NON-MONOTONE QUADRATIC POTENTIAL GAMES WITH SINGLE QUADRATIC CONSTRAINTS

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ABSTRACT

We consider the problem of solving a quadratic potential game with single quadratic constraints, under no monotonicity condition of the game, nor convexity in any of the player's problem. We show existence of Nash equilibria (NE) in the game, and propose a framework to calculate Pareto efficient solutions. Regarding the corresponding non-convex potential function, we show that strong duality holds with its corresponding dual problem, give existence results of solutions and present conditions for global optimality. Finally, we propose a centralized method to solve the potential problem, and a distributed version for compact constraints. We also present simulations showing convergence behavior of the proposed distributed algorithm.

Index Terms— Non-convex quadratic optimization, non-monotone games, Nash equilibrium, Pareto efficiency

1. INTRODUCTION

Quadratic problems have been of great interest due to the strong results that can be derived from them, leading to optimal solvability and efficient computation. For this reason, they have been widely used in the literature as suitable approximations of general models, and form the basis of analysis of trust region methods [1]. In this paper we propose a quadratic potential formulation with single constraints that extends the optimization problem into a game, which is not necessarily monotone, and where the players objective functions are not necessarily convex. We present results regarding existence and computation of global solutions, and provide a framework that can lead to the development of new algorithms for non-monotone games.

Quadratic games have a long history of research, for instance using linear constraints in dynamic games [2], and is fundamental in control theory [3, Ch.4]. They have been used widely as approximating models, such as those summarized in [4, Ch.5.B], in portfolio game optimization [5] and smart grid applications such as in demand-side management [6], among others. Thus, quadratic problems, as well as quadratic games, are pervasive both in models and algorithms in many different applications.

The state of the art in solving quadratic games with quadratic constraints is, as far as we know, that of solving monotone games [7]. This implies, that not only should these games present a convex objective function, i.e., with positive semidefinite Hessian matrix, but also the *coupling* between individual problems should be limited. In our framework, we skip these conditions: allowing non-convex objectives and constraints, and assuming no limitation in the *coupling* among players. Furthermore, our NE existence results are not based

on the monotonicity properties of the game [8], but on the potential problem and its dual.

Being the focus of this paper solving potential games, the analysis of the potential function is critical in devising centralized and distributed algorithms. However, general non-convex quadratic optimization problems with multiple constraints remains an unsolved problem [9], and few cases, such as the quadratic programs with single quadratic constraints have known solution [10, Appx.B]. Given the quadratic potential program that results from our game, we extend the results of problems with single constraints to problems with non-overlapping constraints. In order to do that, we build on the results of previous work [11], which showed global optimality conditions of quadratic problems under the S-property requirement. This property is for instance analyzed in [12]. In our results, we proved that the S-property is satisfied in the potential problem, which is a sufficient condition for strong duality to hold. Finally, this methodology allowed us to propose both a centralized and distributed algorithm to globally solve the quadratic game.

The paper is structured as follows. In Section 2 we introduce the game model and the potential function. We analyze the properties of the player's individual optimization problems and provide existence results of NE. In Section 3 we present results regarding the strong duality of the potential problem, and give existence results of such solutions. Also, we briefly characterize Pareto efficient solutions of the game within the quadratic framework. Finally, in Sections 4 and 5 we present the algorithmic framework to globally solve the game, and simulations to support our results, respectively.

2. QUADRATIC POTENTIAL GAME

Given a set of players $Q = \{1, ..., Q\}$, we introduce the quadratic potential game \mathcal{G}_p where every player $i \in Q$ has to solve

$$\min_{x_i \in \mathbb{R}^n} \quad f_i(x_i, x_{-i}) = x_i^T A_0^{ii} x_i + 2 \sum_{j \neq i} x_j^T A_0^{ij} x_i + 2b_{0i}^T x_i + c_{0i} \tag{1}$$
s.t. $h_i(x_i) = x_i^T A_1^i x_i + 2b_{1i}^T x_i + c_{1i} \le 0$

where A_0^{ii} , $A_1^i \in S^n$, $A_0^{ij} \in \mathbb{R}^{n \times n}$, S^n is the set of symmetric matrices of size n; b_{0i} , $b_{1i} \in \mathbb{R}^n$ are column vectors; and c_{0i} , $c_{1i} \in \mathbb{R}$ are scalar numbers. We do not assume that A_0^{ii} , $A_1^i \succeq 0$, so for every player $i \in Q$, problem (1) is not a convex optimization problem. We refer to the user optimization variables as $x_i \in \mathbb{R}^n$, $x_{-i} = (x_j)_{j \neq i}$ and globally to all of them with $x = (x_i)_{i=1}^Q$. Then, the game is potential if, and only if, its Jacobian given by

$$A_0 = \begin{bmatrix} A_0^{11} & \cdots & A_0^{1N} \\ \vdots & \ddots & \vdots \\ A_0^{N1} & \cdots & A_0^{NN} \end{bmatrix},$$

is symmetric, i.e., $A_0^{ij} = (A_0^{ji})^T$. This can be readily seen since we can express every user's objective in the form

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$$f_i(x_i, x_{-i}) = V(x) + u_i(x_{-i})$$
(2)

where

$$V(x) = x^{T} A_{0} x + \sum_{j} \left(2b_{0j}^{T} x_{j} + c_{0j} \right)$$
(3)

is the potential function associated to the game, and

$$u_i(x_{-i}) = \sum_{j \neq i} \sum_{r \neq i} -x_r^T A_0^{rj} x_j - 2b_{0j}^T x_j - c_{0j}$$
(4)

is a term proper of *dummy games* [13, Prop.1]. The potential function V(x) is useful because the global maximum is itself an NE of the game; we will analyze some of its properties in Section 3. Before that, we further analyze the game and existence of solutions.

2.1. Analysis of the player's optimization problem

We make the following assumption throughout the whole paper:

Assumption 1. There
$$\exists \bar{x}_i \in \mathbb{R}^n \mid h_i(\bar{x}_i) < 0$$
 for every $i \in Q$.

Given Assumption 1, known as Slater's condition, strong duality holds for each player's primal problem (1), and can be reformulated in the form of its dual problem [10, Appx.B.1]. The dual function takes the form

$$g_i(\lambda_i, x_{-i}) = \begin{cases} -(b_{gi} + \lambda_i b_{1i})^T (A_0^{ii} + \lambda_i A_1^i)^\dagger (b_{gi} + \lambda_i b_{1i}) \\ +c_{0i} + \lambda_i c_{1i} & \text{if } A_0^{ii} + \lambda_i A_1^i \succeq 0 \\ \\ \text{and } (b_{gi} + \lambda_i b_{1i}) \in \mathcal{R}(A_0^{ii} + \lambda_i A_1^i) \\ -\infty & \text{otherwise} \end{cases}$$

where $b_{gi} = b_{0i} + \sum_{j \neq i} A_0^{ij} x_j$, Z^{\dagger} is the Moore-Penrose pseudoinverse of Z, $\mathcal{R}(Z)$ represents the range of Z and $\lambda_i \geq 0$. Then, the maximum of $g_i(\lambda_i, x_{-i})$ characterizes the solution of (1) for given strategies x_{-i} . The previous formulation allows us to establish lower and upper bounds on λ_i , namely

$$\{\lambda_i^{\min} | \lambda_i^{\max}\} = \arg\{\min | \max\} \lambda_i$$

s.t. $A_0^{ii} + \lambda_i A_1^i \succeq 0, \quad \lambda_i \ge 0$ (5)

which effectively define the convex region

$$\Lambda_i = \left\{ \lambda_i \in \mathbb{R}_+ \, | \, \lambda_i^{\min} \le \lambda_i \le \lambda_i^{\max} \right\}$$

where \mathbb{R}_+ represents the nonnegative real numbers, and λ_i^{\max} may be unbounded. It is also useful to define

$$X_i = \left\{ x_i \in \mathbb{R}^n \,|\, h_i(x_i) \le 0 \right\} \quad \text{and} \quad X_{-i} = \prod_{j \ne i} X_j.$$

Throughout the paper we make the following assumption:

Assumption 2. For all $\lambda_i \in \operatorname{relint}(\Lambda_i)$ and for all $x_{-i} \in X_{-i}$, it is satisfied that $(b_{0i} + \sum_{j \neq i} A_0^{ij} x_j + \lambda_i b_{1i}) \in \mathcal{R}(A_0^{ii} + \lambda_i A_1^i), \forall i.$

Assumption 2 guarantees that $g_i(\lambda_i, x_{-i})$ is continuous. The condition is not difficult to satisfy in practice, in fact, $\mathcal{R}(A_i^{0i} + \lambda_i A_i^1)$ does not vary for $\lambda_i \in \mathbf{relint}(\Lambda_i)$, so the only requisite is that the other coefficients belong to such subspace.

We can now present the following theorem. A definition of coercivity can be found in [8, Eq.2.1.4].

Theorem 1. Suppose Assumptions 1 and 2 are satisfied and the set relint Λ_i is nonempty. Then, it follows:

- 1. Functions $g_i(\lambda_i, x_{-i})$ are coercive in λ_i for all $i \in Q$.
- 2. The set of maxima of $g_i(\lambda_i, x_{-i})$ is nonempty and compact for all $x_{-i} \in X_{-i}$.
- 3. The set of solutions of problem (1) is nonempty $\forall x_{-i} \in X_{-i}$.

Proof. See Appendix A.

2.2. Existence results of NE

The previous result showed existence of solutions for the dual problems, but it did not account for the existence of NE in the game. We will use the following definition: a point x^* is an NE if

$$f_i(x_i^*, x_{-i}^*) \le f_i(x_i, x_{-i}^*) \quad \forall x_i \in X_i$$
 (6)

is satisfied for every player $i \in Q$. In other words, a strategy x_i^* minimizes the user's objective function f_i given the fixed strategies x_{-i}^* of other players, for all players $i \in Q$.

The state of the art in establishing existence of NE in general potential games is normally guaranteed through the existence of solution of the equivalent potential problem [14], which we introduce below in (7). In the specific case of quadratic games with single quadratic constraints we can establish the following result:

Proposition 1. Suppose Assumptions 1 and 2 are satisfied. Then, an NE exists if and only if a solution to the potential problem exists.

For lack of space we will provide a formal proof in an extension of this paper. Suffice it to say that the KKT conditions derived from the game and from the potential problem are equal and, therefore, the conditions for the existence of solutions for both problems is the same. In Section 3 we analyze sufficient conditions for the existence of solutions of the potential problem and, thus, for the game.

3. QUADRATIC OPTIMIZATION PROBLEM WITH NO OVERLAPPING CONSTRAINTS

Solving the potential function (3) provides an NE solution of the game. Therefore, in this section we analyze some properties of the potential problem. We introduce the following notation:

$$b_0 = (b_{0i})_{i=1}^Q, \ b_1 = (b_{1i})_{i=1}^Q, \ c_0 = (c_{0i})_{i=1}^Q$$

$$A_1 = \operatorname{diag}[A_1^1, \dots, A_1^i, \dots, A_1^Q]$$

$$D(\boldsymbol{\lambda}) = \operatorname{diag}[\boldsymbol{\lambda}] \otimes I_{n \times n}, \ c_1 = (c_{1i})_{i=1}^Q$$

where "diag" is the block diagonal matrix operator and " \otimes " is the Kronecker product. Now we can express the potential problem as

$$\min_{x} \quad V(x) = x^{T} A_{0} x + 2b_{0}^{T} x + \mathbf{1}_{n \times 1}^{T} c_{0}
\text{s.t.} \quad x_{i}^{T} A_{1}^{i} x_{i} + 2b_{1i}^{T} x_{i} + c_{1i} \leq 0 \quad \forall i \in \mathcal{Q}.$$
(7)

The dual function of (7) is given by:

$$q(\boldsymbol{\lambda}) = \begin{cases} -(b_0 + D(\boldsymbol{\lambda})b_1)^T (A_0 + D(\boldsymbol{\lambda})A_1)^{\dagger} (b_0 + D(\boldsymbol{\lambda})b_1) \\ + \mathbf{1}_{n \times 1}^T c_0 + \boldsymbol{\lambda}^T c_1 & \text{if } A_0 + D(\boldsymbol{\lambda})A_1 \succeq 0 \\ & \text{and } (b_0 + D(\boldsymbol{\lambda})b_1) \in \mathcal{R}(A_0 + D(\boldsymbol{\lambda})A_1) \\ -\infty & \text{otherwise.} \end{cases}$$

Hence, the dual problem of (7) is

$$\max_{\boldsymbol{\lambda} \ge 0} \quad q(\boldsymbol{\lambda}). \tag{8}$$

The following theorem shows strong duality between the problems: **Theorem 2.** Suppose Assumptions 1 and 2 hold. Then, the primal problem (7) and the dual problem (8) have zero duality gap.

We can identify the (convex) feasibility region of $q(\lambda)$

$$\Gamma = \left\{ \boldsymbol{\lambda} \in \mathbb{R}^Q_+ \, | \, A_0 + D(\boldsymbol{\lambda}) A_1 \succeq 0 \right\}$$

and give some conditions on the existence of solutions of (8):

Theorem 3. Suppose Assumptions 1 and 2 are satisfied and the set relint Γ is nonempty: Then, the following holds:

- 1. Function $q(\boldsymbol{\lambda})$ is coercive in $\boldsymbol{\lambda}$.
- 2. The set of maxima of $q(\lambda)$ is nonempty and compact.
- 3. The set of solutions of problem (7) is nonempty.

Proof. See Appendix A.

3.1. Pareto efficient problem formulation

Before finishing this section, we briefly consider the problem of finding Pareto efficient points of game \mathcal{G}_p . One method to find these, is to solve the scalarization problem [10, Sec.4.7.5], i.e., solving

$$\min_{\substack{x_i \in \mathbb{R}^n \,\forall i \\ \text{s.t.}}} \sum_{i=1}^{Q} \beta_i f_i(x_i, x_{-i}) \tag{9}$$
s.t. $h_i(x_i) \leq 0, \quad \forall i \in \mathcal{Q}$

where $\beta_i > 0$ weights the relative importance of objective function *i* with the rest. These weights define the inward normal vector to the optimal trade-off surface described by problem (9), and yield a particular Pareto efficient solution. Problem (9) is quadratic with no overlapping constraints and has a form similar to (7). This allows (9) to be solved with the algorithms from Section 4.

Finally, consider the potential quadratic game with form

$$\begin{array}{ll} \min_{x_i \in \mathbb{R}^n} & V(x_i, x_{-i}) \\ \text{s.t.} & h_i(x_i) \le 0 \end{array} \quad \forall i \in \mathcal{Q}, \end{array}$$

i.e., eq. (2) with $u_i(x_{-i}) = 0$ for every $i \in Q$. Then, the maximum solution of the potential (7) is both an NE and a Pareto efficient point.

4. ALGORITHMIC FRAMEWORK

A centralized solution of game \mathcal{G}_p is straightforward when solving the dual problem of the potential function. Since (8) is concave (all dual problems are concave) and, if Γ is nonempty, one can simply use convex optimization solvers and find an optimal dual solution λ^* . Then, in order to obtain the primal solution of (7), solve

$$\min_{x} \quad x^{T} (A_{0} + D(\boldsymbol{\lambda}^{*})A_{1})x + (b_{0}^{T} + b_{1}^{T}D(\boldsymbol{\lambda}))x, \qquad (10)$$

which is convex since $A_0 + D(\lambda^*)A_1 \succeq 0$, and obtain

$$\mathbf{x}^* \in -(A_0 + D(\boldsymbol{\lambda}^*)A_1)^{\dagger} (b_0^T + b_1^T D(\boldsymbol{\lambda})) + \mathcal{N}(A_0 + D(\boldsymbol{\lambda}^*)A_1)$$

where x^* is an NE of \mathcal{G}_p , and $\mathcal{N}(Z)$ represents the nullspace of Z. Note that x^* belongs to a subspace if the solution is not unique.

For a distributed algorithm we focus on the case in which all A_1^i are positive definite. This implies that regions X_i are compact and that a solution to potential problem (7) and game \mathcal{G}_p exists. Note that A_0^{ii} has no restrictions and the problem remains non-convex. If $A_1^i \neq 0$, a more general distributed algorithm would be necessary, and it will be presented in an extension of this paper. The centralized method is general and is still valid in this case.

We show in Algorithm 1 the steps to solve such problems. In line 4 of Algorithm 1, information of all users is aggregated, and in line 16 the operator $\Pi_{\Gamma}((\lambda_i^k)_{i=1}^Q)$ represents the Euclidean projection onto set Γ . Both operations need to be performed by a central unit, and lines between 5-15 can be performed in parallel or distributed. The rest of the algorithm is based on a bisection



Fig. 1. Mean convergence values over 200 simulations.

scheme where $\overline{\lambda}_i$, $\underline{\lambda}_i$ represent the upper and lower limits, respectively. These values are calculated in lines 7-8 and are guaranteed to be found because of the coercivity of the problem. Finally, lines between 10-15 perform the bisection algorithm until convergence. On the special case in which $\lambda_i \approx 0$, then the loop can finish, since complementarity slackness is satisfied. Otherwise, it is required that $h_i(x_i) \approx 0$ for the same reason.

For lack of space we do not present proof of convergence, but we will include it in a future extension of the paper. To support our results, in Section 5 we performed extensive simulations of Algorithm 1 plotting the convergence behavior, and verified that optimality conditions were satisfied.

Algorithm 1 Distributed Jacobi scheme $(A_1^i \succ 0 \forall i \in \mathcal{Q})$	
1:	Initialize $(x_i^0)_i$. Determine $\lambda_i^{\min} \forall i$. Set $k \leftarrow 0$.
2:	while $ x^k - x^{k-1} \ge \varepsilon_{\text{outer}} \operatorname{do}$
3:	Set $k \leftarrow k+1$.
4:	Calculate $b_{gi} = b_{0i} + \sum_{i \neq i} A_0^{ij} x_j, \forall i$ //Mix strategies
5:	for $i\in\mathcal{Q}$ do
6:	Set $\underline{\lambda}_i = \lambda_i^{\min}$, $\overline{\lambda}_i = 2\lambda_i^{\min} + 1$, and $\overline{x}_i = \hat{x}_i(\overline{\lambda}_i, b_{gi})$.
7:	while $h_i(\overline{x}_i) \ge 0$ do //Find bisection limits
8:	Update $\underline{\lambda}_i = \overline{\lambda}_i$; $\overline{\lambda}_i = 2\underline{\lambda}_i$. Solve $\overline{x}_i = \hat{x}_i(\overline{\lambda}_i, b_{gi})$
9:	Set $\Psi_{\rm cost} \ge \epsilon_{\rm inner}$
10:	while $ \Psi_{\text{cost}} \ge \varepsilon_{\text{inner}} \operatorname{do}$ //Perform bisection steps
11:	Set $\lambda_i^k = \frac{1}{2}(\overline{\lambda}_i + \underline{\lambda}_i)$, determine $x_i^k = \hat{x}_i(\lambda_i^k, b_{gi})$.
12:	if $h_i(\overline{x}_i) \leq 0$, then $\overline{\lambda}_i = \lambda_i$
13:	else, $\underline{\lambda}_i = \lambda_i$.
14:	if $\lambda_i^k > 0$, then $\Psi_{\text{cost}} = h_i(x_i^k)$ //Slackness violation
15:	else, $\Psi_{\text{cost}} = 0$ //case $\lambda_i \approx 0$
16:	Solve $(\lambda_i^k)_{i=1}^Q = \prod_{\Gamma}((\lambda_i^k)_{i=1}^Q)$, update $x_i^k = \hat{x}_i(\lambda_i^k, b_{gi})$.

5. SIMULATIONS

In this section we present convergence results for 200 simulated games of model \mathcal{G}_p with Q = 10 payers, and size of each player's problem n = 4. Local constraints were generated using the model

$$\|(A_1^i)^{1/2}x_i - b_{1i}^{\rm sq}\|^2 \le c_{1i}^{\rm sq} \tag{11}$$

where coefficients of matrix $(A_1^i)^{1/2}$ and vector b_{1i}^{sq} were Gaussian distributed, and c_{1i}^{sq} uniform distributed with positive support. The coefficients of the player's objective function, A_0^{ii} , A_0^{ij} , b_{0i} and c_{0i} were also Gaussian distributed. In Figure 1 we show the mean values of the convergence criteria chosen in Algorithm 1 versus the number of iterations. In all cases, the algorithm converged efficiently in few iterations.

6. CONCLUSIONS

We presented a framework for quadratic potential games with single constraints assuming neither convexity nor monotonicity. We analyzed the existence of NE, studied the potential problem through its dual equivalent, and proved strong duality among them. Furthermore, we characterized the global optimality criteria and gave existence results. We proposed a centralized method and an efficient distributed algorithm to reach these equilibrium points. We also introduced the problem to obtain Pareto efficient points within the same quadratic framework.

Our current work focuses on extending these results to Nash Equilibrium Problems (NEPs) without being necessarily potential, and investigate the impact this framework can originate in nonquadratic games through iterative quadratic approximations.

A. APPENDIX

Proof of Theorem 1. For the first part (coercivity) we need to prove that if $\lambda_i \to \infty$, it necessarily implies that $g_i(\lambda_i, x_{-i}) \to -\infty$, since we are maximizing. We can distinguish two cases: *i*) if A_i^i has at least one negative eigenvalue, and *ii*) all eigenvalues are nonnegative. In the first case, $\lambda_i^{\max} < \infty$, therefore Λ_i becomes compact, and $g_i(\lambda_i, x_{-i}) = -\infty$ for $\lambda_i \notin \Lambda_i$. In the second case λ_i^{\max} is unbounded, so we have $A_0^{ii} + \lambda_i A_1^i \succeq 0$ for all $\lambda_i \geq \lambda_i^{\min}$ and $\lim_{\lambda_i \to \infty} g_i(\lambda_i, x_{-i})$ results in

$$\lim_{\lambda_i \to \infty} \lambda_i \left(c_{1i} - \left(\frac{b_{gi}}{\lambda_i} + b_{1i} \right)^T \left(\frac{1}{\lambda_i} A_0^{ii} + A_1^i \right)^\dagger \left(\frac{b_{gi}}{\lambda_i} + b_{1i} \right) \right) = \\\lim_{\lambda_i \to \infty} \lambda_i \left(c_{1i} - b_{1i}^T (A_1^i)^\dagger b_{1i} \right)$$
(12)

where we have used the fact that the singular values of the pseudoinverse are continuous in λ_i . Considering the case in which $A_1^i \succeq 0$, we can express $h_i(x_i)$ in the following form:

$$h_{i}(x_{i}) = \left\| (A_{1}^{i})^{1/2} x_{i} + b_{1i}^{sq} \right\|^{2} - c_{1}^{sq} = x_{i}^{T} A_{1}^{i} x_{i} + 2b_{i}^{T} x_{i} + c_{1i} \leq 0$$
(13)

with $A_1^i = (A_1^i)^{T/2} (A_1^i)^{1/2}$, $b_{1i}^T = (b_{1i}^{sq})^T (A_1^i)^{1/2}$, $c_{1i} = (b_{1i}^{sq})^T b_{1i}^{sq} - c_1^{sq}$, and, because Slater's condition is satisfied, we necessarily have $c_1^{sq} > 0$. We calculate the previous relation:

$$c_{1i} - b_{1i}^{T} (A_1^i)^{\dagger} b_{1i} = (b_{1i}^{sq})^T b_{1i}^{sq} - c_1^{sq} - (b_{1i}^{sq})^T (A_1^i)^{1/2} (A_1^i)^{\dagger/2} (A_1^i)^{\dagger T/2} (A_1^i)^{T/2} (b_{1i}^{sq}) = -c_1^{sq} < 0^{(14)}$$

and conclude that $\lim_{\lambda_i \to \infty} g_i(\lambda_i, x_{-i}) = -\infty$.

The second part (existence of solution) is immediate if we apply Weierstrass' Theorem [15, Prop.3.2.1]. By strong duality [10, Appx.B1], a solution of the primal also exists. \Box

Proof of Theorem 2. The strong duality case presented in [10, Appx.B.1] does not apply to problem (7), since in our case we have multiple quadratic constraints, rather than a single one. We can however resort to [11, Th.3.1] which states that if the *S-Property* is satisfied on the constraints (as defined in [11, Def.2.2]), then a feasible point of any quadratic problem (QP) is globally optimal if and only if it satisfies the KKT conditions, plus the added requirement

$$A_0 + \sum_{i=1}^Q \lambda_i \widetilde{A}_1^i \succeq 0 \tag{15}$$

where Q is the total number of constraints and \tilde{A}_1^i is the corresponding Hessian matrix of constraint *i*, in our case given by equation (17). Then, in our problem, solving the KKT constraints of problem (7) plus condition (15) is equivalent to solving the dual problem (8), under the assumption that strong duality holds. And thereby, to complete the proof it is only required that (7) satisfies the *S*-Property.

For lack of space we cannot give a complete description of how this property is satisfied, but we will include it in a future extension of the paper. However, we enunciate the steps to achieve such result, since the procedure is similar to the one presented in [10, Appx.B.2 and B.4]. We introduce new notation:

$$\widetilde{b}_{0i} = [0_{n \times 1}^T, \dots, b_{1i}^T, \dots, 0_{n \times 1}^T]^T$$
(16)

$$A_1^i = \operatorname{diag}[0_{n \times n}, \dots, A_1^i, \dots, 0_{n \times n}]$$
(17)

where the previous expressions have nonzero elements in positions i. Because Slater's condition is satisfied (Assumption 1), matrices

$$\overline{A}_{i} = \begin{bmatrix} \widetilde{A}_{1}^{i} & \widetilde{b}_{0i} \\ \widetilde{b}_{0i}^{T} & c_{1i} \end{bmatrix}, \quad \forall i \in \mathcal{Q}$$
(18)

have at least one negative eigenvalue and, therefore,

$$au_i \ge 0 \ \forall i, \ \sum_i \tau_i \overline{A}_i \succeq 0 \Longrightarrow \tau_i = 0 \ \forall i \Longrightarrow \sum_i \tau_i \overline{A}_i = 0.$$
 (19)

Expression (19) is fulfilled in our problem because of the sparsity of matrices \overline{A}_i , due to the no-overlap between the constraints. In general, the above implication does not hold and this is why for QP with more than a single quadratic constraint, strong duality is not met. With expression (19) being fulfilled, we can use the theorem of alternatives presented in [10, Example 5.14], with a slight modification due to $\lambda_i \geq 0$. Then, following similar steps to those described in [10, Appx.B.4], the property can be proven to be satisfied.

Proof of Theorem 3. We need to show that if $\|\boldsymbol{\lambda}\| \to \infty$, then $q(\boldsymbol{\lambda}) \to -\infty$. Let's indicate $M = \{i \in \mathcal{Q} | \lambda_i \to \infty\}$ and $\overline{M} = \{i \in \mathcal{Q} | \lambda_i \to \infty\}$, and require *M* is nonempty. We define

$$D_M = \operatorname{diag}[\boldsymbol{\lambda}_M] \otimes I_{n \times n}$$
 (20)

$$D_{\overline{M}} = \operatorname{diag}[\boldsymbol{\lambda}_{\overline{M}}] \otimes I_{n \times n} \tag{21}$$

where λ_M is a column vector of size Q, which has value entry λ_i in the *i*'th position if $i \in M$, and zero entry if $i \notin M$. Likewise, $\lambda_{\overline{M}}$ has λ_i value in the *i*'th position if $i \in \overline{M}$, and zero entry if $i \notin \overline{M}$. We have omitted the dependence of λ in D_M and $D_{\overline{M}}$ to save space. Also notice that $D_M + D_{\overline{M}} = D(\lambda)$ and $D_M D_{\overline{M}} =$ $D_M D_{\overline{M}}^{\dagger} = D_{\overline{M}} D_M^{\dagger} = 0$. We can distinguish two cases: *i*) if there $\exists i \in M \mid A_1^i \not\succeq 0$, then $q(\lambda) \to -\infty$; and *ii*), if $\forall i \in M$, $A_1^i \succeq 0$, then we can establish the following limit:

$$\lim_{\|\boldsymbol{\lambda}\|\to\infty} (A_0 + D(\boldsymbol{\lambda})A_1)^{\dagger} = \lim_{\|\boldsymbol{\lambda}\|\to\infty} D_M^{\dagger} (D_M^{\dagger}A_0 D_M^{\dagger} D_M + A_1)^{\dagger} + D_M^{\dagger} (D_M^{\dagger}A_0 D_M^{\dagger} D_{\overline{M}} + A_1)^{\dagger} = \lim_{\|\boldsymbol{\lambda}\|\to\infty} D_M^{\dagger} A_1^{\dagger} + D_{\overline{M}}^{\dagger} (D_M^{\dagger} A_0 D_{\overline{M}}^{\dagger} D_{\overline{M}} + A_1)^{\dagger}$$

and combine the previous intermediate result with

$$\lim_{\|\boldsymbol{\lambda}\|\to\infty} q(\boldsymbol{\lambda}) = \lim_{\|\boldsymbol{\lambda}\|\to\infty} D_M c_1 + D_{\overline{M}} c_1$$
$$-(b_0 + (D_M + D_{\overline{M}})b_1)^T (A_0 + D(\boldsymbol{\lambda})A_1)^{\dagger} (b_0 + (D_M + D_{\overline{M}})b_1)$$
$$= \lim_{\|\boldsymbol{\lambda}\|\to\infty} D_M (c_1 - b_1^T A_1^{\dagger} b_1) + \Psi (D_{\overline{M}}) = -\infty$$

where, as shown in proof of Theorem 1 in (14), $c_1 - b_1^T A_1^{\dagger} b_1 < 0$; and $\Psi(D_{\overline{M}})$ combines all terms that do not affect the limit.

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