives the lower bounds on the L_2 -norms for the resampling filters under consideration. Section 4 considers design examples that verify the results and give some further insights. Finally, Section 5 concludes the paper.

2. RESAMPLING FILTERS

Assume that we have an analog (continuous-time) signal $x_a(t)$ with the Fourier transform $X_a(j\omega) = \int_{-\infty}^{\infty} x_a(t)e^{-j\omega t}dt$ that is bandlimited to $\omega_c < \pi/T$. Then assume that we sample $x_a(t)$ uniformly with a sampling period of T (sampling frequency of 1/T), resulting in the digital signal (sequence) $x(n) = x_a(nT)$ with the Fourier transform $X(e^{j\omega T}) = \sum_{n=-\infty}^{\infty} x(n)e^{-j\omega Tn}$. Due to the bandlimitation of $x_a(t)$, the Nyquist criterion for alias-free (error-free) sampling and reconstruction is satisfied and we have the following relation in the frequency domain [10]:

$$X(e^{j\omega T}) = \frac{1}{T} X_a(j\omega), \quad |\omega T| \le \omega_c T < \pi.$$
(1)

Assume next that we wish to resample $x_a(t)$ at the time instances $t_n = nT + d_nT$, where $|d_n| < 1$. This is carried out by convolving x(n) with a time-varying linear digital system with the impulse response $h_n(k)$, which gives¹

$$y(n) = \sum_{k=-N}^{N} x(n-k)h_n(k).$$
 (2)

Utilizing the inverse Fourier transform in (2), and (1), y(n) can be rewritten as

$$y(n) = \frac{1}{2\pi} \int_{-\omega_c T}^{\omega_c T} H_n(e^{j\omega T}) X(e^{j\omega T}) e^{j\omega T n} d(\omega T)$$

$$= \frac{1}{2\pi} \int_{-\omega_c}^{\omega_c} H_n(e^{j\omega T}) X_a(j\omega) e^{j\omega T n} d\omega \qquad (3)$$

where

$$H_n(e^{j\omega T}) = \sum_{k=-N}^N h_n(k) e^{-j\omega Tk}.$$
 (4)

It is seen that the desired result, $y(n) = x_a(nT + d_nT)$, is obtained when

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$$H_n(e^{j\omega T}) = e^{j\omega T d_n}, \quad |\omega T| \le \omega_c T < \pi.$$
 (5)

Hence, as is well known [1, 2, 11], resampling corresponds to time-varying fractional-delay filtering, since each $e^{j\omega T d_n}$ is the frequency response of a fractional-delay filter with a delay of $d_n T$ [s].

As discussed in the introduction, the input x(n) of the resampling filter also contains quantization errors, denoted here as e(n). In many cases, like for the output of an ADC, e(n)can be appropriately modeled as a WSS white-noise process with zero mean and variance σ^2 . Since $h_n(k)$ is time varying, the noise at the output is, however, not WSS and each output sample will have a noise variance that is scaled by the squared L_2 -norm of $H_n(e^{j\omega T})$. Hence, the noise at the output of the resampling filter has a time-varying variance according to

$$\sigma_n^2 = \sigma^2 ||H_n(e^{j\omega T})||_2^2$$
 (6)

where $||H_n(e^{j\omega T})||_2^2$ are the squared L_2 -norms of $H_n(e^{j\omega T})$ given by [2, 3, 12] ²

$$||H_n(e^{j\omega T})||_2^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |H_n(e^{j\omega T})|^2 d(\omega T).$$
(7)

Above, it was assumed that x(n) is a uniformly sampled version of $x_a(t)$. This means that all $h_n(k)$ are free parameters. One can then design $H_n(e^{j\omega T})$ to approximate $e^{j\omega T d_n}$ in the low-frequency region $|\omega T| \leq \omega_c T$ and keep $|H_n(e^{j\omega T})|$ small (ideally zero) in the remaining frequency region $\omega_c T < |\omega T| \leq \pi$. The lower bounds on the L_2 -norm are in this case independent of time and apparently obtained when $H_n(e^{j\omega T})$ are zero for $\omega_c T < |\omega T| \leq \pi$. Hence,

$$||H_n(e^{j\omega T})||_2^2 \ge \frac{\omega_c T}{\pi} < 1, \quad \forall n.$$
(8)

3. LOWER BOUNDS ON THE L₂-NORMS OF RESAMPLING FILTERS WITH ZERO-VALUED INPUT SAMPLES

We now turn our attention to the case where one of the samples in each block of 2N + 1 consecutive samples in the input sequence x(n) is zero. This corresponds to a class of nonuniform sampling which occurs in, e.g., TI-ADCs with missing samples where some sampling instances are reserved for a calibration signal whose samples are used for estimating mismatches between the channel ADCs [4–6]³. Mathematically, a missing sample at, say, $n = n_0$, can be handled by setting the corresponding sample $x(n_0)$ to zero instead of using the value $x_a(n_0T)$.

In the extreme case, when $N = \infty$, the bounds to be derived below hold when there is only one zero-valued sample in the whole input sequence. However, in practice, when FIR resampling filters are used, N is finite in which case x(n) can contain more missing samples. In such cases, the bounds still hold but are not tight. The bounds are tight as long as each block contains only one zero-valued sample. In TI-ADCs with missing samples, every Mth sample in x(n) is zero, and the bounds are then tight when M > 2N.

Under the assumption that one of the samples in x(n) is zero (missing), in each block of 2N + 1 consecutive samples,

¹In the examples of this paper, as well as often in practice, finite-length impulse response (FIR) filters are used, in which case N is a finite number. However, the lower bounds derived in the paper hold in general, thus including the case where $N = \infty$.

²Alternatively, $||H_n(e^{j\omega T})||_2^2$ can be computed as $\sum_{k=-N}^N |h_n(k)|^2$ due to Parseval's relation, i.e., the L_2 -norm equals the l_2 -norm.

³In TI-ADCs, there are also additional small deviations from the desired sampling instances nT due to mismatch errors (typically in the order of a percent of T). However, such small deviations have no essential effect on the L_2 -norms. Details will be considered in a full-length paper under way. Further, the bounds derived in this paper are still applicable, if the small deviations have been compensated for first, before recovering the missing sample or resampling.

(1)–(7) in Section 2 still hold, if we impose the additional restriction that the corresponding tap in $h_n(k)$ is zero. To be precise, this is because computing y(n) in (2) with a zero-valued sample $x(n_0)$, $n_0 \in [n - N, n + N]$, is equivalent to computing y(n) without setting $x(n_0)$ to zero [thus using $x(n_0) = x_a(n_0T)$] but instead setting the corresponding filter tap to zero, i.e., $h_n(k_n) = 0$ for $k_n = n - n_0$.

Expressing $h_n(k)$ in terms of its inverse Fourier transform, we thus have the additional restriction

$$h_n(k_n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} H_n(e^{j\omega T}) e^{j\omega T k_n} d(\omega T) = 0.$$
 (9)

Inserting the desired function $H_n(e^{j\omega T}) = e^{j\omega T d_n}$ in the low-frequency region $|\omega T| \le \omega_c T$, and utilizing that all complex quantities involved are conjugate symmetric, (9) can be rewritten as

$$h_{n}(k_{n}) = \frac{1}{2\pi} \int_{-\omega_{c}T}^{\omega_{c}T} e^{j\omega T(k_{n}+d_{n})} d(\omega T) + \frac{1}{2\pi} \int_{\omega_{c}T < |\omega T| \le \pi}^{\omega_{c}T} H_{n}(e^{j\omega T}) e^{j\omega Tk_{n}} d(\omega T) = I_{1n}(\omega_{c}T, k_{n}, d_{n}) + I_{2n}(\omega_{c}T, k_{n}, d_{n}) = 0$$
(10)

where

$$I_{1n}(\omega_c T, k_n, d_n) = \begin{cases} \frac{\sin(\omega_c T(k_n + d_n))}{\pi(k_n + d_n)}, & k_n + d_n \neq 0\\ \frac{\omega_c T}{\pi}, & k_n + d_n = 0 \end{cases}$$
(11)

and

$$I_{2n}(\omega_c T, k_n, d_n) = \frac{1}{\pi} \int_{\omega_c T}^{\pi} \Re\{H_n(e^{j\omega T})e^{j\omega T k_n}\}d(\omega T).$$
(12)

Due to the conjugate symmetry we also have

$$\frac{1}{2\pi} \int_{\omega_c T < |\omega T| \le \pi} \Im\{H_n(e^{j\omega T})e^{j\omega Tk_n}\} d(\omega T) = 0.$$
(13)

To ensure $h_n(k_n) = 0$, we thus have the restriction

$$I_{2n}(\omega_c T, k_n, d_n) = -I_{1n}(\omega_c T, k_n, d_n).$$
 (14)

Subject to (5) and (14), we now wish to find lower bounds on the squared L_2 -norms of $H_n(e^{j\omega T})$, i.e, the minima of $||H_n(e^{j\omega T})||_2^2$. Noting that

$$||H_n(e^{j\omega T})||_2^2 = ||H_n(e^{j\omega T})e^{j\omega Tk_n}||_2^2,$$
(15)

this amounts to minimizing the quantity

$$\frac{1}{\pi} \int_{\omega_c T}^{\pi} |H_n(e^{j\omega T})e^{j\omega T k_n}|^2 d(\omega T).$$

Considering the real-part contribution alone, i.e.,

$$\frac{1}{\pi} \int_{\omega_c T}^{\pi} (\Re\{H_n(e^{j\omega T})e^{j\omega Tk_n}\})^2 d(\omega T),$$



Fig. 1. Lower bounds of the squared L_2 -norm $||H_n(e^{j\omega T})||_2^2$ (noise gain) versus bandwidth $\omega_c T$.

it can be shown that the solution that minimizes this quantity, subject to (5) and (14), is $\Re\{H_n(e^{j\omega T})e^{j\omega Tk_n}\} = A_n(\omega_c T, k_n, d_n)$, where $A_n(\omega_c T, k_n, d_n)$, for each set of parameter values, $(\omega_c T, k_n, d_n)$, is a constant given by

$$A_n(\omega_c T, k_n, d_n) = -\frac{\pi}{\pi - \omega_c T} I_{1n}(\omega_c T, k_n, d_n).$$
(16)

The proof is omitted here due to the space limitation, but follows that provided in [6] for the special case where $k_n = d_n = 0$, which corresponds to the recovery of a missing sample centered at the mid-tap of the filter.

Further, since (14) is independent of the imaginary part $\Im\{H_n(e^{j\omega T})e^{j\omega Tk_n}\}$, due to (13), the imaginary-part contribution $\frac{1}{\pi}\int_{\omega_c T}^{\pi}(\Im\{H_n(e^{j\omega T})e^{j\omega Tk_n}\})^2 d(\omega T)$ can ideally be zero. It now follows that the lower bounds on $||H_n(e^{j\omega T})||_2^2$ are time dependent and given by

$$||H_n(e^{j\omega T})||_2^2 \geq \frac{\omega_c T}{\pi} + \frac{\pi - \omega_c T}{\pi} A_n^2(\omega_c T, k_n, d_n)$$
$$= \frac{\omega_c T}{\pi} + \frac{\pi}{\pi - \omega_c T} I_{1n}^2(\omega_c T, k_n, d_n).(17)$$

It is noted that, in the special case where $k_n + d_n = 0$, $I_{1n}(\omega_c T, k_n, d_n) = \omega_c T/\pi$, and we obtain

$$||H_n(e^{j\omega T})||_2^2 \ge \frac{\omega_c T}{\pi - \omega_c T}, \quad k_n + d_n = 0,$$
 (18)

which goes to infinity when $\omega_c T$ approaches the whole digital bandwidth π .

As can be seen from Fig. 1, the lower bounds on $||H_n(e^{j\omega T})||_2^2$ increase with the bandwidth $\omega_c T$. Also, the lower bounds on $||H_n(e^{j\omega T})||_2^2$ are generally much larger for the filters whose center tap is located at or around the zero-valued input sample $(k_n = 0)$.

4. NUMERICAL EXAMPLE

In this section, we first provide an example showing that, compared to the recovery of the zero-valued sample $(d_n = k_n = 0)$, the maximum of all the L_2 -norms in the resampling can be reduced by selecting an alternative resampling scheme⁴. We assume that the bandwidth is $\omega_c T = 0.8\pi$. It can be seen from Table 1 that by using a resampling scheme with $d_n = 0.5$, the maximum of all L_2 -norms is reduced as compared with the case where $d_n = k_n = 0^5$. In a full-length paper under way, we address the problem of selecting d_n such that they together minimize the maximum of all the L_2 -norms.

Further, in order to validate the lower bounds derived in Section 3, we consider the case where resampling is carried out using FIR filters of order 2N. Like in Section 3, it is assumed that there exists only one zero-valued sample in each block of 2N + 1 consecutive input samples. Here, we assume that resampling is performed using $d_n = 0.5$ and $k_n = 0$, and that the signal reconstruction error shall be less than -40 dB. The coefficients of each resampling filter $h_n(k)$ are determined such that the corresponding $||H_n(e^{j\omega T})||_2^2$ is minimized subject to the constraint that the signal reconstruction error be less than -40 dB. Figure 2 plots $||H_n(e^{j\omega T})||_2^2$ (in dB) versus the filter order. The figure also includes a horizontal line which is the lower bound (4.20 dB) computed using (17) with $k_n = 0$ and $d_n = 0.5$. Moreover, as can be seen from Fig. 3, which plots, the real and imaginary parts of $H_n(e^{j\omega T})e^{j\omega Tk_n}$, the real part approximates a negative-valued constant in the region $\omega_c T < \omega T \leq \pi$ whereas the imaginary part in the same region approximates zero. This validates the results in Section 3 where we showed that for the $H_n(e^{j\omega T})$ that minimizes $||H_n(e^{j\omega T})||_2^2$, $\Re\{H_n(e^{j\omega T})e^{j\omega Tk_n}\}\$ is equal to a constant $A_n(\omega_c T, k_n, d_n)$ (-3.03 in this example) whereas $\Im\{H_n(e^{j\omega T})e^{j\omega Tk_n}\}$ is equal to zero, in the region $\omega_c T < \omega T \leq \pi$.

5. CONCLUSION

This paper derived lower bounds on the L_2 -norms of resampling filters with zero-valued input samples. Recovering a zero-valued sample is a special case, and the derived bounds are thus applicable for the missing sample recovery problem as well. Using the lower bounds, it was shown that the filter which recovers (resamples) the zero-valued sample has an L_2 -norm that is much larger than unity. With the help of a numerical example, it was then also shown that a reduction of the maximum of all L_2 -norms can be achieved through alternative resampling schemes.



Fig. 2. Plot of squared L_2 -norm $||H_n(e^{j\omega T})||_2^2 = ||H_n(e^{j\omega T})e^{j\omega Tk_n}||_2^2$ versus filter order 2N for the example in Section 4. Here, $\omega_c T = 0.8\pi$, $k_n = 0$, and $d_n = 0.5$.



Fig. 3. Real and imaginary parts of $H_n(e^{j\omega T})e^{j\omega Tk_n}$ for the example in Section 4 where $\omega_c T = 0.8\pi$, $k_n = 0$, $d_n = 0.5$, 2N = 250, and $A_n(\omega_c T, k_n, d_n) = -3.03$.

Table 1. Comparison of the lower bounds on the squared L_2 norms of the reconstruction filters corresponding to the zerovalued sample recovery case $(d_n = k_n = 0)$ and with the resampling case $(d_n = 0.5)$. In both cases, the bandwidth is $\omega_c T = 0.8\pi$.

		$ H_n(e^{j\omega T}) _2^2$ (dB)			$ H_n(e^{j\omega T}) _2^2 \text{ (dB)}$	
	k_n	$d_n = 0$	$d_n = 0.5$	k_n	$d_n = 0$	$d_n = 0.5$
	0	6.02	4.20			
	-1	-0.11	4.20	1	-0.11	-0.57
	-2	-0.39	-0.57	2	-0.39	-0.97
	-3	-0.70	-0.97	3	-0.70	-0.89
	-4	-0.91	-0.89	4	-0.91	-0.85
	-5	-0.97	-0.85	5	-0.97	-0.89

⁴The cases where $d_n = 0$ and $k_n \neq 0$ do not correspond to the recovery of missing samples, but instead to low-pass filtering cases with a zero-tap for $h_n(k_n)$. Then, as $|k_n|$ increases, one reaches the bound in (8).

⁵It is noted that $H_n(e^{j\omega T})$ depend on n via d_n and k_n . Hence, if $d_{n_0} = d_{n_1}$ and $k_{n_0} = k_{n_1}$, then $||H_{n_0}(e^{j\omega T})||_2^2 = ||H_{n_1}(e^{j\omega T})||_2^2$

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