ON PARAMETER ESTIMATION OF SYMMETRIC ALPHA-STABLE DISTRIBUTION

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ABSTRACT

In this paper, a novel approach has been proposed for estimating the parameters of symmetric alpha-stable distribution with $1 \leq \alpha \leq 2$. Alpha-stable distribution can model the statistical behavior of non-Gaussian heavy tailed signals and noises with impulsive components. There are some serious consideration in parameter estimation with stable random variable due to the lack of expression in analytic form for its distribution. In this method, the closed-form expression for parameter estimators is achieved by means of the values of characteristic function (CF) in two different points. We also establish a framework to determine these points in the general case. Performance comparison between the proposed method and the other methods is performed through several simulations and numerical results confirm that our method is efficient and has better performance even with a few number of samples.

***Index Terms*—** symmetric alpha-stable distribution, parameter estimation, characteristic exponent, dispersion, characteristic function

1. INTRODUCTION

In recent years, stable distributions have obtained an important position in the signal processing community as a generalization of the Gaussian distribution, such as SAR return signals, biomedical, underwater and atmospheric environment signal processing [1]. Several works are addressed in the literature to prevail the problem of estimating the parameters of symmetric α-stable ($S\alpha S$) distributions [2]. Classical parameter estimation techniques for $S\alpha S$ random variables can be categorized into six main categories [3]; 1- Characteristic function method (CFM), 2- Quantile method (QM), 3- Maximum likelihood method (ML), 4- Extreme value method (EVM), 5- Fractional lower order moment (FLOM) method [4], 6- Method of log-cumulant (MoLC) [5]. Although some proposed methods of matrix log-cumulants may lead to non-invertible equations and cause precarious performance. In order to overcome such difficulty, a Bayesian-based method to re-estimate the log-cumulants was proposed. However, the burden of these methods is too expensive in terms of time and number of computations or the variances of their estimates are high [6].

1.1. Relation to Prior Works

From all known techniques for estimating the characteristics exponent and dispersion parameter, only FLOM, MoLC and QM yield to a closed form expression for parameter estimators [3, 7]. Also, it is important to note that the QM needs look up tables and linear interpolation which are computationally expensive [8]. For moment based methods like FLOM and MoLC, the selection of proper moment order is a critical bottleneck [9]. Furthermore, the higher order moments greater than α for α-stable distribution are infinite [10]. Through this paper, the general problem of zero mean $S\alpha S$ parameter estimation is done analytically. We develop a new approach to estimate two important parameters of $S\alpha S$ distribution, i.e. α, γ, based on CF. Furthermore, our technique has significantly more accuracy than the aforementioned methods.

The rest of this paper is organized as follows. In Section 2, we state some necessary preliminaries on α-stable distribution and related parameter estimation methods. In Section 3, we present our CF based proposed method for parameter estimation. Section 4 includes the performance evaluation and numerically comparison of proposed approach with other methods. Finally, the paper is concluded in Section 5.

2. PARAMETER ESTIMATION OF SYMMETRIC ALPHA-STABLE DISTRIBUTION

2.1. $S\alpha S$ Distribution

Probability density function (PDF) for α-stable random variate is provided by taking the inverse Fourier transform from its CF [3]. Suppose $-\infty < x < \infty$ and $x \sim S(\alpha, \gamma, \beta, \delta)$, then its PDF is completely determined by four parameters; however, a closed-form formula does not exist for its PDF. α is the *characteristic exponent* and it determines the *shape* of the distribution, $(0 < \alpha \leq 2)$, γ is the *dispersion* or *scale* parameter of the distribution and plays a similar role to the variance of the Gaussian distribution, $(\gamma > 0)$, β is the index of skewness $(-1 \leq \beta \leq 1)$ and δ is the *location* parameter, $(\delta \in \mathbb{R})$. The case of $\beta = 0$ corresponds to the symmetric α-stable distribution. In this study, we focus on zero mean

($\delta = 0$) $S\alpha S$ distribution with $1 \leq \alpha \leq 2$, which is defined in terms of its CF, as the following,

$$\varphi_{\alpha,\gamma}(\omega) = \exp\{-\gamma|\omega|^{\alpha}\}. \tag{1}$$

Analytical formula for PDF expression only exists for two special value of α. The case $\alpha = 2$, $f_G(x) = f(x|\alpha = 2, \gamma) = \frac{1}{2\sqrt{\pi\gamma}} \exp(-\frac{x^2}{4\gamma})$, which is basically the classical Gaussian distribution. The other special case is $\alpha = 1$, $f_C(x) = f(x|\alpha = 1, \gamma) = \frac{\gamma}{\pi(x^2+\gamma^2)}$, corresponds to the Cauchy distribution.

2.2. FLOM

For $p > \alpha$, the p^{th} order moment of α-stable distribution is infinite, therefore classical estimation methods based on integer moments order are unenforceable [9]. Expression for the FLOMs of $S\alpha S$ distributions has previously been given by Ma and Nikias [11]. Kuruoglu [12] developed the theory of FLOM to general α-stable distributions. Based on this method, the characteristic exponent estimation is given as the solution to the following equation,

$$\text{sinc}(\frac{p}{\hat{\alpha}}) = \frac{\tan q}{qA_pA_{-p}}, \quad q = \frac{p\pi}{2}, \tag{2}$$

where sinc$(\cdot)$ is the sinc function and A_p is defined as $A_p = E\{|x|^p\}$. Although, the argument of sinc function in [12] is wrong and π must be omitted. For N independent observed samples, $\frac{1}{N}\sum_{i=1}^{N}|x_i|^p$ may be utilized to estimate A_p. Furthermore, an estimate of γ is given as

$$\hat{\gamma} = \left\{\frac{\Gamma(1-p)\cos q}{\Gamma(1-p/\hat{\alpha})}A_p\right\}^{\hat{\alpha}/p}. \tag{3}$$

2.3. MoLC

Second kind cumulants are constructed according to the same rules as traditional cumulants and the relation between log-moments and log-cumulants is identical to the relation existing between moments and cumulants [13]. The log-cumulants of $S\alpha S$ distribution are given as $\tilde{k}_1 = \psi_0(1-\frac{1}{\alpha}) + \frac{1}{\alpha}\ln\gamma$, $\tilde{k}_2 = \psi_1(\frac{1}{2}+\frac{1}{\alpha^2})$ [14], where $\psi_0 = -0.57721566$ and $\psi_1 = \pi^2/6$ are the values of the Polygamma function, $\psi_{n-1} = \frac{d^n}{dx^n}\ln\Gamma(x)\Big|_{x=1}$. Using empirical log-cumulants, estimate for α and γ can be obtained as the following,

$$\hat{\alpha} = \left(\frac{\hat{\tilde{k}}_2}{\psi_1} - \frac{1}{2}\right)^{-1/2} \quad , \quad \hat{\gamma} = \exp\left((\hat{\tilde{k}}_1 - \psi_0)\hat{\alpha} + \psi_0\right). \tag{4}$$

2.4. QM

The main feature of quantile is that it discovers α, γ and can be expressed by a function of ν_α and ν_β, where $\nu_\alpha = (q_{0.95} - q_{0.05})/(q_{0.75} - q_{0.25})$ and $\nu_\beta = (q_{0.95} + q_{0.05} - 2q_{0.5})/(q_{0.95} - q_{0.05})$. The estimate of parameters α and γ is now established consistently by means of look-up tables and linear interpolation [3]. If the distribution is symmetric then the quantiles are symmetric too.

3. PROPOSED CF BASED APPROACH

In this section, we derive closed-form expression for parameter estimators of $S\alpha S$ distribution which is basically based on CF. The estimation framework we utilize here has a hierarchical rather than a simultaneous structure. First, an approach is proposed for estimating the dispersion of $S\alpha S$ PDF, which does not involve knowledge or simultaneous estimation of the characteristic exponent. In the next step, an algorithm is used for characteristic exponent estimation. In this new scheme, two different points of CF are considered to establish the required framework. One of these points is $\omega = 1$, which is an appropriate choice, because all of $S\alpha S$ CFs have equal values in this point, in other words, $e^{-\gamma|\omega|^{\alpha}}|_{\omega=1} = e^{-\gamma}$ and its value is independent from α. Additionally, we are able to estimate γ with no need to know about α which is a considerable advantage. The other point is ω_0 which we will discuss about the mathematically calculation of it.

3.1. Estimation of Dispersion, γ

We first consider the problem of estimating γ from a set of observed samples. CF of a $S\alpha S$ distribution as illustrated in (1) is inherently an even function versus ω. Since the empirical CF (ECF) used in parameter estimation methods is not necessarily real and even, we take the absolute value of (1) as the following,

$$|\varphi_{\alpha,\gamma}(\omega)| = |E\{e^{j\omega x}\}| = \exp\{-\gamma|\omega|^{\alpha}\}. \tag{5}$$

Note that the right side of (1) contains a real and positive function. In the next step, the natural logarithm is taken from both sides of the above equation,

$$\ln E\{e^{j\omega x}\} = -\gamma|\omega|^{\alpha}, \tag{6}$$

then the value of (6) is appraised at $\omega = 1$. So we have,

$$\gamma = -\ln|E\{e^{jx}\}|, \tag{7}$$

γ estimation now is obtained and can be empirically achieved as the following

$$\hat{\gamma} = -\ln\frac{1}{N}\left|\sum_{i=1}^{N}\exp(jx_i)\right|. \tag{8}$$

3.2. Estimation of Characteristic Exponent, α

In addition to $\omega = 1$, another point of CF is needed to estimate α. Fig. 1 shows the behaviour of CF for $S\alpha S$ distribution for

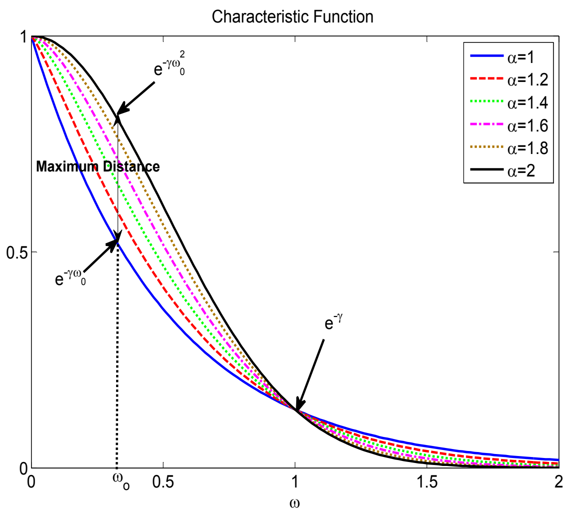

Fig. 1. CF of $S\alpha S$ distribution; $\alpha = 1, 1.2, 1.4, 1.6, 1.8, 2$.

six different values of $\alpha \in [1, 2]$. We propose to calculate ω_0 by determining the point that the maximum distance between $e^{-\gamma\omega}$ and $e^{-\gamma\omega^2}$ occurs; i.e. $\frac{d}{d\omega}(e^{-\gamma\omega^2} - e^{-\gamma\omega}) = 0$. After simplification, we may write,

$$2\omega e^{-\gamma\omega^2} = e^{-\gamma\omega}. \tag{9}$$

To solve (9), which is an exponential equation, one solution is to rewrite the equation in the logarithmic form and solve it for the variable ω.

$$\ln 2\omega - \gamma\omega^2 = -\gamma\omega. \tag{10}$$

Assume an estimate of γ is achievable, by isolating γ, the value of ω_0 can straightforwardly obtained as one of the solutions of the following equation which is nonlinear.

$$\left.\frac{\ln 2\omega}{\omega^2 - \omega}\right|_{\omega_0} = \hat{\gamma}. \tag{11}$$

It is of great importance to note that for $\alpha \in [1, 2]$ and $\gamma > 0$, ω_0 takes the only values between 0 to 0.5 . Since $\omega_0 > 0$, substituting it into (6) and dividing the resultant by (7), the following equation is achieved.

$$\omega_0^\alpha = \frac{\ln |E\{e^{j\omega_0 x}\}|}{\ln |E\{e^{jx}\}|}. \tag{12}$$

All we need to do now is finding the solution of the above equation to get an estimate of α. After this, the characteristic exponent estimate is expressed as the following,

$$\begin{aligned}\hat{\alpha} &= \frac{\ln \frac{\ln |E\{e^{j\omega_0 x}\}|}{\ln |E\{e^{jx}\}|}}{\ln \omega_0} \\ &= \frac{\ln \ln |E\{e^{j\omega_0 x}\}| - \ln \ln |E\{e^{jx}\}|}{\ln \omega_0} \\ &= \log_{\omega_0} \ln |E\{e^{j\omega_0 x}\}| - \log_{\omega_0} \ln |E\{e^{jx}\}|. \end{aligned} \tag{13}$$

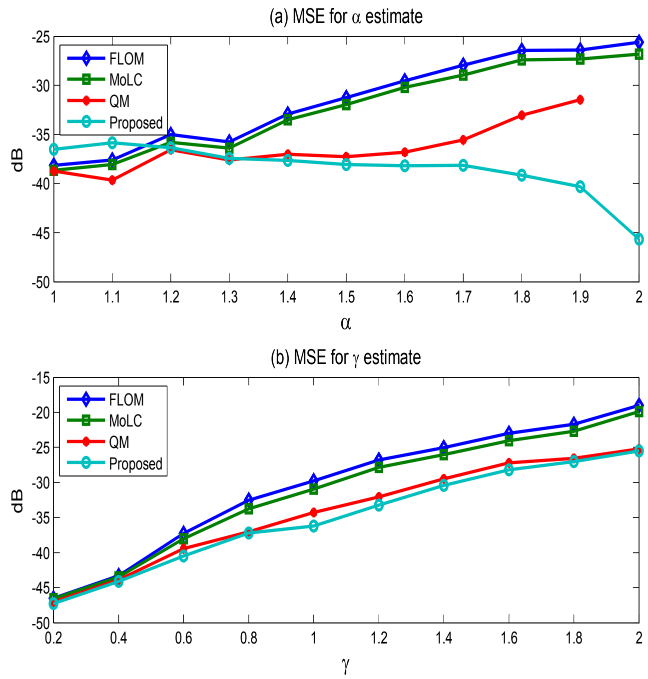

Fig. 2. (a) MSE for characteristic exponent estimation versus $\alpha \in [1, 2]$, $\gamma = 1$, $p = 0.25$, N=10000. (b) MSE for dispersion estimation versus $\gamma \in [0.2, 2]$, $\alpha = 1.55$, $p = 0.25$, N=10000.

4. SIMULATION RESULTS

In this section, we assess the performance of our proposed parameter estimation scheme with other methods used for parameter estimation of $S\alpha S$ distribution. We call M as the number of times an experiment is repeated and in all the simulations and M is set to be 100. Furthermore, moment order selection plays an important role in FLOM technique. In [12] the choice $p = \alpha/4$ and the choice $p = 0.2$ have been suggested for this purpose. Through several simulations, $p = 0.25$ was determined as the best choice for order of moment. In the next section, we verify the mean squared error (MSE) of different estimators as a comparison criterion.

4.1. MSE of Parameter Estimators

We first evaluate the performance of our proposed characteristic exponent estimator with three other methods which have closed-form expression for their estimates, i.e. FLOM, MoLC and QM. We generate $N = 10,000$ i.i.d samples of the $S\alpha S$ random variables with $\gamma = 1$. Fig. 2(a) illustrates MSE as a function of $1 \leq \alpha \leq 2$. The performance of our method for $1 \leq \alpha \leq 1.2$ in not very good, because the nature of samples are very impulsive and 10,000 samples are not sufficient to have a good result. Increasing the number of samples

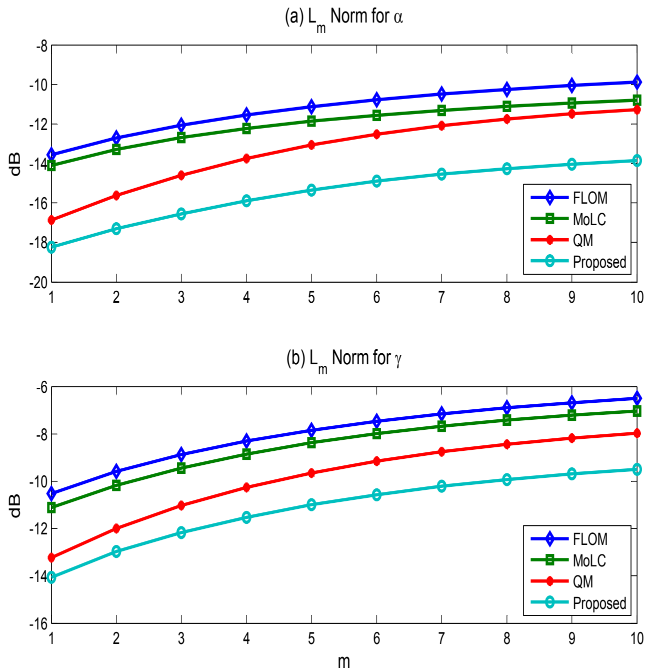

Fig. 3. $\mathcal{L}_m$-Norm versus m; $\alpha = 1.75$, $\gamma = 2$, $p = 0.25$. (a)$\mathcal{L}_m$-Norm of α estimate. (b)$\mathcal{L}_m$-Norm of γ estimate.

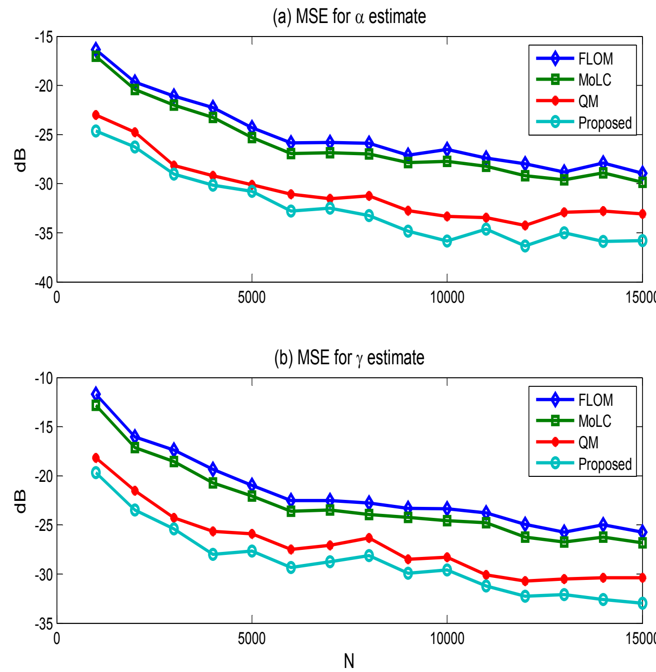

Fig. 4. MSE versus number of samples (N); $\alpha = 1.6$, $\gamma = 1.5$, $p = 0.25$. (a) MSE of α estimate. (b) MSE of γ estimate.

or the number of times an experiment is repeated we may have a better output. Our approach has better performance for $\alpha \geq 1.3$ specifically for $\alpha \geq 1.5$. Fig. 2(b) displays MSE as a function of different values of γ. In a similar way, in this case $N = 10,000$ samples with $\alpha = 1.55$ are employed. Since the behaviour of samples is very heavy-tailed, our proposed method performance is a little better rather than QM. For higher values of α the difference between these two methods increases and our estimator has better performance.

4.2. $\mathcal{L}_m$-Norm

The validity of the MSE criterion has been investigated through the $\mathcal{L}_m$-Norm. We carry out simulations to measure the $\mathcal{L}_m$-Norm between α, γ and their estimates. For a real number $m \geq 1$, $\mathcal{L}_m$-Norm is defined by,

$$||\theta - \hat{\theta}||_m = \left(\frac{1}{M} \sum_{k=1}^{M} (\theta - \hat{\theta}_k)^m \right)^{\frac{1}{m}}. \qquad (14)$$

In Fig. 3 the simulation results for four different techniques are given for $N = 10,000$ i.i.d samples of $S\alpha S$ distribution with $\alpha = 1.75, \gamma = 2$. Numerical results depicted in Fig. 3 confirm that the proposed method has better consistency with different distances criterion defined as $\mathcal{L}_m$-Norm. Also, comparing Fig. 3(a) and Fig. 3(b), it can be realized that the α estimate has lower MSE regarding to γ estimate. In the same manner with previous results, developed method has the best and the FLOM method has the worst performance for estimating the characteristic exponent and dispersion parameters.

4.3. Performance Evaluation as a Function of N

In order to demonstrate the effectiveness of our approach, we establish a scenario with different number of available samples. For this purpose, MSE for different sample size, N, generated from a $S\alpha S$ distribution with $\alpha = 1.6$, $\gamma = 1.5$, is plotted in Fig. 4. This figure shows that the corresponding estimation MSE of the proposed approach remains smaller than the other methods. Moreover, the values of α and γ estimated by the FLOM and MoLC diverges greatly in the case of $N < 2000$.

5. CONCLUSION

We developed a new approach based on the $S\alpha S$ CF for parameter estimation. The main feature of our proposed method is its simplicity and efficiency. Performance of the proposed characteristic exponent estimator and dispersion estimator through several simulations was compared with FLOM method, MoLC and QM. Numerical results verify that even with a few number of samples our proposed method is more accurate and has better efficiency.

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