A NOVEL SUB-NYQUIST FOURIER TRANSFORM ESTIMATOR BASED ON ALIAS-FREE HYBRID STRATIFIED SAMPLING

Bashar I. Ahmad

SigProC Lab, Engineering Department
University of Cambridge
Trumpington Street, Cambridge, CB2 1PZ
Email: bia23@cam.ac.uk

Andrzej Tarczynski

Faculty of Science and Technology
University of Westminster
New Cavendish Street, London, W1W 6UW
Email: tarczya@wmin.ac.uk

ABSTRACT

This paper introduces a novel method of estimating the Fourier transform of deterministic continuous-time signals from a finite number N of their nonuniformly spaced measurements. These samples, located at a mixture of deterministic and random time instants, are collected at sub-Nyquist rates since no constraints are imposed on either the bandwidth or the spectral support of the processed signal. It is shown that the proposed estimation approach converges uniformly for all frequencies at the rate N^{-5} or faster. This implies that it significantly outperforms its alias-free-sampling-based predecessors, namely stratified and antithetical stratified estimates, which are shown to uniformly convergence at a rate of N^{-1}. Simulations are presented to demonstrate the superior performance and low complexity of the introduced technique.

Index Terms— Fourier transform estimation, uniform convergence, nonuniform sampling.

1. INTRODUCTION

Fourier Transform (FT) estimation is an important signal processing task with diverse application areas such as astronomy, seismology, biomedical applications and communications, to name a few. The majority of digital signal processing techniques that estimate the FT of a continuous-time signal $x(t)$ use equidistant data samples. The uniform sampling rate f_{US} has to exceed the Nyquist frequency f_{Nyq}, which is twice the total single-sided bandwidth of the processed signal; or its truncated bandwidth when $x(t)$ is observed over a finite-duration window $\mathcal{T}$ as common in practice [1, 2]. Otherwise, the aliasing phenomenon can render certain processing tasks, e.g. detection and accurate signal recovery, unresolvable.

If the spectral support $\mathcal{F}$ of the treated signal is not known *a priori* and the processed range(s) of frequencies (i.e. bandwidth $\mathcal{B}$) is wide, the required f_{US} is proportional to $|\mathcal{B}|$ and can be prohibitively high; $|.|$ is the Lebesgue measure. Bandpass sampling [3] or demodulation with low rate data acquisition [4] cannot be effectively applied since the location of the signal spectral components, i.e. $\mathcal{F}$, is unknown. Examples of such scenarios can be found in instrumentation (e.g. analysing multiband signals with unknown central frequencies as in spectrum analysers), astronomy (e.g. detecting periodic signals in noise [5]) and wireless communications (e.g. wideband spectrum sensing for cognitive radio [4]).

Using random nonuniform sampling brings new opportunities for mitigating the aliasing effects and the notion of *alias-free* sampling was introduced in [6]. Its definition was revised in various subsequent studies such as [7] and [8]. They consider the problem of suppressing spectrum aliasing to allow the accurate estimation of the power spectral density of the underlying stationary continuous-time signal, albeit sampling at arbitrary low, sub-Nyquist, rates. Such approaches generally employ non-equidistant samples, whose distribution depends on the statistical characteristics of the sampling process. More recently, alias-free-based FT estimators were reported in [9, 10, 11]. They are distinct from prior work and consider deterministic signals, whose FTs are complex functions.

In this paper, we tackle the problem of estimating the FT of a deterministic continuous-time signal $x(t)$ from a finite number N of its samples and propose the Hybrid Stratified Sampling (HySt) estimator. Since no constraints are imposed on either the bandwidth, or the spectral support of $x(t)$, the introduced HySt scheme can be classified as sub-Nyquist. It is shown that the HySt estimation mean square error uniformly converges to zero at a rate of N^{-5} or faster for all frequencies, if $x(t)$ has a continuous third order derivative. Hence, the proposed approach significantly outperforms its alias-free-based predecessors, namely Total Random Sampling (ToRa) [9], the Stratified Sampling (StSa) [10] and Antithetical Stratified Sampling (AnSt) [11], whose uniform convergence rates are N^{-1} as per Theorem 1. The latter provides a generic recipe to establish the uniform convergence rates of a broad class of stratification-based FT estimators. Simulations are presented to illustrate the HySt superior performance.

If the target spectrum is sparse, e.g. $x(t)$ is a multiband signal of unknown spectral support $\mathcal{F} \subset \mathcal{B}$, approaches such as universal sampling [12, 13] (uses deterministic nonuniform sampling) and/or compressed sensing (CS) [14, 15] can offer solutions that permit reconstructing $x(t)$ from a small number of its measurements collected at sub-Nyquist rates. Such rates are tied to the spectrum sparsity level $|\mathcal{F}|$, rather than the overall processed frequency range(s) or bandwidth $\mathcal{B}$, assuming $|\mathcal{F}| \ll |\mathcal{B}|$. The approaches in [12, 13, 14, 15] entail devising specialised processing algorithms utilising advanced and computationally demanding optimization techniques to determine $x(t)$ or its characteristics. In this paper, however, the objective is estimating FT of deterministic signals, not signal recovery, using random nonuniform sampling. A low complexity linear estimator is introduced without constraining $\mathcal{F}$, unlike in [12, 13, 14, 15]. Most importantly, the simple HySt estimator does not involve solving an optimisation problem and yet delivers competitive FT estimation results compared with CS as demonstrated in Section 5.

2. PROBLEM FORMULATION

The target Fourier transform is defined by

$$X(f) = \int_{\mathcal{T}} x(t)w(t)e^{-j2\pi ft}dt, \tag{1}$$

for the finite time window $\mathcal{T} = [0, H]$ of width $|\mathcal{T}| = H$. The windowing function $0 \leqslant w(t) \leqslant 1$ aims to keep $X(f)$ smooth [1, 2]. Let $\lambda(t, f) = w(t)e^{-j2\pi ft}$, $x_{k,max} = \sup_{t\in\mathcal{T}} |x^{(k)}(t)|$ and $\lambda_{k,max}(f) = \sup_{t\in\mathcal{T}} |\lambda^{(k)}(t, f)|$ such that $x^{(k)}(t)$ and $\lambda^{(k)}(t, f)$ are the k^{th} order derivatives of $x(t)$ and $\lambda(t, f)$ with respect to t, respectively.
The range of frequencies for which (1) is estimated is arbitrary and no assumptions are made about the spectral support or bandwidth of $x(t)$. The estimation quality is measured by the mean square error (MSE): $\mathcal{E}_N(f) = \mathbb{E}[|\hat{X}_N(f) - X(f)|^2]$ where $\hat{X}_N(f)$ denotes the FT estimator constructed from the N samples. For an unbiased estimator, $\mathcal{E}_N(f) = \sigma^2[\hat{X}_N(f)]$.

3. FT STRATIFICATION: AN OVERVIEW

ToRa is one of the early alias-free-based FT estimators, with a convergence rate of N^{-1} [9]. It motivated the stratification strategies in [10, 11] for higher *point-wise* convergence rates.

3.1. Stratification in FT Estimation

Stratification entails selecting $L_N + 1$ time instants such that $0 < t_{N,0} < t_{N,1} < ... < t_{L_N} = H$ and defining the non-overlapping strata $\mathcal{T}_{N,l} = [t_{N,l}, t_{N,l+1}]$, $l = 0, 1, ..., L_N - 1$. The l^{th} stratum is of width $\Delta_{N,l} = t_{N,l+1} - t_{N,l}$ and its centre $c_{N,l} = 0.5(t_{N,l+1} + t_{N,l})$. The FT can be written as $X(f) = \sum_{l=0}^{L_N-1} I_{N,l}(f)$ and $I_{N,l}(f) = \int_{\mathcal{T}_{N,l}} x(t)w(t)e^{-j2\pi ft}dt$. The estimator $\hat{X}_N(f)$ of (1) can be expressed by the sum of individual estimators, one for each $I_{N,l}(f)$, as per

$$\hat{X}_N(f) = \sum_{l=0}^{L_N-1} \hat{I}_{N,l}(f). \tag{2}$$

Here and similar to [10, 11], the strata are constructed via the stratifying function $g(t)$, continuous in $\mathcal{T}$, separated from zero by $g_{min}(t) \leqslant g(t)$ such that $\int_{\mathcal{T}} g(t)dt = H$. The strata boundaries $\{t_{N,l}\}_{l=0}^{L_N}$ are the solution to the equation: $\int_0^{t_{N,l}} g(t)dt = lH/L_N$, implying that $\Delta_{N,l} \leqslant H/(g_{min}L_N)$.

Theorem 1[1]*:* For estimator $\hat{X}_N(f) = \sum_{l=0}^{L_N-1} \hat{I}_{N,l}(f)$ with $\hat{I}_{N,l} = \sum_{r=1}^{S} a_{N,l,r}(f)x(\tau_{N,l,r})$, using S samples in $\mathcal{T}_{N,l}$, if:
(A.1) $\mathbb{E}\{\hat{X}_N(f)\} = X(f)$, i.e. an unbiased estimator;
(A.2) $\hat{I}_{N,l}$, $l = 1, 2, 3...L_N$, are independent from each other;
(A.3) For any l and f, there exists A_r, $r = 1, 2..., S$, such that $|a_{N,l,r}(f)| \leqslant \Delta_{N,l}A_r$;
(A.4) There exist $D > 0$ such that $\sum_{l=0}^{L_N-1} \Delta_{N,l}A_r \leqslant N^{-1}D$ then, for any f and N and $D > 0$, we then have

$$\mathcal{E}_N(f) \leqslant N^{-1}\hat{\kappa} \tag{3}$$

Sketch of the Proof: It follows from $\hat{I}_{N,l}$ definition that the MSE $\mathcal{E}_N(f) \leqslant \sum_{l=0}^{L_N-1} \mathbb{E}\left\{(\sum_{s=1}^{S} |a_{N,l,r}(f)||x(\tau_{N,l,r})|)^2\right\}$ utilising (A.1) and (A.2). This can be shown to lead to (3) where $\hat{\kappa} = N^{-1}x_{0,max}^2 D \sum_{r=1}^{S} A_r^2$, given (A.3) and (A.4).

Theorem 1 sets sufficient conditions for $\hat{X}_N(f)$ to be guaranteed to uniformly converge to $X(f)$ at rate N^{-1} or faster.

3.2. Stratified and Antithetical Stratified Estimates

With StSa in [10] the sampling instants $\{\tau_n\}_{n=1}^{N}$ in $\hat{X}_{\text{StSa},N}(f)$ are independent random variables (one per statra, $S = 1$ and $N = L_N$) located as per the probability density function (pdf) $p_{\text{StSa}}(\tau) = \Delta_{N,l}^{-1}$ if $\tau \in \mathcal{T}_{N,l}$ and zero otherwise. The estimator of $I_{N,l}(f)$ is: $\hat{I}_{N,l}(f) = \Delta_{N,l}x(\tau_{N,l})\lambda(\tau_{N,l}, f)$. In the antithetical stratified estimator $\hat{X}_{\text{AnSt},N}(f)$, two samples per strata are collected, i.e. $N = 2L_N$ and $S = 2$ [11]. Whilst the first sample is selected randomly similar to StSa, $\tau_{N,2l}$ in the l^{th} stratum, the second one is taken in an antithetical manner, $\tau_{N,2l+1} = 2c_{N,l} - \tau_{N,2l}$. Its estimator is given by: $\hat{I}_{N,l}(f) = 0.5\Delta_{N,l}[x(\tau_{N,2l})\lambda(\tau_{N,2l}, f) + x(\tau_{N,2l+1})\lambda(\tau_{N,2l+1}, f)]$.
It can be shown that both StSa and AnSt satisfy all the assumptions of Theorem 1; (A.4) is fulfilled since $N = aL_N + b$ ($a \geqslant 1$, $b \geqslant 0$). Thus, both estimates uniformly converge at a rate equal or greater than N^{-1}. Let $\mathcal{P}_{\text{StSa}}$ and $\mathcal{P}_{\text{AnSt}}$ be the set of all rates at which the *unbiased* StSa and AnSt estimators can be guaranteed to uniformly converge to $X(f)$ for any f,

$$\sigma^2[\hat{X}_{\text{StSa},N}(f)] < N^{-p_1}\kappa; \;\; \sigma^2[\hat{X}_{\text{AnSt},N}(f)] < N^{-p_2}\kappa \tag{4}$$

such that $p_1 \in \mathcal{P}_{\text{StSa}}$, $p_2 \in \mathcal{P}_{\text{AnSt}}$ and $\kappa > 0$ is a constant.

Corollary 1[1]*:* $\mathcal{P}_{\text{StSa}} \in (0, 1]$ and $\mathcal{P}_{\text{AnSt}} \in (0, 1]$.
Proof Sketch: Given Theorem 1, showing that for $p > 1$, (4) does not hold for $x(t) = w(t) = g(t) = 1$ at $f = N$ suffices.

[1]The full proof will be published in an extended version of this paper [16].

This corollary combined with (3) stipulates that N^{-1} is the fastest uniform convergence rate achievable by the StSa and AnSt estimators. This is despite the reported expedited *point-wise* (i.e. frequency dependent) convergence rate of N^{-5} for AnSt, if $x(t)$ has a continuous second-order derivative [11].

4. HYBRID STRATIFIED FT ESTIMATION

Here, we introduce the HySt method and show its notably fast uniform convergence compared with ToRa, StSa and AnSt.

4.1. Sampling Scheme and Unbiased Estimator

In the HySt technique, the sampling instants are a mixture of deterministic, i.e. the strata boarders $\{t_l\}_{l=0}^{L_N}$, and random time instants, i.e. $\{\tau_l\}_{l=0}^{L_N-1}$ that are selected in the same manner as in StSa. For example, the l^{th} random sample is located in $\mathcal{T}_l$ according to the pdf: $p(\tau_l) = \Delta^{-1}$ if $\tau_l \in \mathcal{T}_l$ and zero elsewhere (the subscript N is discarded for the brevity of notation). Hence, the total number of processed signal samples is $N = 2L_N + 1$ and it follows that the strata width is upper bounded by $\Delta_l \leqslant 3H/Ng_{min}$, assuming $N \geq 3$. Similar to StSa and Ant, $N = aL_N + b$ implies that the HySt fulfills assumption (A.4) of Theorem 1.
The HySt estimator of $X(f)$ is given by

$$\hat{X}_{N,\text{HySt}}(f) = \sum_{l=0}^{L_N-1} \hat{I}_l(f) = \sum_{l=0}^{L_N-1} \alpha_l(f)x(t_l) + \beta_l(f)x(\tau_l) + \gamma_l(f)x(t_{l+1}), \quad (5)$$

where the estimator of FT in the l^{th} stratum, i.e. $I_l(f) = \int_{\mathcal{T}_l} x(t)w(t)e^{-j2\pi ft}dt$, is a linear combination of $x(t_l)$, $x(\tau_l)$ and $x(t_{l+1})$ such that
$\alpha_l = \Delta_l^{-1}\int_{\mathcal{T}_l}(t_{l+1} - t)\lambda(t,f)dt - (t_{l+1} - \tau_l)\lambda(\tau_l, f)$, $\gamma_l = \Delta_l^{-1}\int_{\mathcal{T}_l}(t - t_l)\lambda(t,f)dt - (\tau_l - t_l)\lambda(\tau_l, f)$, and $\beta_l = \Delta_l\lambda(\tau_l, f)$. Since $X(f) = \sum_{l=0}^{L_N-1} I_l(f)$ and the random sampling instants in separate strata (τ_k and τ_l for $k \neq l$) are independent, showing that $\hat{I}_l(f)$ is an unbiased estimator of $I_l(f)$ implies that $\mathbb{E}[\hat{X}_{N,\text{HySt}}(f)] = X(f)$.
We start with $\mathbb{E}[\alpha_l(f)x(t_l)] = \Delta_l^{-1}\int_{\mathcal{T}_l}(t_{l+1} - t)\lambda(t,f)dt - \mathbb{E}[(t_{l+1} - \tau_l)\lambda(\tau_l, f)] = 0$ since the pdf of τ_l is $p(\tau) = \Delta_l^{-1}$ if $\tau \in \mathcal{T}_l$ and zero elsewhere. Similarly, $\mathbb{E}[\beta_l(f)x(\tau_l)] = \int_{\mathcal{T}_l}\lambda(\tau, f)x(\tau)d\tau = I_l(f)$ and $\mathbb{E}[\gamma_l(f)x(t_{l+1})] = \Delta_l^{-1}\int_{\mathcal{T}_l}(t - t_l)\lambda(t,f)dt - \mathbb{E}[(\tau_l - t_l)\lambda(\tau_l, f)] = 0$. Noting that $\mathbb{E}[\hat{I}_{N,l}(f)] = \mathbb{E}[\alpha_l(f)x(t_l)] + \mathbb{E}[\beta_l(f)x(\tau_l)] + \mathbb{E}[\gamma_l(f)x(t_{l+1})]$, we conclude that $\hat{X}_{N,\text{HySt}}(f)$ is an unbiased estimator of $X(f)$ and it accordingly satisfies assumption (A.1) of Theorem 1.

4.2. Fast Uniform Convergence Rate

We start with confirming that Theorem 1 applies to the proposed HySt estimator. We recall that the random instants τ_k and τ_l are independent for $l \neq k$, thus, $\hat{I}_l(f)$ and $\hat{I}_k(f)$ are independent, i.e. estimator in (5) fulfils assumption (A.2). We note that $S = 3$ for the hybrid stratified sampling scheme. Since $\lambda_{0,max} = 1$ given $w(t) = 1$ for some $t \in \mathcal{T}$, it can be shown that $|\alpha_l| \leqslant \Delta_l^{-1}\int_{\mathcal{T}_l}(t_{l+1} - t)|\lambda(t,f)|dt + \Delta_l = 1.5\Delta_l$. Similarly, it can be easily seen that $|\alpha_l| \leqslant \Delta_l$ and $|\gamma_l| \leqslant 1.5\Delta_l$. Hence, $\hat{X}_{N,\text{HySt}}(f)$ satisfies assumption (A.3) of Theorem 1 ($A_1 = A_3 = 1.5$ and $A_2 = 1$) as well as (A.1), (A.2) and (A.4). This concludes the proof that HySt estimator uniformly converges to $X(f)$ at rate at least equal to N^{-1}.
Let $\mathcal{T}_B$ be some open time interval comprising $\mathcal{T}$ such that $\mathcal{T} \subset \mathcal{T}_B$. The next theorem states the main feature of the proposed HySt estimates:

Theorem 2[2]*:* If the signal $x(t)$ has continuous third order derivative in $\mathcal{T}_B$, the hybrid stratified sampling estimator converges uniformly to $X(f)$ in (1) at the rate N^{-5} or faster.

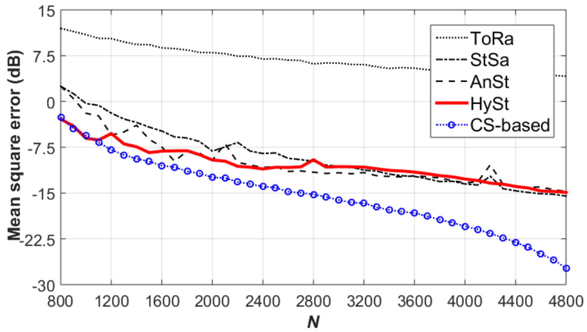
Sketch of Proof : Let $\Psi_l(f) = \hat{I}_l(f) - I_l(f)$ and $\mathcal{E}_N(f) = \sum_{l=0}^{L_N-1}\mathbb{E}\left\{|\Psi_l(f)|^2\right\}$. The second order Taylor expansion of $x(t)$ about c_l is: $x(t) = x(c_l) + (t - c_l)x^{(1)}(c_l) + 0.5(t - c_l)^2x^{(2)}(c_l) + r_l(t)$. Since $x^{(3)}(t)$ is bounded in $\mathcal{T}_B$, it can be shown that $|r_l(t)| \leqslant \Delta_l^3 x_{3,\max}/48$ where $t \in \mathcal{T}_l$, $r_l(t) = (t - c_l)^3x^{(3)}(\tilde{t})/6$ and $\tilde{t} \in \mathcal{T}_l$. After several manipulations, we reach $|\Psi_l(f)| \leqslant N^{-3}C$ and $\mathcal{E}_N(f) = \sigma^2\{\hat{X}_{N,\text{HySt}}(f)\} \leqslant N^{-5}C^2/2$; $C = \frac{45H^3}{8g_{\min}^3}x_{2,\max} + \frac{45H^4}{16g_{\min}^4}x_{3,\max}$.

This theorem illustrates that the HySt estimator provides a frequency-independent upper bound on the Fourier transform estimation errors that decays to zero at the rate N^{-5} (or faster) and the achieved accelerated convergence transpires simultaneously across all frequencies, unlike StSa and AnSt.

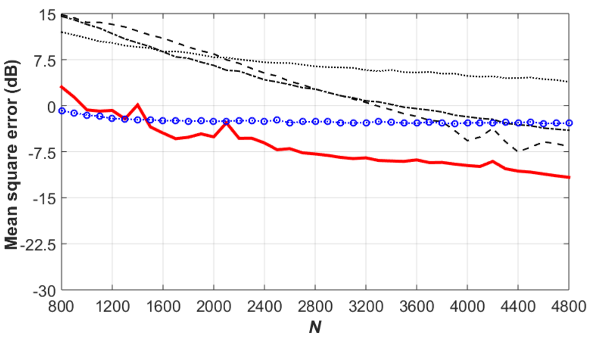
5. SIMULATIONS

We present an example to demonstrate the gains attained by the proposed hybrid stratified FT estimator and compare its performance to ToRa [9], StSa [10], AnSt [11] and a compressed sensing FT estimator. The latter uses random partial Fourier sensing matrices and the efficient greedy subspaces pursuit method [14, 17]; CS is not considered in [16]. Assume that the overall double-sided processed bandwidth is $\mathcal{B} = [-250, 250]$ KHz, without prior knowledge of the spectral support of the present signal $x(t)$, i.e., $f_{\text{Nyq}} = 500$ KHz. Let $x(t)$ be a multiband signal comprising four components of distinct magnitudes, bandwidths and central frequencies such that $x(t) = \sum_{m=1}^{4} A_m\text{sinc}\left(B_m(t - d)\right)\cos\left(2\pi f_m(t - d)\right)$ where $A_1 = 10^4$, $B_1 = 4$ KHz, $f_1 = 10$ KHz, $A_2 = 10^3$, $B_2 = 8$ KHz, $f_2 = 40$ KHz, $A_3 = 40$, $B_3 = 0$ (sinusoid), $f_3 = 70$ KHz, $A_4 = 2 \times 10^3$, $B_4 = 10$ KHz, $f_4 = 210$ KHz and $d = H/2$. A Hanning window of width $H = 10$ *m*s is

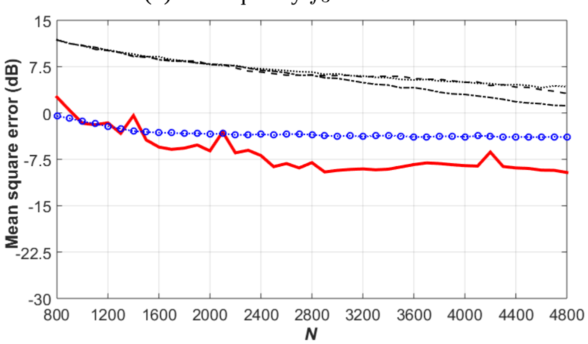
[2]The full proofs of all stated theorems will be published in an extended version of this paper, analysing the HySt estimates in more details; see [16].



(a) At frequency $f_1 = 10$ KHz.



(b) At frequency $f_3 = 70$ KHz.



(c) At frequency $f_4 = 210$ KHz.

Fig. 1: MSE (in dB) of the FT estimation for ToRa, StSa, AnSt, HySt and CS-based techniques at selected frequency points in $\mathcal{B}$.

employed; $N_{\text{Nyq}} = 5000$. For CS recovery, the sparsity level is set to $K = 200$ (to avoid prohibitively high time-memory requirements of inverting larger matrices as N increases).
Fig. 1 depicts the MSE, $\mathcal{E}_N(f)$, of the FT estimates as a function of the number of utilised signal samples, $N < N_{\text{Nyq}}$ (i.e. for sub-Nyquist rates), at the selected frequency points f_1, f_3 and f_4. Whereas, Fig. 2 exhibits the average ratio of the run time, $T_{\text{CS}}/T_{\text{HySt}}$, of an unoptimised MATLAB implementations of the compressed sensing and HySt estimators on a standard PC (Intel i7 CPU, 3.4 GHz) for various values of N. All plots in Figs. 1 and 2 are obtained from averaging the outcome of 2000 independent Monte Carlo simulations.
Fig.1 reveals that the HySt significantly outperforms its alias-free sampling predecessors as the assessed frequency in-

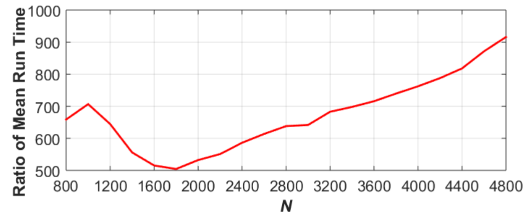


Fig. 2: Average relative run time, $T_{\text{CS}}/T_{\text{HySt}}$, of CS and HySt.

creases, namely for f_3 and f_4, and for $N \ll N_{\text{Nyq}}$. It can reduce the estimation MSE by $\approx$ 15 dB. Unlike the HySt method, it can be noticed that StSa and AnSt estimators are sensitive to the frequency for which the FT is estimated (for f_4 their results are similar to the basic ToRa). Their reported point-wise expedited convergence rates can only be observed for relatively low frequencies or excessive N. Conversely, the HySt uniform and point-wise convergence are the same, i.e. N^{-5}, leading to a consistent performance albeit f (for the low frequency f_1, HySt and AnSt produce similar results).
Compared to the CS-based estimation, HySt delivers a competitive performance; it achieves more accurate estimation results for f_3 and f_4. The fact that CS produces better results in Fig.1a is not surprising since the component at f_1 has the highest magnitude spectrum. Additionally, $X(f)$, which is identical to the signal's DFT at the examined frequencies, is notably sparse where $N \gg K$. Nevertheless, Fig. 2 shows that CS is immensely more computationally demanding than the HySt approach, with its run time being 500 to 920 times longer (the brief increase in the complexity ratio for low N can be explained by the CS suffering from instability/low-convergence in such cases). This highlights the computational cost of the CS technique, which can be excessive, especially for a portable platform with limited power, memory and processing capabilities (setting higher K values leads to even higher complexity/run-time of the CS recovery, however, it can produce more accurate results). Conversely, the HySt utilises a simple linear estimator that can be implemented using a modified FFT-type process. A complete comparison between the HySt estimator and other CS-based FT estimation methods is outside the scope of this paper.

6. CONCLUSION

A simple sub-Nyquist hybrid-stratified-sampling method to estimate the FT of a deterministic continuous-time signal, at arbitrary frequencies, from a number of its samples is introduced. It does not impose constraints on the bandwidth of the processed signal. The HySt estimator mean square error is shown to uniformly converge to zero at a rate of at least N^{-5}, enabling it to deliver more consistent estimates compared with its alias-free-type predecessors. This paper serves to motivate further research into using alias-free sampling for sub-Nyquist processing, e.g., for spectrum sensing [18, 19].

7. REFERENCES

[1] A Andreas, "Digital signal processing: Signals, systems, and filters," 2006.

[2] Fredric J Harris, "On the use of windows for harmonic analysis with the discrete Fourier transform," *Proceedings of the IEEE*, vol. 66, no. 1, pp. 51–83, 1978.

[3] Rodney G Vaughan, Neil L Scott, and D White, "The theory of bandpass sampling," *IEEE Trans. on Signal Processing,*, vol. 39, no. 9, pp. 1973–1984, 1991.

[4] Erik Axell, Geert Leus, Erik G Larsson, and H Vincent Poor, "Spectrum sensing for cognitive radio: State-of-the-art and recent advances," *IEEE Signal Processing Magazine*, vol. 29, no. 3, pp. 101–116, 2012.

[5] Jeffrey D Scargle, "Studies in astronomical time series analysis. ii-statistical aspects of spectral analysis of unevenly spaced data," *The Astrophysical Journal*, vol. 263, pp. 835–853, 1982.

[6] Harold S Shapiro and Richard A Silverman, "Alias-free sampling of random noise," *Journal of the Society for Industrial and Applied Mathematics*, vol. 8, no. 2, pp. 225–248, 1960.

[7] Elias Masry, "Alias-free sampling: An alternative conceptualization and its applications," *IEEE Trans. on Information Theory*, vol. 24, no. 3, pp. 317–324, 1978.

[8] Ivars Bilinskis and AK Mikelson, *Randomized signal processing*, Prentice-Hall, Inc., 1992.

[9] Andrzej Tarczynski and Najib Allay, "Spectral analysis of randomly sampled signals: suppression of aliasing and sampler jitter," *IEEE Trans. on Signal Processing*, vol. 52, no. 12, pp. 3324–3334, 2004.

[10] Elias Masry, "Random sampling of deterministic signals: statistical analysis of Fourier transform estimates," *IEEE Trans.on Signal Processing*, vol. 54, no. 5, pp. 1750–1761, 2006.

[11] Elias Masry and Aditya Vadrevu, "Random sampling estimates of Fourier transforms: antithetical stratified monte carlo," *IEEE Trans. on Signal Processing*, vol. 57, no. 1, pp. 194–204, 2009.

[12] Pang Feng and Yoram Bresler, "Spectrum-blind minimum-rate sampling and reconstruction of multiband signals," in *IEEE Int. Conf. on Acoustics, Speech and Signal Processing (ICASSP '96)*, 1996, vol. 3, pp. 1688–1691.

[13] Raman Venkataramani and Yoram Bresler, "Perfect reconstruction formulas and bounds on aliasing error in sub-Nyquist nonuniform sampling of multiband signals," *IEEE Trans. on Information Theory*, vol. 46, no. 6, pp. 2173–2183, 2000.

[14] Marco F Duarte and Yonina C Eldar, "Structured compressed sensing: From theory to applications," *IEEE Trans. on Signal Processing*, vol. 59, no. 9, pp. 4053–4085, 2011.

[15] Moshe Mishali and Yonina C Eldar, "Blind multiband signal reconstruction: Compressed sensing for analog signals," *IEEE Trans. on Signal Processing*, vol. 57, no. 3, pp. 993–1009, 2009.

[16] Andrzej Tarczynski and Bashar I. Ahmad, "Estimation of Fourier transform using alias-free hybrid-stratified sampling," *IEEE Transactions on Signal Processing (submitted-revised)*, 2016.

[17] Wei Dai and Olgica Milenkovic, "Subspace pursuit for compressive sensing signal reconstruction," *Information Theory, IEEE Transactions on*, vol. 55, no. 5, pp. 2230–2249, 2009.

[18] Bashar Ahmad and Andrzej Tarczynski, "A SARS method for reliable spectrum sensing in multiband communication systems," *IEEE Transactions on Signal Processing*, vol. 59, no. 12, pp. 6008–6020, 2011.

[19] Bashar Ahmad and Andrzej Tarczynski, "Spectral analysis of stratified sampling: A means to perform efficient multiband spectrum sensing," *IEEE Transactions on Wireless Communications*, vol. 11, no. 1, pp. 178–187, 2012.