SPARSITY-BASED RECONSTRUCTION METHOD FOR SIGNALS WITH FINITE RATE OF INNOVATION

Guoxing Huang, Ning Fu, Member, IEEE, Jingchao Zhang, Liyan Qiao, Member, IEEE

Dept. of Automatic Test and Control, Harbin Institute of Technology
Harbin, 150080, Heilongjiang Province, P.R.China
Email: funinghit_paper@163.com

ABSTRACT

In the last decade, it was shown that it is possible to reconstruct signals with finite rate of innovation (FRI signals) from the samples of their filtered versions. However, when noise is present, the present reconstruction algorithms tend to be low accuracy. In this work, a new sparsity-based reconstruction method for FRI signals is put forward. The streams of Diracs and exponential reproducing kernel are considered. Firstly, the analog time axis is quantified and aligned to grids. Secondly, selecting a finite subset of time delay parameters, the measurement vector is represented as a sparse linear combination of the amplitude parameters. Finally, the sparse solution is calculated by solving an optimization problem under L0 norm. The position of non-zero elements is approximation to the time delays, and the value of non-zero elements is the amplitude. Extensive numerical simulations demonstrate the accuracy and robustness of our method.

***Index Terms*—**finite rate of innovation (FRI), sparsity, streams of Diracs, exponential splines, L0 norm.

1. INTRODUCTION

It is widely known from the Nyquist-Shannon sampling theorem [1-2] that a continuous signal bandlimited to $\Omega_{\max}$ can be sampled with no loss of information and perfectly reconstructed from its samples if the sampling rate $f_s \geq 2\Omega_{\max}$. However, there exist other signals with not a limited bandwidth, e.g. streams of Diracs, nonuniform splines, and piecewise polynomials. These later signals cannot obviously be sampled with a finite sampling rate if no information has to be lost. But they all have a finite number of degrees of freedom per unit of time and can be called as signals with finite rate of innovation (FRI signals) [3-5]. Even though these FRI signals are not bandlimited, the authors in [6] showed that they can be sampled uniformly at the rate of innovation using an appropriate kernel and then be perfectly reconstructed.

Regarding the reconstruction methods, the algorithm proposed in [6] is based on the use of annihilating filters, and is adapted to work either with periodic or aperiodic signals. But this algorithm needs to solve the polynomial roots, which lead to a heavy computation when the signal has high degree of freedom. The bigger drawback is that the reconstruction accuracy drastically worsen when the samples are distorted by noise. In order to improve the reconstruction accuracy in the presence of noise, the authors in [7] proposed a new reconstruction scheme based on the state space method [8] and named subspace-based algorithm. Unfortunately, Although this method reduces the estimation error from noisy samples, the procedure could run into computational problems when considering the case of aperiodic FRI signals. The iterative method proposed in [9] is an improved algorithm for annihilating filters, which can also improve the reconstruction accuracy in the presence of noise. However, this method needs more samples and operation time. There are also some optimization-based methods, such as genetic algorithm in [10] and unconstrained optimal algorithm in [11], which perform effective only in the case of low degree of freedom and are always unstable. Other efficient but more time expensive methods in the literature are the stochastic algorithms in [12-13], which are only suitable for finite-length streams of Diracs.

In this paper, we propose a novel reconstruction method for FRI signals, which is based on the sparsity of time delay parameters and shows to have high accuracy in the presence of noise. Firstly, the analog time axis is quantified and aligned to grids. Then the measurement vector is sparse represented, and the time delay and amplitude parameters of the input FRI signal are estimated by solving an optimization problem under L0 norm. In this work, the streams of Diracs is considered as the input signal, and the exponential reproducing kernel is used for sampling kernels.

The paper is organized as follows: Section II introduce the sampling framework of FRI signals. Then, in Section III, the proposed method is presented. Finally, Section IV shows some simulation results and in Section V we conclude with a brief summary.

This work was supported by National Natural Science Foundation of China (NSFC, No.61102148).

2. FINITE RATE OF INNOVATION SAMPLING FRAMEWORK

2.1. Signals with finite rate of innovation

FRI signals are the signals which can be determined by a finite number of parameters per unit time period, and the rate of innovation means the number of parameters per unit time. In this work, our interest focuses on the reconstruction of finite-length streams of Diracs, which is a typical FRI signal:

$$x(t)=\sum_{l=0}^{L-1} a_l \delta(t-t_l), \qquad t_l \in [0,T) \tag{1}$$

Where T is the time length of signal *x(t),* and K is the number of Diracs pulses. Clearly, the only free parameters in the signal *x(t)* are the amplitudes a_l and the time delays t_l. It is, therefore, natural to introduce a counting function $C_x(t_a,t_b)$ which counts the number of free parameters of signal *x(t)* over the interval $\tau=[t_a,t_b]$, then the rate of innovation ρ of the signal *x(t)* can be defined as:

$$\rho=\lim_{\tau\to\infty}\frac{1}{\tau}C_x(-\frac{\tau}{2},\frac{\tau}{2}). \tag{2}$$

If ρ is finite, then the signal *x(t)* is said to have a finite rate of innovation, widely named as signals with finite rate of innovation (FRI signals).

Assuming that time delays t_l located at distinct instants $t_l \in [0,T)$ and the number of Diracs pulses K of signal *x(t)* is known, the rate of innovation ρ can be calculated as:

$$\rho=\frac{2L}{T}. \tag{3}$$

As in [6] showed, the signal *x(t)* can be sampled uniformly at the rate $f_s=\rho$ using an appropriate kernel and then be perfectly reconstructed.

2.2. Sampling setup

In the typical sampling setup depicted in Fig.1, the original continuous-time FRI signal *x(t)* is filtered before being uniformly sampled with sampling period $t_s=1/f_s=1/\rho$, *h(t)* is the impulse response of the acquisition device, the sampling kernel $\varphi(t)$ is the scaled and time-reversed version of *h(t)*.

If we denote with $y(t)=x(t)*h(t)$ the filtered version of FRI signal *x(t)*, the following expression for the samples y_k can be deduced:

$$\begin{aligned} y_k &= x(t)*h(t)|_{t=kt_s} \\ &=<\sum_{l=0}^{L-1} a_l\delta(t-t_l),\varphi(t/t_s-k)>, \\ &=\sum_{l=0}^{L-1} a_l\varphi(t_l/t_s-k) \end{aligned} \tag{4}$$

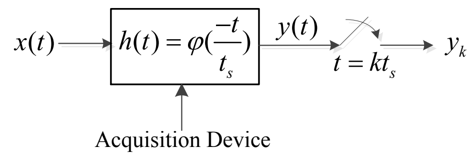

Fig.1. Sampling setup. Here, *x(t)* is the continous-time FRI signal, *h(t)* is the impulse response of the acquisition device, and t_s is the sampling period.

where $k=1,2,\cdots,K$ is the number of samples, $K=T/t_s$ is the total number of samples. Having achieved the samples from the setup described above, the key problem then is to find the best way to reconstruct FRI signal *x(t)* from the given samples $y_k(k=1,2,\cdots,K)$. This includes the type of sampling kernels that can be employed and also the reconstruction techniques that are required.

2.3. Sampling kernels

Unlike the classical sampling schemes, FRI sampling schemes provide a larger choice of kernels that allow perfect reconstruction of the input signal. The sinc and Gaussian sampling kernels are proposed in [6, 14], which have an infinite support and are therefore physically unrealizable. The other finite support and more stable sampling kernels are polynomial reproducing kernels, exponential reproducing kernels and rational kernels [15].

In this work, the exponential reproducing kernel, which tend to be more stable than other kernels, was considered, and other classes of kernels can also be employed. Any kernel $\varphi(t)$ that together with its shifted versions can reproduce real or complex exponentials of the form $e^{\alpha_m x}$ with $\alpha_m=\alpha_0+m\lambda$ and $m=0,1,\cdots,M$ is called an exponential reproducing kernel of order *M.* That is any kernel satisfying the following property:

$$\sum_{k\in Z} c_{m,k}\varphi(t-k)=e^{\alpha_m t}. \tag{5}$$

The coefficients $c_{m,k}$ in the above equation are given by the following expression:

$$c_{m,k}=\int_{-\infty}^{\infty} e^{\alpha_m t}\tilde{\varphi}(t-k)dt, \tag{6}$$

where $\tilde{\varphi}(t)$ is chosen to form $\varphi(t)$ with a quasi orthonormal set.

3. SPARSITY-BASED RECONSTRUCTION METHOD FOR FRI SIGNALS

3.1. Reconstruction problem

Having gone through the sampling stage, we will now discuss the reconstruction process. According to the definition of exponential reproducing kernel, the kernel function of order *M* can reproduce *M+1* complex exponentials $e^{\alpha_m t}$ with $m=0,1,\ldots,M$. Consider the weighted sum of the K samples y_k with $k=1,2,\ldots K$ and $(M+1)\times K$ coefficients $c_{m,k}$, $M+1$ measurement values can be calculated as follows:

$$\begin{aligned}\tau_m &= \sum_k c_{m,k} y_k \\ &= \sum_{l=0}^{L-1} a_l \sum_k c_{m,k} \varphi(t_l / t_s - k) \\ &= \sum_{l=0}^{L-1} a_l e^{\alpha_m (t_l/t_s)}, \qquad with\ m = 0,1,\cdots,M \end{aligned} \quad . \quad (7)$$

The reconstruction problem of FRI signals is to estimate the unknown parameters: the time delays t_l and the amplitudes a_l, with $l=0,1,\ldots,L-1$. Obviously, these parameters can be exactly recovered, provided that the number of measurements $M+1 \geq 2L$.

3.2. Sparsity-Based reconstruction method

We begin by quantizing the analog time axis with a resolution step of Δ, thus, the analog time t can be approximated to $t = n\Delta$ with $n=0,1,\ldots,N-1$ and $N = T/\Delta$, and unknown parameters of time delays t_l can be approximated to $t_l = n_l \Delta$. Then equation (7) can be approximated as:

$$\tau_m \approx \sum_{l=0}^{L-1} a_l e^{\alpha_m (n_l \Delta / t_s)}, \qquad with\ m = 0,1,\cdots,M-1 \quad (8)$$

where $n_l \in [0, N-1)$ is the discrete numeric value of time delays t_l. After that, equation (8) can be rewritten in the form of matrices:

$$\begin{bmatrix} \tau_0 \\ \tau_1 \\ \vdots \\ \tau_M \end{bmatrix} = \begin{bmatrix} e^{\alpha_0 (n_0 \Delta/t_s)} & e^{\alpha_0 (n_1 \Delta/t_s)} & \cdots & e^{\alpha_0 (n_{L-1} \Delta/t_s)} \\ e^{\alpha_1 (n_0 \Delta/t_s)} & e^{\alpha_1 (n_1 \Delta/t_s)} & \cdots & e^{\alpha_1 (n_{L-1} \Delta/t_s)} \\ \vdots & \vdots & \ddots & \vdots \\ e^{\alpha_M (n_0 \Delta/t_s)} & e^{\alpha_M (n_1 \Delta/t_s)} & \cdots & e^{\alpha_M (n_{L-1} \Delta/t_s)} \end{bmatrix} \cdot \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_{L-1} \end{bmatrix}. \quad (9)$$

Considering that the time domain of FRI signal $x(t)$ is limited to $t_l \in [0,T)$, A complete set of the analog time can be obtained as $U = \{0, \Delta, 2\Delta, \cdots, (N-1)\Delta\}$ with $N = T/\Delta$, in the condition of ignoring quantization error. Thus the time delays parameters set of $x(t)$ is $V = \{n_0\Delta, n_1\Delta, \cdots, n_{L-1}\Delta\}$, which constitute a smaller subset of the set U, that is $V \subset U$ with $L \ll N$. Then (9) can be rewritten as:

$$\begin{bmatrix} \tau_0 \\ \tau_1 \\ \vdots \\ \tau_M \end{bmatrix} = \begin{bmatrix} 1 & e^{\alpha_0 (\Delta/t_s)} & \cdots & e^{\alpha_0 [(N-1)\Delta/t_s]} \\ 1 & e^{\alpha_1 (\Delta/t_s)} & \cdots & e^{\alpha_1 [(N-1)\Delta/t_s]} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & e^{\alpha_M (\Delta/t_s)} & \cdots & e^{\alpha_M [(N-1)\Delta/t_s]} \end{bmatrix} \cdot \begin{bmatrix} x_0 \\ x_1 \\ \vdots \\ x_{N-1} \end{bmatrix}, \quad (10)$$

where $[x_0, x_1, \cdots, x_{N-1}]^T$ is a $N \times 1$ vector, formed by L amplitude parameters $\{a_l\}_{l=0}^{L-1}$ and N-L zero values. Our goal is to find the nonzero entries of vector $[x_0, x_1, \cdots, x_{N-1}]^T$ from the measurements values. For simplicity, (10) may be written as:

$$\Gamma = AX, \quad (11)$$

where $\Gamma = [\tau_0, \tau_1, \cdots \tau_M]^T \in R^{(M+1)\times 1}$ is the measurement vector, A is a $(M+1)\times N$ matrix formed by the set of $\{e^{\alpha_m (n\Delta/t_s)}\}_{n=1}^{N}$, $X = [x_0, x_1, \cdots, x_{N-1}]^T \in R^{N\times 1}$ is an L-sparse vector with nonzero entries at indices $\{n_l\}_{l=0}^{L-1}$, and the corresponding nonzero element values are $\{a_l\}_{l=0}^{L-1}$.

The most direct way to solve (11) is by solving a L0 norm optimization problem described as (12):

$$\hat{X} = \arg\min \| X \|_0 \quad such\ that\ \ \Gamma = AX, \quad (12)$$

where L0 norm $\| X \|_0$ means the number of non-zero coefficients in vector X. Solving (12) is an NP-hard problem. An approximate solution can be found, for example, by using the well-known OMP algorithm [16]. The algorithm iteratively finds the nonzero entries of X by seeking the maximal correlations between Γ and the columns of A, while maintaining an orthogonalization step at the end of each iteration [17].

Once the nonzero entries $n_l(l=0,1,\ldots,L-1)$ of vector X are found, the time delays are directly calculated as $\hat{t}_l = n_l \Delta$, and the amplitudes are estimated via $\hat{a}_l = X(n_l)$. As the signal of interest is a stream of Dirac pulses, $x(t)$ will be totally reconstructed if we know the time instants and the amplitudes of these pulses. Finally, the reconstruction signal of $x(t)$ can be described as:

$$\hat{x}(t) = \sum_{l=0}^{L-1} \hat{a}_l \delta(t - \hat{t}_l), \qquad \hat{t}_l \in [0,T). \quad (13)$$

3.3. The process of algorithm

After the FRI signals with the form of (1) are sampled as (4), the specific steps of the proposed method are as follows:

Step 1, Initialization. Assume that there are K samples $y_k(k=1,2,\ldots,K)$, exponential reproducing kernel of order M is used, with coefficients $c_{m,k}$ and $m = 0,1,\cdots,M$.

Step 2, Measurement values. $M+1$ measurement values $\tau_m (m = 0,1,\cdots,M)$ are calculated as (7).

Step 3, Mesh grid. The analog time t is approximated to $t = n\Delta$ with $n = 0,1,\cdots,N-1$ and $N = T/\Delta$, and the time delays parameters t_l are approximated to $t_l = n_l \Delta$

Step 4, Sparse representation. Measurement vector Γ is represented as a sparse linear combination of the amplitude parameters as (10), and simplified to (11).

Step 5, Sparse solution. OMP algorithm is employed to solve the sparse solution X by solving an optimization problem under L0 norm as (12).

Step 6, Parameter estimation. Once the nonzero entries $n_l(l=0,1,\ldots,L-1)$ of X are found, the time delay parameters are estimated as $\hat{t}_l = n_l\Delta$, and the amplitudes parameters are $\hat{a}_l = X(n_l)$.

Step 7, Output. Finally, the reconstruction signal $\hat{x}(t)$ can be estimated as (13).

4. THE SIMULATION EXPERIMENT RESULTS

In this section, we provide several experiments to evaluate the performance of the proposed method in the presence of white Gaussian noise, and compare to other techniques. For this purpose, we have simulated signals consisting of streams of Diracs with the following set up: The amplitudes of the pulses are independently generated from a uniform distribution $a_l \sim U[0,1]$; The time spacing between pulses are randomly selected within the range $t_l \in [0,T)$ with $T=1$ second; The analog time axis is quantized with the same step of $\Delta = 0.001$ second.

In order to numerical evaluate the performance of the reconstruction methods, mean squared error (MSE) is considered as the evaluation index. For comparison, the logarithm of MSE is considered:

$$MSE[dB] = 10 \times \log_{10}\left(\frac{1}{L}\sum_{l=0}^{L-1}(t_l - \hat{t}_l)^2\right), \tag{14}$$

where L is the number of pulses, t_l is the true values and $\hat{t}_l$ is the estimation parameters of time delays. Because the error in the amplitudes is proportional to the error in the time instants, here we will use the MSE in the time instants to measure the efficiency of the method.

Firstly, we demonstrate the performance of our method from different number of samples, in the presence of white Gaussian noise with SNR levels increase from 0 to 100. The input signal is consist of $L=2$ Diracs with t_l=[0.256,0.38] and a_l=[0.8,1]. So the rate of innovation is $\rho = 2L/T = 4$. Experiment was carried out 100 times, and the average reconstruction result is shown in fig.2. From fig.2 it can be seen that our method performs well in the presence of noise, when the number of samples K is 4, 8 and 16. The corresponding sampling rate is 4Hz, 8 Hz and 16Hz, which is greater than or equal to the rate of innovation ρ. And we can also conclude that the reconstruction accuracy improved with the number of samples K increasing.

Next, we compare our method to the one in [15] and [18]. This approach, which is based on B-splines [19] and E-splines [20] sampling kernels, operates at a rate higher than the rate of innovation. The input signal is $L=4$ Diracs with t_l=[0.213,0.452,0.664,0.754] and a_l=[1,0.9,0.7,0.6]. So the rate of innovation is $\rho = 2L/T = 8$. For all algorithms $K=24$ samples are used, with sampling rate three times of the rate of innovation ρ. Experiment was carried out 100 times, and the average estimation error of the time-delays versus SNR is depicted in Fig. 3. From the figure it can be seen that our scheme exhibits better noise robustness than

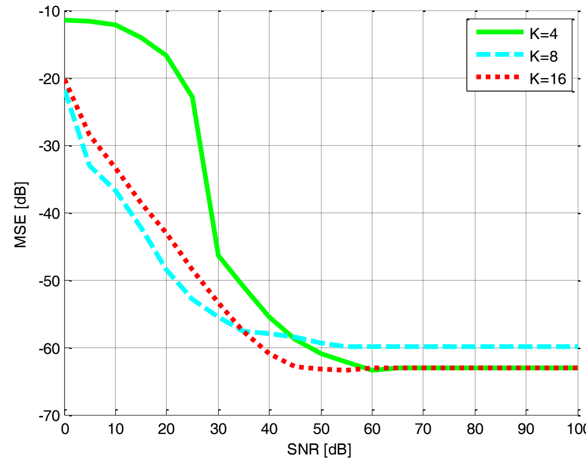

Fig.2. Performance in the presence of noise, using different number of samples

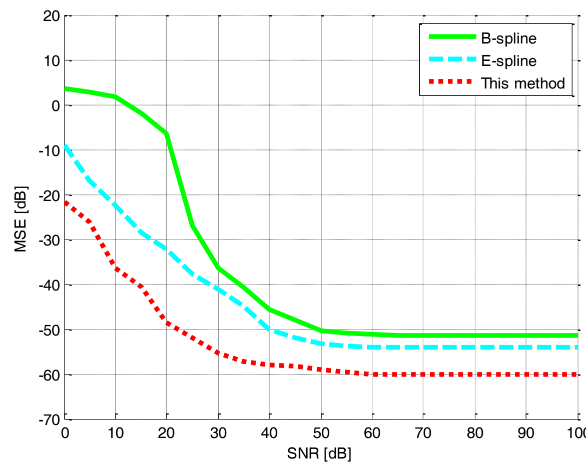

Fig.3. Performance in the presence of noise, using different reconstruction methods

both B-spline and E-spline based methods. Above all, this sparsity-based reconstruction method for signals with finite rate of innovation has high reconstruction accuracy and strong anti-noise interference ability.

5. CONCLUSIONS

In this work, a new sparsity-based reconstruction method for FRI signals is put forward. The proposed scheme is based on sparsity and correlation, which means that the measurement vector can be represented as a sparse linear combination of the amplitude parameters, in condition the analog time axis quantifying and aligning to grids. In contrast to previous reconstruction algorithms which perform deteriorates in the presence of noise, the proposed method work well at low SNR values. The experiment result shows that the proposed method has high reconstruction accuracy and strong anti-noise interference ability.

6. REFERENCES

[1] H. Nyquist, "Certain topics in telegraph transmission theory," *Proceedings of the IEEE*, vol. 90, no. 2, pp. 280-305, 2002.

[2] M. Unser, "Sampling - 50 Years after Shannon," *Proceedings of the IEEE,* vol. 88, no. 4, pp. 569-587, 2000.

[3] X. Wei and P. L. Dragotti, "Universal sampling of signals with finite rate of innovation," *IEEE International Conference on Acoustics, Speech and Signal Processing (ICASSP)*, pp. 1803-1807, 2014.

[4] M. S. Kotzagiannidis and P. L. Dragotti, "Sparse graph signal reconstruction and image processing on circulant graphs," *IEEE Global Conference on Signal and Information Processing (GlobalSIP)*, pp. 923-927, 2014.

[5] A. Nair and P. Marziliano, "Fetal heart rate detection using VPW-FRI," *IEEE International Conference on* Acoustic*, Speech and Signal Processing (ICASSP)*, pp. 4438-4442, 2014.

[6] M. Vetterli, P. Marziliano, and T. Blu, "Sampling signals with finite rate of innovation," *IEEE Transactions on Signal Processing*, vol. 50, no. 6, pp. 1417-1428, 2002.

[7] A. Erdozain and P. Crespo, "Reconstruction of aperiodic FRI signals and estimation of the rate of innovation based on the state space method," *Signal Processing*, vol. 91, no.1, pp.1709-1718, 2011.

[8] I. Maravic and M.Vetterli, "Sampling and reconstruction of signals with finite rate of innovation in the presence of noise," *IEEE Transactions on Signal Processing*, vol. 53, no. 8, pp. 2788-2805, 2005.

[9] T. Blu, P. L. Dragotti, and M. Vetterli, "Sparse Sampling of Signal Innovations: Theory, Algorithms and Performance Bounds," *IEEE Signal Processing Magazine*, vol. 25, no. 2, pp. 31-40, 2008.

[10] A. Erdozain and P. M. Crespo, "A new stochastic algorithm inspired on genetic algorithms to estimate signals with finite rate of innovation from noisy samples," *ELSEVIER Signal Processing*, vol. 90, no. 1, pp. 134-144, 2010.

[11] T. Michaeli and Y. C. Eldar, "Xampling at the Rate of Innovation," *IEEE Transactions on Signal Processing*, vol. 60, no. 3, pp. 1121-1133, 2012.

[12] V. Y. Tan and V. K. Goyal, "Estimating Signals With Finite Rate of Innovation From Noisy Samples: A Stochastic Algorithm," *IEEE Transactions on Signal Processing*, vol. 56, no. 10, pp. 5135-5146, 2008.

[13] A. Erdozain and P. M. Crespo, "A new stochastic algorithm inspired on genetic algorithms to estimate signals with finite rate of innovation from noisy samples," *ELSEVIER Signal Processing*, vol. 90, no. 1, pp. 134-144, 2010.

[14] H. Johansson and P. Lowenborg, "Reconstruction of nonuniformly sampled bandlimited signals by means of time-varying discrete-time FIR filters," *EURASIP Journal on Advances in Signal Processing*, vol. 6, no. 1, pp. 97-100, 2006.

[15] P. L. Dragotti, M. Vetterli and T. Blu, "Sampling Moments and Reconstructing Signals of Finite Rate of Innovation: Shannon Meets Strang-Fix," *IEEE Transactions on Signal Processing*, vol. 55, no. 5, pp. 1741-1757, 2007.

[16] R. G. Baraniuk, "Compressive sensing," *Signal Processing Magazine, IEEE*, vol. 24, no. 4, pp. 118-121, 2007.

[17] J. A. Tropp and A. C. Gilbert, "Signal recovery from random measurements via orthogonal matching pursuit," *Information Theory, IEEE Trans. on*, vol. 53, no. 12, pp. 4655-4666, 2007.

[18] J.A. Uriguen, T. Blu and P.L. Dragotti, "FRI Sampling with Arbitrary Kernels", *IEEE Transactions on Signal Processing*, vol. 61, no. 21, pp. 5310-5323, 2013.

[19] M. Unser, "Splines: A perfect fit for signal and image processing," *IEEE Trans. Signal Process*., vol. 16, no. 6, pp. 22-38, 1999.

[20] M. Unser and T. Blu, "Cardinal exponential splines: Part I-Theory and filtering algorithms," *IEEE Trans. Signal Process*., vol. 53, no. 4, pp. 1425-1438, 2005.