ON PARAMETRIC LOWER BOUNDS FOR DISCRETE-TIME FILTERING

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ABSTRACT

Parametric Cramér-Rao lower bounds (CRLBs) are given for discrete-time systems with non-zero process noise. Recursive expressions for the conditional bias and mean-square-error (MSE) (given a specific state sequence) are obtained for Kalman filter estimating the states of a linear Gaussian system. It is discussed that Kalman filter is conditionally biased with a non-zero process noise realization in the given state sequence. Recursive parametric CRLBs are obtained for biased estimators for linear state estimators of linear Gaussian systems. Simulation studies are conducted where it is shown that Kalman filter is not an efficient estimator in a conditional sense.

Index Terms— Parametric CRLB, Cramér-Rao lower bound, biased estimator, nonlinear filtering, state estimation.

1. INTRODUCTION

Consider the dynamic system represented by the state space model given as

$$x_{k+1} = f_k(x_k) + w_k, \qquad y_k = h_k(x_k) + v_k, \qquad (1)$$

where $x_k \in \mathbb{R}^n$ is the state and $y_k \in \mathbb{R}^m$ are the measurements, $f_k(\cdot)$ and $h_k(\cdot)$ are nonlinear mappings. The terms $w_k \sim p(w_k) = \mathcal{N}(w_k; 0, Q_k)$ and $v_k \sim p(v_k) = \mathcal{N}(v_k; 0, R_k)$ represent the zero-mean Gaussian white process and measurement noises which are independent of each other. The initial state is usually assumed to be random with pdf $x_0 \sim p(x_0)$ and independent of w_k and v_k.

The aim in (non-)linear filtering is to estimate the current state x_k of the system, based on all available measurements $y_{0:k}$. The corresponding estimator is denoted as $\hat{x}_k(y_{0:k})$ in the following. In the literature, a plethora of different (non-)linear filters can be found to solve this estimation task [1–7]. In order to compare the performance of the different filters to each other, one usually simulates N state and measurement sequences $x_{0:k}^{(i)} = [x_0^{(i),T}, \dots, x_k^{(i),T}]^T, y_{0:k}^{(i)} = [y_0^{(i),T}, \dots, y_k^{(i),T}]^T$, $i = 1, \dots, N$ of the stochastic system (1), from which a Monte-Carlo (MC) average of the mean square error (MSE) matrix for the estimator $\hat{x}_k(y_{0:k})$ is then computed:

$$\mathcal{M}(\hat{x}_k(y_{0:k})) = \mathbb{E}_{x_{0:k}, y_{0:k}}\{(\hat{x}_k(y_{0:k}) - x_k)(\cdot)^T\}, \qquad (2)$$

where $(A)(\cdot)^T$ is a short-hand notation for $(A)(A)^T$ for any matrix A and $\mathbb{E}_x$ denotes mathematical expectation with respect to (w.r.t.) the pdf $p(x)$. In order to know how "good" the estimator is, lower bounds on the MSE matrix have been established [6, 8]. Perhaps the most popular lower bound that is widely used in the literature is the Bayesian Cramér-Rao lower bound (BCRLB),

$$\mathcal{M}(\hat{x}_k(y_{0:k})) \geq J_k^{-1}, \qquad (3)$$

where the inequality means that the difference $\mathcal{M} - J_k^{-1} \geq 0$ is positive semi-definite [9], and J_k is some Bayesian information matrix, see [10–13] for different BCRLBs and their tightness.

In the Bayesian setup, both the state and measurement sequences are random quantities. Hence, the BCRLB shall be interpreted as an average bound over all possible state (and measurement) sequences. In the majority of practical setups, however, only a single state sequence such as a trajectory of an aircraft or ground vehicle is of interest, from which one wants to determine a lower bound on the estimation performance. In these situations, the estimator performance shall be evaluated based on the MSE matrix conditioned on a specific state sequence

$$\mathcal{M}(\hat{x}_k(y_{0:k})|x_{0:k}) = \mathbb{E}_{y_{0:k}}\{(\hat{x}_k(y_{0:k}) - x_k)(\cdot)^T|x_{0:k}\}, \; (4)$$

where the expectation is now evaluated w.r.t. $p(y_{0:k}|x_{0:k})$. In practice, this means that one generates (or uses) a *single* state sequence $x_{0:k}$, and then generates N measurement sequences $y_{0:k}^{(i)}$, $i = 1, \dots, N$, given the single state sequence $x_{0:k}$, from which an MC average of (4) for each estimator $\hat{x}_k(y_{0:k})$ is computed. In general, two types of state sequences can be used in the evaluation of the conditional MSE matrix. The first type originates from a deterministic state dynamic, i.e.

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the process noise w_k is zero. The second, often more realistic type assumes uncertainties in the modeling of the system and therefore adds process noise. For either type of state sequence, it can be easily checked that the BCRLB does not provide a lower bound for the conditional MSE matrix, and thus one has to resort to alternative approaches.

The aim of this paper is to develop lower bounds for the MSE matrix conditioned on state sequences when there is non-zero process noise. Since these bounds depend on a specific state sequence realization, i.e. a deterministic parameter and not a random quantity, these bounds are referred to as parametric bounds and the corresponding state sequence as deterministic state sequence.

2. RELATED WORK AND EXTENSIONS

Previous work on parametric bounds can be attributed to Taylor [14], who derived the parametric CRLB for unbiased estimators for continuous-time nonlinear deterministic system dynamics with discrete nonlinear measurements in additive Gaussian white noise. These results were later on adopted in [6] to the discrete-time nonlinear filtering problem and are nowadays widely used [15–17]. It has to be stressed that for the derivation of the parametric CRLB in [6,14], it is assumed that the state sequence $x_{0:k}$ contains no process noise. The results of [6] can be generalized to deterministic state sequences that contain process noise, as follows:

Theorem 1. *For nonlinear systems given in* (1) *and under suitable regularity conditions, the MSE of any unbiased estimator $\hat{x}_k(y_{0:k})$ conditioned on a process noise affected deterministic state sequence $x_{0:k}$ is bounded from below by*

$$\begin{aligned}\mathcal{M}(\hat{x}_k(y_{0:k})|x_{0:k}) &= \mathbb{E}_{y_{0:k}|x_{0:k}}\{(\hat{x}_k(y_{0:k}) - x_k)(\cdot)^T\} \\ &\geq [J_k(x_{0:k})]^{-1},\end{aligned}$$

with auxiliary Fisher information submatrix

$$\begin{aligned}J_k(x_{0:k}) =& [F_{k-1}^{-1}(x_{k-1})]^T J_{k-1}(x_{0:k-1})[F_{k-1}^{-1}(x_{k-1})] \\ &+ H_k^T(x_k)R_k^{-1}H_k(x_k)\end{aligned}$$

and Jacobian matrices $F_{k-1}(x_{k-1}) = [\nabla_{x_{k-1}} f_{k-1}^T(x_{k-1})]^T$ and $H_k(x_k) = [\nabla_{x_k} h_k^T(x_k)]^T$. □

Proof: *Due to space restrictions, the proof is given in the companion technical report [18].* □

It is worth noting that the parametric CRLB recursions are initiated by the statistical information contained in the pdf $p(x_0)$, i.e. $J_0 = P_{0|0}^{-1}$. Since the state sequence $x_{0:k}$ of the system is deterministic, the initial system state x_0 is fixed and the estimator has to be randomly initialized, which can be seen as an additional fictitious measurement y_0 at time $k = 0$, that is available to the estimator, see also [14, 19].

By a closer inspection of the above findings it becomes apparent, that the bound given in Theorem 1 is equivalent to the parametric CRLB derived in [6], namely the covariance matrix propagation of the extended Kalman filter (EKF), in the absence of the process noise covariance matrix Q_k. But in contrast to the previous work, the Jacobian matrices are now evaluated using the state sequence with non-zero process noise. With this new interpretation, it is possible to derive new BCRLBs. For instance, the unconditional MSE (2) can be bounded from below by

$$\mathcal{M}(\hat{x}_k(y_{0:k})) \geq \mathbb{E}_{x_{0:k}}\{[J_k(x_{0:k})]^{-1}\}, \tag{5}$$

which can be easily computed using MC averaging. Another possibility is to use MC averaging in the information domain according to

$$\mathcal{M}(\hat{x}_k(y_{0:k})) \geq [\mathbb{E}_{x_{0:k}}\{J_k(x_{0:k})\} + J_{x_{0:k}}]^{-1}, \tag{6}$$

where $J_{x_{0:k}}$ denotes the information submatrix of the prior $p(x_{0:k})$, see also [20] for a similar approach where the realization of the process noise sequence is treated as nuisance parameter. The parametric CRLB has been reported to be overly optimistic (i.e. it is not a tight bound), and therefore cannot be achieved by any unbiased estimator. Further, the parametric CRLBs derived above and in [6, 14] only hold for unbiased estimators. Especially in nonlinear systems, estimators such as the EKF or unscented KF are rarely assessed in terms of bias and generally lack analytical bias expressions. Due to the inherent approximations, these estimators generally exhibit a (small) bias, thus making the estimator performance comparison to the parametric CRLB for unbiased estimators inappropriate.

In the following, we restrict our attention to the derivation of parametric bounds for linear Gaussian systems. The bias analysis in these systems is much easier and often analytical expressions for the bias are available.

3. PARAMETRIC LOWER BOUNDS

Consider the following discrete-time linear Gaussian system

$$x_{k+1} = F\,x_k + w_k, \qquad y_k = H\,x_k + v_k, \tag{7}$$

where H are F time-invariant mapping matrices of appropriate size. For such systems, the celebrated Kalman filter [1, 2] is generally used to perform the estimation task. This is due to its favorable properties such as: It is the optimal filter that minimizes the MSE, and the KF is unbiased, hence it is also the minimum variance unbiased estimator (MVUE). These properties, however, only hold under the Bayesian paradigm. In particular, state estimates $\hat{x}_{k|k}(y_{0:k})$ of the KF are unbiased when we consider the state as a random variable. In mathematical terms this means that the following equality holds

$$\mathbb{E}_{x_k,y_{0:k}}\{\hat{x}_{k|k}(y_{0:k})\} = \mathbb{E}_{x_k}\{x_k\}. \tag{8}$$

On the other hand, given a specific deterministic state sequence $x_{0:k}$, we do not necessarily have the unbiasedness property for the same estimate, i.e.,

$$\mathbb{E}_{y_{0:k}}\{\hat{x}_{k|k}(y_{0:k})|x_{0:k}\} \neq x_k, \tag{9}$$

where the expected value is taken w.r.t. the conditional pdf $p(y_{0:k}|x_{0:k})$. In the following, we define the conditional bias in the KF estimate $\hat{x}_{k|k}(y_{0:k})$ by

$$b_k(x_{0:k}) \triangleq \mathbb{E}_{y_{0:k}}\{\hat{x}_{k|k}(y_{0:k})|x_{0:k}\} - x_k. \tag{10}$$

Here, the bias b_k is written as a function of $x_{0:k}$ to emphasize the dependence of the bias on the specific state sequence. For systems as given by (7), the conditional bias can be evaluated recursively according to the following lemma:

Lemma 1. *For linear Gaussian systems, the conditional bias in the KF estimate can be evaluated recursively as follows.*

$$\begin{aligned} b_{k+1}(x_{0:k+1}) =& (I - K_{k+1}H)F\, b_k(x_{0:k}) - (I - K_{k+1}F) \\ & \times (x_{k+1} - Fx_k), \end{aligned}$$

where K_k denotes the Kalman filter gain, and I is the identity matrix. □

Proof: *Due to space restrictions, the proof is given in the companion technical report [18].* □

It is worth noting that in the above lemma the quantity $(x_{k+1} - Fx_k)$ can be replaced by w_k according to the state dynamics in order to highlight the dependence of the conditional bias on the specific realization of the process noise w_k in the given state sequence $x_{0:k}$. Several important conclusions can be drawn from Lemma 1: If the filter is initialized with $b_0(x_0) = 0$, then the filter estimates remain conditionally unbiased, when there is no process noise in the given specific state sequence $x_{0:k}$. Hence Kalman filter estimates are conditionally unbiased (in addition to being unbiased in a Bayesian sense) when there is no process noise. In the case that there is non-zero process noise in the given specific state sequence $x_{0:k}$, the filter estimates would still remain conditionally unbiased provided that the filter gains $\{K_l\}_{l=0}^{k}$ satisfy $(I - K_l H)F = 0$, for $l = 1, \ldots, k$. These findings illustrate, that, except in the above stated cases, the KF estimates are conditionally biased. Hence, applying the parametric CRLB introduced in Theorem 1 is inappropriate. Rather, it is required to derive a lower bound on the conditional MSE for biased estimators. The main drawback of such an approach is that the bound becomes estimator dependent, a property that is not desired, since the bound generally should hold for any estimator. However, due to the popularity of the Kalman filter and its widespread use, it is believed that these results are of high value, especially for practitioners working in the field.

Lemma 2. *The conditional MSE matrix for any biased estimator $\hat{x}_{0:k}(y_{0:k})$ of the deterministic state sequence is bounded from below as follows*

$$\begin{aligned} \mathcal{M}(\hat{x}_{0:k}(y_{0:k})|x_{0:k}) \geq & (I + B_k)[J_{0:k}(x_{0:k})]^{-1}(I + B_k)^T \\ & + b_{0:k}(x_{0:k})b_{0:k}^T(x_{0:k}), \end{aligned}$$

where $B_k \triangleq \nabla_{x_{0:k}}^T b_{0:k}(x_{0:k})$ is the bias Jacobian matrix of the specific estimator; $J_{0:k}(x_{0:k})$ is the auxiliary Fisher information matrix for the entire state sequence and $b_{0:k}(x_{0:k}) \triangleq [b_0^T(x_0), \ldots, b_k^T(x_{0:k})]^T$. □

Proof: *Due to space restrictions, the proof is given in the companion technical report [18].* □

In the filtering framework, we are interested in a bound only for the current deterministic state x_k at each time step k i.e., we want to establish a lower bound C_k on $\mathcal{M}(\hat{x}_k(y_{0:k})|x_{0:k})$, which is the $n \times n$ lower-right partition of the matrix $\mathcal{M}(\hat{x}_{0:k}(y_{0:k})|x_{0:k})$ in Lemma 2. It is clear that C_k would be the $n \times n$ lower-right partition of the right hand side of the inequality in Lemma 2. One way to calculate C_k would be finding the whole right hand side of the inequality in Lemma 2 first and then taking the lower right partition of the resulting large matrix. However this comes at the cost of having to invert at each time step the $n(k+1) \times n(k+1)$ matrix $J_{0:k}$, which grows with time k. Hence, a recursion for the parametric CRLB C_k for biased estimators with finite memory and computation requirements is of interest. Such a recursion is provided by the following theorem.

Theorem 2. *In the case that Kalman filter is used for obtaining $\hat{x}_{0:k}(y_{0:k})$ (i.e., when the bias expression of Lemma 1 applies), the $n \times n$ lower-right partition C_k of the right hand side of the inequality in Lemma 2 can be calculated as below.*

$$C_k = \widetilde{C}_k + b_k(x_{0:k})b_k^T(x_{0:k}) \tag{11}$$

where

$$\begin{aligned} \widetilde{C}_{k+1} =& \left[\ (I - K_{k+1}H)F \quad K_{k+1}H\ \right] \\ & \times \begin{bmatrix} \widetilde{C}_k - \Psi_k J_k \left(J_k^{-1} - \widetilde{J}_k^{-1}\right) J_k \Psi_k^T & -\Psi_k J_k \widetilde{J}_k^{-1} F^T \\ -F\widetilde{J}_k^{-1} J_k \Psi_k^T & J_{k+1}^{-1} \end{bmatrix} \\ & \times \left[\ (I - K_{k+1}H)F \quad K_{k+1}H\ \right]^T, \end{aligned} \tag{12}$$

$$\widetilde{J}_k = J_k + F^T H^T R^{-1} H F, \tag{13}$$

$$\Psi_{k+1} = -(I - K_{k+1}H)F\Psi_k J_k^{-1} \widetilde{J}_k^{-1} F^T + K_{k+1} H J_{k+1}^{-1}. \tag{14}$$

and K_k is the Kalman gain at time k. The recursion for the bias term $b_k(x_{0:k})$ in (11) *is provided in Lemma 1. The recursions in* (12) *and* (14) *are initialized with*

$$\widetilde{C}_0 = \left[I + \nabla_{x_0}^T b_0(x_0)\right] J_0^{-1} \left[I + \nabla_{x_0}^T b_0(x_0)\right]^T, \tag{15}$$

$$\Psi_0 = \left[I + \nabla_{x_0}^T b_0(x_0)\right] J_0^{-1}. \tag{16}$$

In (12) *to* (16)*,* J_k *denotes the auxiliary Fisher information matrix for unbiased estimates whose recursion is defined in Theorem 1.* □

Proof: *Due to space restrictions, the proof is given in the companion technical report [18].* □

Theorem 2 provides a recursive bound C_k for the conditional MSE $\mathcal{M}(\hat{x}_{k|k}(y_{0:k})|x_{0:k})$ of the Kalman filter. The conditional MSE $\mathcal{M}(\hat{x}_{k|k}(y_{0:k})|x_{0:k})$ of the Kalman filter can be calculated recursively as given in the following lemma.

Lemma 3. *For linear Gaussian systems, the conditional MSE* $\mathcal{M}(\hat{x}_{k|k}(y_{0:k})|x_{0:k})$ *of the Kalman filter can be evaluated recursively as follows.*

$$\begin{aligned}
&\mathcal{M}(\hat{x}_{k+1|k+1}(y_{0:k+1})|x_{0:k+1})\\
&=(I-K_{k+1}H)\Big[F\mathcal{M}(\hat{x}_{k|k}(y_{0:k})|x_{0:k})F^T\\
&\quad - Fb_k(x_{0:k})(x_{k+1}-Fx_k)^T\\
&\quad - (x_{k+1}-Fx_k)b_k^T(x_{0:k})F^T\\
&\quad + (x_{k+1}-Fx_k)(x_{k+1}-Fx_k)^T\Big](I-K_{k+1}H)^T\\
&\quad + K_{k+1}RK_{k+1}^T.
\end{aligned}$$

Proof: *Due to space restrictions, the proof is given in the companion technical report [18].* □

As a final remark, it must be emphasized here that although the results given so far are derived for the Kalman filter, they equivalently apply to any linear state estimator with the update equation

$$\hat{x}_{k+1|k+1} = F\hat{x}_{k|k} + K_{k+1}(y_{k+1} - HF\hat{x}_{k|k}) \qquad (17)$$

where K_k is an arbitrary filter gain independent of the measurements $y_{0:k-1}$.

4. SIMULATIONS

We consider the system given as

$$x_{k+1} = \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix} x_k + w_{k+1}, \quad y = \begin{bmatrix} 1 & 0 \end{bmatrix} x_k + v_k$$

where $x_k \triangleq [\mathsf{p}_k, \mathsf{v}_k]^T$ is the state composed of the scalar position $\mathsf{p}_k \in \mathbb{R}$ and velocity $\mathsf{v}_k \in \mathbb{R}$ variables. The terms $w_k \sim \mathcal{N}\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \sigma_w^2 \begin{bmatrix} \frac{T^2}{2} \\ T \end{bmatrix} \begin{bmatrix} \frac{T^2}{2} \\ T \end{bmatrix}^T\right)$ and $v_k \sim \mathcal{N}(0, \sigma_v^2)$ represent white Gaussian process and measurement noises, respectively, which are independent of each other. The quantities $\sigma_w = 1\,\text{m/s}^2$ and $\sigma_v = 1\,\text{m}$ are the process and measurement noise standard deviations. $T = 1\,\text{s}$ is the sampling period.

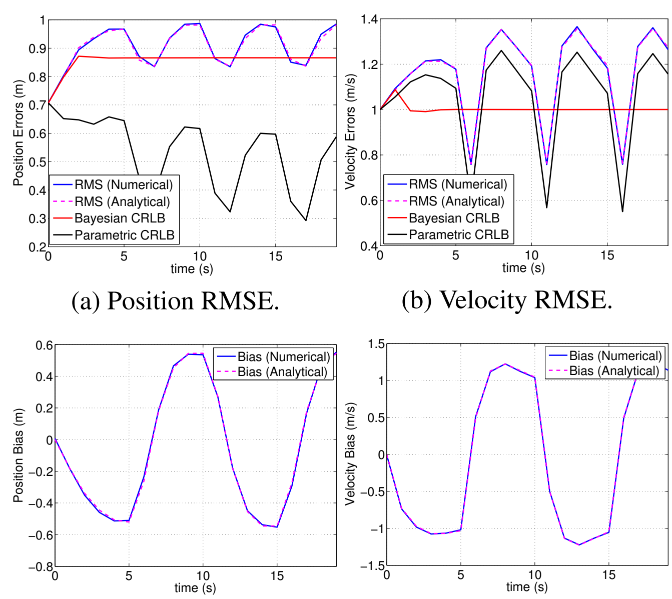
We consider the specific state sequence of length 20 (i.e., $k = 0, \ldots, 19$) starting from $x_0 = [0, 0]^T$ and obtained by the following process noise selection.

$$w_k = \begin{cases} 1, & 1 \le k \le 5 \text{ or } 11 \le k \le 15 \\ -1, & \text{otherwise} \end{cases}. \qquad (18)$$

A total of 10000 noisy measurement sets $y_{0:19}$ are generated from the true state sequence described above. A Kalman filter using the true model parameters is used to estimate the states from each of the measurement sequences. The initial state of the Kalman filter is selected randomly as $\hat{x}_0 \sim \mathcal{N}([0, 0]^T, I)$ for each measurement set.

The conditional RMSE calculated over the MC runs (RMSE Numerical), conditional RMSE calculated analytically using the result of Lemma 2 (RMS Analytical), (root-)Bayesian CRLB of Tichavsky et al. [10] and (root-)parametric CRLB calculated using Theorem 2 are shown for position and velocity variables in Figures 1 (a) and 1 (b) respectively. The conditional bias values calculated numerically over the MC runs (Bias Numerical) and conditional bias values calculated analytically using the result of Lemma 1 are shown for position and velocity variables in Figures 1 (c) and 1 (d) respectively.

It is seen in the figures that the analytical expressions for the bias and MSE predict the numerical quantities rather well. It is evident that the Bayesian CRLB is not a bound for the conditional MSE which is bounded below by parametric CRLB. There is a significant gap between the RMSE values and the parametric CRLB especially in the position estimation. Hence in a conditional sense, Kalman filter is not efficient.



(a) Position RMSE. (b) Velocity RMSE.

(c) Position Bias. (d) Velocity Bias.

Fig. 1. Parametric CRLBs, conditional bias and conditional MSE values obtained in the simulation.

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