ACCURATE ASYMPTOTIC ANALYSIS FOR JOHN'S TEST IN MULTICHANNEL SIGNAL DETECTION

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ABSTRACT

John's test, which is also known as the locally most invariant test for sphericity of Gaussian variables, is one of the most frequently used methods in multichannel signal detection. The application of John's test requires closed-form and accurate formula to set threshold according to a prescribed false alarm rate. Asymptotic expansion is a powerful method in deriving the threshold expressions of detectors for large samples. However, the existing asymptotic analysis of John's test in the real-valued Gaussian case is not accurate, causing the obtained false alarm rate to deviate from the preset value. This work first corrects a miscalculation in the existing results. Then this accurate approach is extended to the complex-valued case. In this scenario our result is as accurate as the state-of-the-art scheme but enjoys higher computational efficiency.

***Index Terms*—** John's test, sphericity, decision threshold, asymptotic expansion

1. INTRODUCTION

The test of sphericity has wide applications in a variety of research areas, including source detection [1], spectrum sensing [2–4] and image processing [5]. In [6], the author derived the generalized likelihood ratio test (GLRT) for this test. This problem was revisited by [7] and [8], where the locally most powerful invariant test (LMPIT) for sphericity, i.e., the John's test, was derived. As inherently designed for detecting small deviations from the null hypothesi, John's test shows good performance in the low signal-to-noise ratio (SNR) regime, thereby being a preference in many practical applications. However, despite its detection power, the practical implementation of this detector requires accurate and closed-form formula of the decision threshold. Due to the difficulties in deriving the exact null distribution, Nagao [9] derived an asymptotic distribution in the real Gaussian case up to order $\mathcal{O}(n^{-2})$, with n being the sample size, which could result in a simple and closed-form threshold formula. Unfortunately, this null distribution includes some errors. Consequently, the authors of [4] found that this null distribution is not as accurate as predicted by the remnant's order, therefore they turned to the Beta-approximation method to evaluate null distribution and threshold in the complex Gaussian case. However, although the approximated Beta distribution is very accurate, the threshold formula is not of closed-form and can only be calculated by numerically inversing the Beta cumulative density function (CDF). The calculation of Beta CDF could be a highly complicated procedure, thereby leading to difficulties in real-time processing.

One advantage of asymptotic expansion over Beta approximation is that the resulted threshold expression is of closed-form and offers computational efficiency. In this work, a correction is given to result in [9], which turns out to be more accurate. Then this result was generalized to the complex Gaussian case. The resulted threshold expression requires only one Chi-square table for each p, which is more easily implemented comparing to that in [4].

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2. JOHN'S TEST

Suppose we collect n samples $\mathbf{X} = [\mathbf{x}(1), \dots, \mathbf{x}(n)]$ from a $p-$ variate Gaussian population $\mathcal{N}(\mathbf{0}, \mathbf{\Sigma})$. Our aim is to test the sphericity hypothesis:

$$\mathcal{H}_0 : \mathbf{\Sigma} = \sigma^2 \mathbf{I}_p \tag{1}$$

against the alternative:

$$\mathcal{H}_1 : \mathbf{\Sigma} \neq \sigma^2 \mathbf{I}_p \tag{2}$$

with σ^2 being an unspecified value and $\mathbf{I}_p$ being $p \times p$ identity matrix.

The John's test, derived in [7, 8] is given as

$$T_J = \frac{\mathrm{tr}(\mathbf{R}^2)}{\mathrm{tr}^2(\mathbf{R})} \tag{3}$$

where

$$\mathbf{R} = \frac{1}{n}\sum_{i=1}^{n}\mathbf{x}(i)\mathbf{x}^H(i) \tag{4}$$

is the sample covariance matrix (SCM).

To utilize the asymptotic expansion method, the test statistic needs to be modified to a monotonic form that follows asymptotic Chi-square distribution. Let $\mathbf{S} = n\mathbf{R}$, then one proper form is,

$$T = \frac{np^2}{2}\mathrm{tr}(\mathbf{S}/\mathrm{tr}(\mathbf{S}) - p^{-1}\mathbf{I}_p)^2. \tag{5}$$

3. CORRECTIONS TO PREVIOUS RESULT

Due to the difficulties in deriving the exact distribution of T, [9] uses a transformation $\mathbf{Y} = \sqrt{n/2}\log(\mathbf{S}/n)$ to expand the characteristic function of T.

The asymptotic distribution of $\mathbf{Y}$ under large n could be approximated by:

$$\begin{aligned} f_Y(\mathbf{Y}) = c \times \mathrm{etr}\left[\frac{n-p+1}{2}\sqrt{\frac{2}{n}}\mathbf{Y} - \sqrt{\frac{n}{2}}\mathbf{e}^{\sqrt{\frac{2}{n}}\mathbf{Y}}\right] \\ \times \Big[1 + \frac{p-1}{2}\sqrt{\frac{n}{2}}\mathrm{tr}(\mathbf{Y}) + \frac{3p^2-6p+2}{12n}\mathrm{tr}^2(\mathbf{Y}) \\ + p\,\mathrm{tr}(\mathbf{Y}^2) + \mathcal{O}(n^{-\frac{3}{2}})\Big] \end{aligned} \tag{6}$$

where

$$c = \frac{(n/2)^{p(2n-p-1)/4}\pi^{-p(p-1)/4}}{\prod_{i=1}^{p}\Gamma(\frac{1}{2}(n+1-i))} \tag{7}$$

As stated in [9], the characteristic function of T, after some calculations, can be expressed as:

$$\begin{aligned} C(t) = c_1\phi^{-\frac{f}{2}}\mathbb{E}\Big[1 + \frac{1}{n}\Big\{\frac{p}{12}\mathrm{tr}(\mathbf{Y}^2) - \frac{1}{12}\mathrm{tr}(\mathbf{Y})^2 - \frac{6it}{p^3}\mathrm{tr}^4(\mathbf{Y}) \\ + \frac{14it-1}{12}\mathrm{tr}(\mathbf{Y}^4) - \frac{14it}{3p}\mathrm{tr}(\mathbf{Y})\mathrm{tr}(\mathbf{Y}^3) - \frac{5it}{2p}\mathrm{tr}^2(\mathbf{Y}^2) \\ + \frac{12it}{p^2}\mathrm{tr}^2(\mathbf{Y})\mathrm{tr}(\mathbf{Y}^2) + \Big((it - \frac{1}{6})\mathrm{tr}(\mathbf{Y}^2) + \frac{2it}{p^2}\mathrm{tr}^3(\mathbf{Y}) \\ - \frac{3it}{p}\mathrm{tr}(\mathbf{Y})\mathrm{tr}(\mathbf{Y}^2)\Big)^2\Big\}\Big] \end{aligned} \tag{8}$$

where $f = \frac{1}{2}p(p+2) - 1$, $\phi = (1-2it)^{-1}$ and constant c_1 could be replaced by its Stirling's approximation [10]:

$$\begin{aligned} c_1 =& c \times (2\pi)^{\frac{1}{4}p(p+1)}2^{-\frac{1}{4}p(p-1)}\exp\left[-\frac{1}{2}pn\right] \\ =& 1 - \frac{p}{24}(2p^2+3p-1) + \mathcal{O}(n^{-2}) \end{aligned} \tag{9}$$

The expectation in (8) is taken under a normal distribution with mean zero and covariance matrix $\mathbf{R}$, which is described as

$$\mathrm{COV}(\mathbf{Y}_{i,j}, \mathbf{Y}_{k,l}) = \frac{\phi(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})}{2} + \frac{(1-\phi)\delta_{ij}\delta_{kl}}{p} \tag{10}$$

where δ_{ij} is the Kronecker delta function.

Furthermore, the moments in (8) are listed as follows,

$$\begin{aligned} &\mathbb{E}[\mathrm{tr}(\mathbf{Y}^2)] = \phi(\frac{p^2}{2} + \frac{p}{2} - 1) + 1 \\ &\mathbb{E}[\mathrm{tr}^2(\mathbf{Y})] = p \\ &\mathbb{E}[\mathrm{tr}(\mathbf{Y}^4)] = \phi^2(\frac{p^3}{2} + \frac{5p^2}{4} - \frac{7p}{4} - 3 + \frac{3}{p}) + \phi(3p + 3 - \frac{6}{p}) + \frac{3}{p} \\ &\mathbb{E}[\mathrm{tr}(\mathbf{Y}^3)\mathrm{tr}(\mathbf{Y})] = \phi(\frac{3}{2}p^2 + \frac{3}{2}p - 3) + 3 \\ &\mathbb{E}[\mathrm{tr}^2(\mathbf{Y}^2)] = \phi^2(\frac{1}{4}p^4 + \frac{1}{2}p^3 + \frac{1}{4}p^2 - 1) + \phi(p^2 + p - 2) + 3 \\ &\mathbb{E}[\mathrm{tr}(\mathbf{Y}^2)\mathrm{tr}^2(\mathbf{Y})] = \phi(\frac{1}{2}p^3 + \frac{1}{2}p^2 - p) + 3p \\ &\mathbb{E}[\mathrm{tr}^4(\mathbf{Y})] = 3p^2 \\ &\mathbb{E}[\mathrm{tr}^2(\mathbf{Y}^3)] = \phi^3(\frac{3}{4}p^3 + \frac{9}{4}p^2 - 6p - 9 + 12/p) + \phi^2(\frac{9}{4}p^3 \\ &\qquad + \frac{9}{2}p^2 + \frac{9}{4}p - \frac{9}{p}) + \phi(9p + 9 - \frac{18}{p}) + \frac{15}{p} \\ &\mathbb{E}[(\mathrm{tr}(\mathbf{Y}^2)\mathrm{tr}(\mathbf{Y}))^2] = \phi^2(\frac{1}{4}p^5 + \frac{1}{2}p^4 + \frac{1}{4}p^3 - p) \\ &\qquad + \phi(3p^3 + 3p^2 - 6p) + 15p \\ &\mathbb{E}[\mathrm{tr}^6(\mathbf{Y})] = 15p^3 \\ &\mathbb{E}[\mathrm{tr}(\mathbf{Y}^3)\mathrm{tr}(\mathbf{Y}^2)\mathrm{tr}(\mathbf{Y})] = \phi^2(\frac{3}{4}p^4 + \frac{3}{2}p^3 + \frac{3}{4}p^2 - 3) \\ &\qquad + \phi(6p^2 + 6p - 12) + 15 \\ &\mathbb{E}[\mathrm{tr}(\mathbf{Y}^3)\mathrm{tr}^3(\mathbf{Y})] = \phi(\frac{9}{2}p^3 + \frac{9}{2}p^2 - 9p) + 15p \\ &\mathbb{E}[\mathrm{tr}(\mathbf{Y}^2)\mathrm{tr}^4(\mathbf{Y})] = \phi(\frac{3}{2}p^4 + \frac{3}{2}p^3 - 3p^2) + 15p^2 \end{aligned} \tag{11}$$

Substituting (9) and (11) into (8) yields

$$C(t) = \phi^{\frac{f}{2}}\left[\sum_{i=0}^{3}h_i\phi^i + \mathcal{O}\left(n^{-2}\right)\right] \tag{12}$$

where

$$\begin{aligned} h_0 &= 1 + \frac{1}{24n}(-2p^3 - 3p^2 + p + \frac{4}{p}) \\ h_1 &= \frac{1}{4n}(p^3 + 2p^2 - p - 2) \\ h_2 &= \frac{1}{8n}(-2p^3 - 5p^2 + 7p + 12 - \frac{12}{p}) \\ h_3 &= \frac{1}{12n}(p^3 + 3p^2 - 8p - 12 + \frac{16}{p}) \end{aligned} \tag{13}$$

This in turn signifies the result calculated in [9] is in error (p^{-1} terms in $h_0 - h_3$). Consequently, Theorem 5.1 in [9] should be corrected as follows:

The null distribution of T can be approximated asymptotically up to order $\mathcal{O}(n^{-2})$ by

$$\Pr(T \leqslant \gamma) = \sum_{i=0}^{3} h_i \Pr(\chi^2_{f+2i} \leqslant \gamma) + \mathcal{O}(n^{-2}) \tag{14}$$

where χ^2_f denotes a Chi-square distributed random variable with f degrees-of-freedom.

4. GENERALIZATION TO COMPLEX CASE

In the complex case, it follows from [11] that

$$T' = np^2 \mathrm{tr}(\mathbf{S}/\mathrm{tr}(\mathbf{S}) - p^{-1}\mathbf{I}_p)^2 \tag{15}$$

is asymptotically Chi-square distributed. On the other hand, the transformation should be $\mathbf{Y} = \sqrt{n}\log(\mathbf{S}/n)$, whose asymptotic distribution under large n is

$$\begin{aligned} f_{\mathbf{Y}}(\mathbf{Y}) = c^* \times \mathrm{etr}\left(\frac{n-p+1}{\sqrt{n}}\mathbf{Y} - n\mathbf{e}^{\frac{1}{\sqrt{n}}\mathbf{Y}}\right) \\ \times \left[1 + \frac{p-1}{\sqrt{n}}\mathrm{tr}(\mathbf{Y}) + \frac{6p^2-12p+5}{12n}\mathrm{tr}^2(\mathbf{Y}) \right. \\ \left. + \frac{p}{12n}\mathrm{tr}(\mathbf{Y}^2) + \mathcal{O}(n^{-\frac{3}{2}})\right] \end{aligned} \tag{16}$$

where

$$c^* = \frac{N^{p(N-\frac{p}{2})}\pi^{-\frac{p(p-1)}{2}}}{\prod_{k=1}^{p}(\Gamma(n+1-k))} \tag{17}$$

The complete proof can be found in [12]. Similarly, the asymptotic expansion of T' can be expressed as:

$$\begin{aligned} C'(t) = c_1^* \phi^{-\frac{f'}{2}} \mathbb{E}\left[1 + \frac{1}{n}\left\{\frac{p}{12}\mathrm{tr}(\mathbf{Y}^2) - \frac{1}{12}\mathrm{tr}(\mathbf{Y})^2 - \frac{3it}{p^3}\mathrm{tr}^4(\mathbf{Y}) \right.\right. \\ + \frac{14it-1}{24}\mathrm{tr}(\mathbf{Y}^4) - \frac{7it}{3p}\mathrm{tr}(\mathbf{Y})\mathrm{tr}(\mathbf{Y}^3) - \frac{5it}{4p}\mathrm{tr}^2(\mathbf{Y}^2) \\ + \frac{6it}{p^2}\mathrm{tr}^2(\mathbf{Y})\mathrm{tr}(\mathbf{Y}^2) + \left((\frac{6it-1}{12})\mathrm{tr}(\mathbf{Y}^3) + \frac{it}{p^2}\mathrm{tr}^3(\mathbf{Y}) \right. \\ \left.\left.\left. - \frac{3it}{2p}\mathrm{tr}(\mathbf{Y})\mathrm{tr}(\mathbf{Y}^2)\right)^2\right\}\right] \end{aligned} \tag{18}$$

with $f' = p^2 - 1$, and

$$c_1^* = c^*(2\pi)^{\frac{p^2}{2}} 2^{-\frac{m(m-1)}{2}} = 1 - \frac{2p^3-p}{12n} + \mathcal{O}\left(n^{-2}\right). \tag{19}$$

The expectation in (18) is taken under a complex circular Gaussian distribution with mean zero and covariance matrix $\mathbf{R}'$, which is described as

$$\mathrm{COV}\left(\mathbf{Y}_{i,j}\mathbf{Y}_{k,l}\right) = \delta_{il}\delta_{jk}\phi + p^{-1}(1-\phi)\delta_{ij}\delta_{kl} \tag{20}$$

with the moments in (18) calculated as,

$$\begin{aligned} &\mathbb{E}[\mathrm{tr}(\mathbf{Y}^2)] = \phi(p^2-1)+1 \\ &\mathbb{E}[\mathrm{tr}^2(\mathbf{Y})] = p \\ &\mathbb{E}[\mathrm{tr}(\mathbf{Y}^4)] = \phi^2(2p^3-5p+\frac{3}{p}) + \phi(6p-\frac{6}{p}) + \frac{3}{p} \\ &\mathbb{E}[\mathrm{tr}(\mathbf{Y}^3)\mathrm{tr}(\mathbf{Y})] = \phi(3p^2-3)+3 \\ &\mathbb{E}[\mathrm{tr}^2(\mathbf{Y}^2)] = \phi^2(p^4-1)+\phi(2p^2-2)+3 \\ &\mathbb{E}[\mathrm{tr}(\mathbf{Y}^2)\mathrm{tr}^2(\mathbf{Y})] = \phi(p^3-p)+3p \\ &\mathbb{E}[\mathrm{tr}^4(\mathbf{Y})] = 3p^2 \\ &\mathbb{E}[\mathrm{tr}^2(\mathbf{Y}^3)] = \phi^3(3p^3-15p+\frac{12}{p}) + \phi^2(9p^3-\frac{9}{p}) \\ &\qquad + \phi(18p-\frac{18}{p}) + \frac{15}{p} \\ &\mathbb{E}[(\mathrm{tr}(\mathbf{Y}^2)\mathrm{tr}(\mathbf{Y}))^2] = \phi^2(p^5-p) + \phi(6p^3-6p) + 15p \\ &\mathbb{E}[\mathrm{tr}^6(\mathbf{Y})] = 15p^3 \\ &\mathbb{E}[\mathrm{tr}(\mathbf{Y}^3)\mathrm{tr}(\mathbf{Y}^2)\mathrm{tr}(\mathbf{Y})] = \phi^2(3p^4-3) + \phi(12p^2-12) + 15 \\ &\mathbb{E}[\mathrm{tr}(\mathbf{Y}^3)\mathrm{tr}^3(\mathbf{Y})] = \phi(9p^3-9p) + 15p \\ &\mathbb{E}[\mathrm{tr}(\mathbf{Y}^2)\mathrm{tr}^4(\mathbf{Y})] = \phi(3p^4-3p^2) + 15p^2 \end{aligned} \tag{21}$$

Utilizing these moment expressions the asymptotic expansion of the null distribution of T' are calculated as

$$\Pr(T' \leqslant \gamma) = \sum_{i=0}^{3} g_i \Pr(\chi^2_{f+2i} \leqslant \gamma) + \mathcal{O}(n^{-2}) \tag{22}$$

with

$$\begin{aligned} g_0 &= 1 + \frac{1}{12n}(-2p^3+p+\frac{1}{p}) \quad & g_1 &= \frac{1}{2n}(p^3-p) \\ g_2 &= \frac{1}{4n}(-2p^3+5p+12-\frac{3}{p}) \quad & g_3 &= \frac{1}{6n}(p^3-5p+\frac{4}{p}) \end{aligned} \tag{23}$$

It follows from [13] that for the asymptotic distribution of T' given in (22) and a prescribed false alarm probability, the decision threshold can be approximated by

$$\begin{aligned} \gamma(P_{\mathrm{fa}}) = u + \frac{2g_6 u}{f(f+2)(f+4)}\left[u^2 + (f+4)u \right. \\ \left. + (f+2)(f+4)\right] + \frac{2g_4 u}{f(f+2)}(u+f+2) + \frac{2g_2 u}{f} \\ + \mathcal{O}\left(n^{-2}\right). \end{aligned} \tag{24}$$

where $\Pr(\chi^2_f \leqslant u) = 1 - P_{\mathrm{fa}}$.

It is manifested that the threshold can be easily evaluated by looking up Chi-square tables. And only one Chi-square table for each p needs to be established at the receiver, therefore the resulting additional storage could be small. As a result, our expression is more suitable for applications with strict real-time requirements comparing with [4].

5. SIMULATION

We present simulation results to confirm our theoretical calculations, including the accuracy of the derived null distributions as well as threshold formulas. Each result represents an average of 10^6 independent Monte Carlo trials.

First we compare the accuracy of Nagao's result and our corrected expression (14) by examining their goodness of fit to the simulated results. Fig 1 plots the false alarm probability as a function of decision threshold, where the numbers of antennas and snapshots are 2 and 20, respectively. It is seen from Fig. 1 that our result surpasses previous result in terms of fitting the simulated one. This in turn confirms confirmed that the previous result is in error. Furthermore, this error is particularly large in the low false alarm rate regime, which is usually more interested in practical applications. Therefore our correction can sharply reduce the threshold error comparing to the previous result.

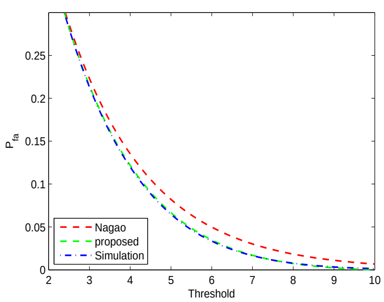


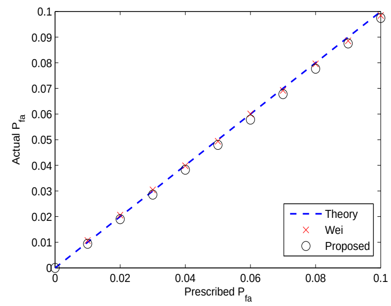
Fig. 1. False alarm probability versus threshold at $p = 2$, $n = 20$

In the complex Gaussian case, we consider both the accuracy and processing time of threshold selection. More specifically, we plot actual P_{fa} versus prescribed P_{fa} for the condition of $p = 4$ and $n = 50$. For comparison, the Beta approximation approach provided in [4] is also included. When calculating the Beta CDF, we take Q points in the interval $[1/p, 1]$. Although this Beta CDF is very accurate, there will be an error introduced by the numerical inverse, therefore we need to set Q large enough. However, this will result in additional processing time. It is seen in Fig. 2 that (24) can yield false alarm rates that align very well with the prescribed value. In contrast, Beta approximation is accurate only when Q is large.

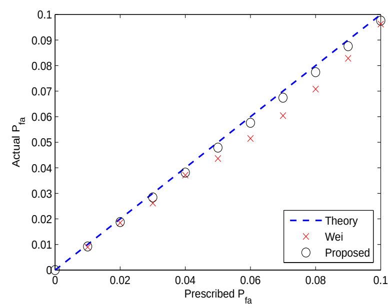
Table 1 lists the elapsing times for both methods in a number of settings. It is seen the time consumed by Beta approximation increases dramatically with Q and p, and is much larger than our method. This thereby signifies our closed-form threshold expression is comparable to Beta approximation in accuracy but requires much less computation load.

Table 1. Elapsing time comparison (second)

(p, n)	(4,50)	(4,80)	(7,80)
Beta($Q = 10^3$)	1.50	1.51	6.60
Beta($Q = 10^4$)	14.87	15.29	67.89
Proposed	4.21×10^{-5}	4.21×10^{-5}	4.37×10^{-5}



(a) $p = 4$, $n = 50$, $Q = 10^4$



(b) $p = 4$, $n = 50$, $Q = 10^3$

Fig. 2. Actual P_{fa} versus prescribed P_{fa} at $p = 4$, $n = 50$

6. CONCLUSION

The asymptotic expansion is a conventional approach to determine the distributions and decision threshold of test statistics. In this work, we have obtained accurate null distribution and decision threshold for John's test in both real and complex Gaussian cases. For the real case, we correct error parameters in previous work regarding the asymptotic expansion of its null distribution. Then we extend our development to the complex scenario and obtain a threshold formula which is different from the existing result. The new expression requires much less computation load therefore better meets the requirements of real-time applications.

7. REFERENCES

[1] D. B. Williams and D. H. Johnson, “Using the sphericity test for source detection with narrow-band passive arrays,” *IEEE Transactions on Acoustics, Speech and Signal Processing*, vol. 38, no. 11, pp. 2008-2014, 1990.

[2] Q. T. Zhang, ”Advanced detection techniques for cognitive radio,” *in Proc. IEEE International Conference on Communications (ICC’09)* , vol. 60, no.5, pp. 1268–1277, 2012.

[3] L. Wei and O. Tirkkonen, ”Spectrum sensing in the presence of multiple primary users,” *IEEE Trans. Commun.*, pp. 1-5, 2009.

[4] L. Wei, P. Dharmawansa and O. Tirkkonen, ”Multiple primary user spectrum sensing in the low SNR regime,” *IEEE Trans. Commun.*, vol. 16, no.5, pp. 1720–1731, 2013.

[5] Kakarala, Ramakrishna and P. O. Ogunbona, ”Signal analysis using a multiresolution form of the singular value decomposition,” *IEEE Trans. Image Process.*, vol. 10, no.5, pp. 724–735, 2001.

[6] J. W. Mauchly, “Significance test for sphericity of a normal n-variate distribution,” *Ann. Math. Statist.,* vol. 11, no. 2, pp. 204–209, 1940.

[7] S. John, “Some optimal multivariate tests,” *Biometrika*, vol. 58, no. 1, pp. 123–127, 1971.

[8] N. Sugiura, ”Locally best invariant test for sphericity and the limiting distributions,” *Ann. Math. Statist.,* vol. 43, no. 4, pp.1312-1316, 1972.

[9] H. Nagao, “On some test criteria for covariance matrix,” *Ann. Statist.*, vol. 1, no. 4, pp. 700–709, 1973.

[10] G. Marsaglia and J. C. W. Marsaglia, “A new derivation of Stirling’s approximation to $n!$,” *Amer. Math.*, vol. 97, no. 9, pp. 826–829, 1990.

[11] D. Ramírez, J. Vía, I. Santamaría, and L. L. Scharf, “Locally most powerful invariant tests for correlation and sphericity of Gaussian vectors,” *IEEE Trans. Inf. Theory*, vol. 59, no. 4, Apr. 2013.

[12] Y. Xiao, L. Huang, J. Xie, and H. C. So, ”Accurate distribution of locally most powerful invariant test for independence: Complex case,” *IEEE Trans. Inf. Theory*, submitted, 2015.

[13] Y. Fujikoshi, “Asymptotic expansions of the distributions of test statistics in multivariate analysis,” *J. Sci. Hiroshima Univ. Ser. A-I*, vol. 34, no. 1, pp. 73–144, 1970.