ITERATIVELY REWEIGHTED TENSOR SVD FOR ROBUST MULTI-DIMENSIONAL HARMONIC RETRIEVAL

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ABSTRACT

In this paper, parameter estimation for multi-dimensional sinusoids in additive impulsive noise is addressed. Our underlying idea is to minimize the ℓ_p-norm of the residual error tensor, where $1 < p < 2$, and transform this problem to an iterative ℓ_2-norm minimization. In doing so, we can utilize the tensorial structure of the received data and then apply iteratively reweighted tensor singular value decomposition, referred to as IR-t-SVD, to recover the subspace or the signal tensor. After the recovery step, standard subspace techniques can be applied for parameter estimation. Based on the numerical results, IR-t-SVD outperforms several state-of-the-art methods in terms of mean square frequency error under α-stable noise.

Index Terms— Harmonic Retrieval, Parameter Estimation, tensor Singular Value Decomposition, ℓ_p-norm

1. INTRODUCTION

Multi-dimensional harmonic retrieval (HR) is an important problem in communications and signal processing, and representative application examples include wireless communication channel estimation [1], nuclear magnetic resonance spectroscopy [2] as well as multiple-input multiple-output radar imaging [3]. In this work, we tackle the problem of R-dimensional (R-D) HR with multiple snapshots in additive noise. When the noise is white Gaussian distributed, the optimum solution can be obtained using an ℓ_2-norm minimization, and solutions include maximum likelihood (ML) [4], iterative quadratic ML (IQML) [5] and subspace approaches such as multiple signal classification (MUSIC) [6], unitary estimation of signal parameters via rotational invariance techniques (U-ESPRIT or UE) [7] and principal-singular-vector utilization for modal analysis (PUMA) [8].

The ML-based methods are only feasible for 2-D HR due to their extremely high computational requirement, while the subspace methodology involves a smaller complexity. Its underlying principle is to separate the data into signal and noise subspaces, usually via a eigenvalue decomposition (EVD) or singular value decomposition (SVD), which is optimum for ℓ_2-norm minimization. The parameters of interest can then be extracted from the corresponding signal or noise subspaces. Furthermore, with the development of tensor algebra and higher-order SVD (HOSVD), the subspace methods have been extended to their multi-dimensional variants such as unitary tensor ESPRIT (UTE) [9], tensor PUMA [10] and tensor eigenvector (TEV) approach [11].

It is worth noting that in practice the noise often has non-Gaussian properties [12], and one frequently encountered process is the impulsive noise, such as the symmetric α-stable (SαS) variables. Compared to Gaussian distribution, the probability density function (PDF) of impulsive noise exhibits heavier tails, which corresponds to outliers, therefore the perfomance of existing ℓ_2-norm minimization based techniques may be severely degraded. To overcome this problem, techniques such as robust covariation-based MUSIC (ROC-MUSIC) [13], sign covariance matrix MUSIC (SCM-MUSIC) and Kendalls tau covariance matrix MUSIC (TCM-MUSIC) [14] are proposed. Other methods include the ℓ_1-norm minimization based robust iterative algorithm [15], ℓ_p-MUSIC [16] with $1 < p < 2$ and its tensor version [17]. These algorithms adopt the ℓ_p-norm of the residual fitting error data as the objective function for subspace decomposition, and develop the iteratively reweighted SVD (IR-SVD) or IR-HOSVD to solve the ℓ_p-norm minimization problem. This approach outperforms several existing outlier-resistant HR approaches in terms of resolution capability and estimation accuracy [16].

Recently, a new tensor decomposition method, named tensor-SVD (t-SVD) [18], is proposed for image processing [19] and tensor completion [20]. Like SVD of a matrix, the t-SVD gives a good performance in ℓ_2-norm based 3-D tensor factorization. The t-SVD has almost the same form as the matrix based SVD except that it uses tensor product and tensor transpose instead of matrix multiplication and matrix transpose. In this work, we propose to combine ℓ_p-norm minimization and t-SVD techniques to achieve R-D HR with low computational complexity and/or high accuracy.

The rest of this paper is organized as follows. In Section 2, the notation and problem formulation are provided. In Section 3, we present the 3-D t-SVD HR estimators, namely, IR-t-SVD-UE and IR-t-SVD-MUSIC, and extent it to more general situation. Numerical examples are included to demonstrate the effectiveness of the proposed algorithms in Section 4. Finally, conclusions are drawn in Section 5.

2. NOTATION, DATA MODEL AND T-SVD

2.1. Notation and Data Model

Scalars, vectors, matrices and tensors are denoted by italic, bold lower-case, bold upper-case and bold calligraphic symbols, respectively. The transpose and conjugate transpose of a vector or a matrix are written as ^{T} and ^{H}, and the $i \times i$ identity matrix is symbolized as $\mathbf{I}_i$. To refer to the $(m_1, m_2, \ldots, m_R)$ entry of a R-D tensor $\boldsymbol{\mathcal{A}} \in \mathbb{C}^{M_1 \times M_2 \times \cdots \times M_R}$, we use $a_{m_1,m_2,\ldots,m_R}$. The element wise multiplication operator between two tensors is defined as $\odot$, while the

The work described in this paper was in part supported by the National Natural Science Foundation of China under Grants (No. 61501300, U1501253 and 61271420), by the Guangdong Natural Science Foundation under Grant (No. 2015A030311030), by the Shenzhen Key Laboratory Establishment Foundation under Grants (No. ZDSYS201507081625213 and KC2015ZDYF0023A), and by the Shenzhen University garnt (No. 836-00008340).

outer product of $\mathcal{A} \in \mathbb{C}^{M_1 \times M_2 \times \cdots \times M_P}$ and $\mathcal{B} \in \mathbb{C}^{N_1 \times N_2 \times \cdots \times N_Q}$ is written as $\circ$ and $\mathcal{C} = \mathcal{A} \circ \mathcal{B} \in \mathbb{C}^{M_1 \times M_2 \times \cdots \times M_P \times N_1 \times N_2 \times \cdots \times N_Q}$ where $c_{m_1,m_2,\ldots,m_P,n_1,n_2,\ldots,n_Q} = a_{m_1,m_2,\ldots,m_P} \cdot b_{n_1,n_2,\ldots,n_Q}$. The symbol $\sqcup$ represents the concatenation operator where $\mathcal{A} = \mathcal{A}_1 \sqcup_r \mathcal{A}_2$ is obtained by stacking $\mathcal{A}_2$ to the end of $\mathcal{A}_1$ along the rth dimension. Furthermore, we define the n_3th frontal slice of a 3-D tensor $\mathcal{A} \in \mathbb{C}^{M_1 \times M_2 \times M_3}$ as $\mathcal{A}(:,:,n_3)$ for $n_3 = 1, 2, \cdots, M_3$, that is, $\mathcal{A} = \mathbf{A}_1 \sqcup_3 \mathbf{A}_2 \sqcup_3 \cdots \sqcup_3 \mathbf{A}_{M_3}$, and define $\mathcal{I}_{M_1 M_1 M_3}$ as the $M_1 \times M_1 \times M_3$ identity tensor whose first frontal slice is the $M_1 \times M_1$ identity matrix while the other frontal slices are all zeros.

We consider the problem of R-D HR with multiple snapshots in the presence of additive noise. The noise-free signal tensor $\mathcal{X} \in \mathbb{C}^{M_1 \times M_2 \times \cdots \times M_R \times N}$ has entries of the form:

$$x_{m_1,m_2,\ldots,m_R,n} = \sum_{f=1}^{F} \gamma_f(n) \prod_{r=1}^{R} e^{j\omega_{r,f} m_r} \tag{1}$$

where $n = 1, 2, \ldots, N$, $m_r = 1, 2, \ldots, M_r$, $r = 1, 2, \ldots, R$, and $f = 1, 2, \ldots, F$. The M_r, N and R are the data length of the rth dimension, number of snapshots and dimension number, respectively. Therefore the total dimension is $(R+1)$ together with the snapshots. The number of frequencies is denoted by F and is assumed known *a priori*. The $\gamma_f(n)$ denotes the unknown complex amplitude of the fth signal at the nth snapshot with power $\mathbb{E}\{|\gamma_f(n)|^2\} = \sigma_f^2$ while $\omega_{r,f} \in (-\pi, \pi)$ are the R-D frequencies. Define $\mathbf{a}_{r,f} = \begin{bmatrix} e^{j\omega_{r,f}} & e^{j\omega_{r,f} 2} & \cdots & e^{j\omega_{r,f} M_r} \end{bmatrix}$ and $\boldsymbol{\gamma}_f = \begin{bmatrix} \gamma_f(1) & \gamma_f(1) & \cdots & \gamma_f(N) \end{bmatrix}$, $\mathcal{X}$ can be written as $\mathcal{X} = \sum_{f=1}^{F} \mathbf{a}_{1,f} \circ \mathbf{a}_{2,f} \circ \cdots \circ \mathbf{a}_{R,f} \circ \boldsymbol{\gamma}_f$ and the observed signal, denoted by $\mathcal{Y} \in \mathbb{C}^{M_1 \times M_2 \times \cdots \times M_R \times N}$, is:

$$\mathcal{Y} = \mathcal{X} + \mathcal{Q} \tag{2}$$

where $\mathcal{Q}$ is the impulsive noise component.

2.2. t-SVD

Before going to the main result, we need to present the notation of t-SVD used in this paper. We follow the notations in [18] [19] [20] and define the t-product $*$ between two tensors $\mathcal{A}$ and $\mathcal{B}$ as

$$\mathcal{C} = \mathcal{A} * \mathcal{B} = \mathrm{fold}(\mathrm{circ}(\mathcal{A})\mathrm{MatVec}(\mathcal{B})) \tag{3}$$

where

$$\mathrm{circ}(\mathcal{A}) = \begin{bmatrix} \mathbf{A}_1 & \mathbf{A}_{M_3} & \mathbf{A}_{M_3-1} & \cdots & \mathbf{A}_2 \\ \mathbf{A}_2 & \mathbf{A}_1 & \mathbf{A}_{M_3} & \cdots & \mathbf{A}_3 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \mathbf{A}_{M_3} & \mathbf{A}_{M_3-1} & \mathbf{A}_{M_3-2} & \cdots & \mathbf{A}_1 \end{bmatrix} \tag{4}$$

$$\mathrm{MatVec}(\mathcal{A}) = \begin{bmatrix} \mathbf{A}_1^T & \mathbf{A}_2^T & \cdots & \mathbf{A}_{M_3}^T \end{bmatrix}^T \tag{5}$$

and $\mathrm{fold}(\cdot)$ is the inverse process of $\mathrm{MatVec}(\cdot)$. Furthermore, the tensor conjugate transpose of a 3-D tensor $\mathcal{A} \in \mathbb{C}^{M_1 \times M_2 \times M_3}$ is $\mathcal{A}^H \in \mathbb{C}^{M_2 \times M_1 \times M_3}$, which is computed by conjugate transposing each of the frontal slice of $\mathcal{A}$ and then reversing the order of transposed frontal slices 2 through M_3. According to [18], the operation of (3) can be computed using fast Fourier transform (FFT) and inverse fast Fourier transform(IFFT) as follows: Firstly, do the FFT along the 3rd dimension of $\mathcal{A}$ and $\mathcal{B}$, denoted as $\mathcal{A}_F = \mathrm{fft}(\mathcal{A}, [\,], 3)$ and $\mathcal{B}_F = \mathrm{fft}(\mathcal{B}, [\,], 3)$; secondly, compute $\mathcal{C}_F$ with the frontal slices $\mathcal{C}_F(:,:,m_3) = \mathcal{A}_F(:,:,m_3)\mathcal{B}_F(:,:,m_3)$; In the end, perform the IFFT along the 3rd dimension of $\mathcal{C}_F$ and get $\mathcal{C} = \mathrm{ifft}(\mathcal{C}, [\,], 3)$.

Analogous to the SVD of a 2-D matrix, the t-SVD of a complex-valued 3-D tensor $\mathcal{Y} \in \mathbb{C}^{M_1 \times M_2 \times M_3}$ can be written as

$$\mathcal{Y} = \mathcal{U} * \mathcal{S} * \mathcal{V}^H \tag{6}$$

The $\mathcal{U} \in \mathbb{C}^{M_1 \times M_1 \times M_3}$ is the left orthogonal singular tensor where $\mathcal{U} * \mathcal{U}^H = \mathcal{U}^H * \mathcal{U} = \mathcal{I}_{M_1 M_1 M_3}$ and $\mathcal{V} \in \mathbb{C}^{M_2 \times M_2 \times M_3}$ being the right orthogonal singular tensor with the same properties. The $\mathcal{S} \in \mathbb{C}^{M_1 \times M_2 \times M_3}$ is defined as a rectangular f-diagonal tensor, that is, all the frontal slices of $\mathcal{S}$ are diagonal matrices. Therefore, the entries in $\mathcal{S}$ are named the singular values of $\mathcal{Y}$. Table 1 [19] shows the detailed steps of the t-SVD, which uses the FFT, IFFT and matrix SVD to compute the tensor decomposition.

Furthermore, by defining $\mathfrak{f}_k = \mathcal{S}(k,k,:) \in \mathbb{C}^{1 \times 1 \times M_3}$, $k = 1, 2, \cdots, \min(M_1, M_2)$, we can decompose the tensor $\mathcal{S}$ into $\min(M_1, M_2)$ singular value tubes. As the operation of t-SVD contains a set of SVDs, we may say that the dominant $F < \min(M_1, M_2)$ singular value tubes of a t-SVD are the same as the dominant singular values of a SVD, although this definition is not a straightforward one [20].

(i) $\mathcal{Y}_F = \mathrm{fft}(\mathcal{Y}, [\,], 3)$.
for $m_3 = 1, 2, \cdots, M_3$
(ii) Perform SVD of $\mathcal{Y}_F(:,:,m_3)$ and get
$\mathcal{Y}_F(:,:,m_3) = \mathbf{USV}^H$.
(iii) Assign $\mathcal{U}_F(:,:,m_3) = \mathbf{U}, \mathcal{S}_F(:,:,m_3) = \mathbf{S}$
and $\mathcal{V}_F(:,:,m_3) = \mathbf{V}$.
end
(iv) Compute $\mathcal{U} = \mathrm{ifft}(\mathcal{U}_F, [\,], 3), \mathcal{S} = \mathrm{ifft}(\mathcal{S}_F, [\,], 3)$
and $\mathcal{V} = \mathrm{ifft}(\mathcal{V}_F, [\,], 3)$.

Table 1: t-SVD Algorithm

3. PROPOSED ESTIMATOR

3.1. 3-D Harmonic Retrieval

For 3-D harmonic retrieval, the noise-free tensor $\mathcal{X}$ is reduced to

$$\mathcal{X} = \sum_{f=1}^{F} \mathbf{a}_{1,f} \circ \mathbf{a}_{2,f} \circ \boldsymbol{\gamma}_f \tag{7}$$

As an extension of the matrix rank, we define F as the tensor rank of $\mathcal{X}$ here. According to [18], we can find two 3-D tensors $\mathcal{P}$ and $\mathcal{Q}$ that satisfy

$$\mathcal{X} = \mathcal{P} * \mathcal{Q} \tag{8}$$

where

$$\mathcal{P} = \sum_{f=1}^{L} \mathbf{u}_f \circ \mathbf{v}_f \circ \mathbf{k}_1 \tag{9}$$

$$\mathcal{Q} = \sum_{f=1}^{F} \mathbf{b}_f \circ \mathbf{a}_{2,f} \circ \boldsymbol{\gamma}_f \tag{10}$$

with $\mathbf{A}_1 = \begin{bmatrix} \mathbf{a}_{1,1} & \mathbf{a}_{1,2} & \cdots & \mathbf{a}_{1,F} \end{bmatrix} = \mathbf{UE}$ and $\mathbf{V}^H\mathbf{B} = \mathbf{E}$. The $\mathbf{u}_f$, $\mathbf{v}_f$ and $\mathbf{b}_f$ are the fth columns of $\mathbf{U}$, $\mathbf{V}$ and $\mathbf{E}$, and $\mathbf{k}_1$ is the first column of $\mathbf{I}_N$. Defining $\mathbf{U} = \mathbf{A}_1$ and $\mathbf{E} = \mathbf{I}_f$, we can prove that $L = F$ in (9), indicating that the tensor $\mathcal{X}$ has low tensor rank property and can be decomposed into two tensors $\mathcal{P} \in \mathbb{C}^{M_1 \times F \times N}$ and $\mathcal{Q} \in \mathbb{C}^{F \times M_2 \times N}$ of rank F. Since the received data are always noisy in practice, we get the approximate decomposition

$$\mathcal{Y} \approx \mathcal{P} * \mathcal{Q} \tag{11}$$

It is well known that the least squares (LS) based approach corresponding to ℓ_2-norm and is not robust to outliers. To achieve robust estimation in the presence of impulsive noise, one key idea [21] is to replace the squared residuals in the LS methodology by another function which emphasizes large samples less than the square, and ℓ_p-norm with $1 < p < 2$ is one common choice. The ℓ_p-norm of a complex-valued 3-D tensor is defined as

$$||\boldsymbol{\mathcal{Y}}||_p = (\sum_{m_1=1}^{M_1}\sum_{m_2=1}^{M_2}\sum_{n=1}^{N}|y_{m_1,m_2,n}|^p)^{1/p} \tag{12}$$

and the ℓ_p-norm based objective function is

$$J(\boldsymbol{\mathcal{P}},\boldsymbol{\mathcal{Q}}) = ||\boldsymbol{\mathcal{Y}} - \boldsymbol{\mathcal{P}} * \boldsymbol{\mathcal{Q}}||_p^p \tag{13}$$

We define the residual error tensor as $\boldsymbol{\mathcal{R}} = \boldsymbol{\mathcal{Y}} - \boldsymbol{\mathcal{P}} * \boldsymbol{\mathcal{Q}}$, then its ℓ_p-norm can be expressed as

$$\begin{aligned}||\boldsymbol{\mathcal{R}}||_p^p =& \sum_{m_1=1}^{M_1}\sum_{m_2=1}^{M_2}\sum_{n=1}^{N}|r_{m_1,m_2,n}|^{(p-2)}|r_{m_1,m_2,n}|^2\\ =& ||\boldsymbol{\mathcal{D}} \odot \boldsymbol{\mathcal{Y}} - \boldsymbol{\mathcal{D}} \odot (\boldsymbol{\mathcal{P}} * \boldsymbol{\mathcal{Q}})||_2^2\end{aligned} \tag{14}$$

where $\boldsymbol{\mathcal{D}}$ is the weighting matrix with $d_{m_1,m_2,n} = |r_{m_1,m_2,n}|^{(p-2)/2}$. Equation (14) indicates that the ℓ_p-norm minimization can be converted to ℓ_2-norm minimization, and the solution of this optimization can be computed through the t-SVD of $\boldsymbol{\mathcal{D}} \odot \boldsymbol{\mathcal{Y}}$:

$$\boldsymbol{\mathcal{D}} \odot \boldsymbol{\mathcal{Y}} = \boldsymbol{\mathcal{U}}_s * \boldsymbol{\mathcal{S}}_s * \boldsymbol{\mathcal{V}}_s^H + \boldsymbol{\mathcal{U}}_n * \boldsymbol{\mathcal{S}}_n * \boldsymbol{\mathcal{V}}_n^H \tag{15}$$

where $\boldsymbol{\mathcal{S}}_s \in \mathbb{C}^{F\times F\times M_3}$ contains the F dominant singular value tubes, and $\boldsymbol{\mathcal{U}}_s \in \mathbb{C}^{M_1\times F\times M_3}$ and $\boldsymbol{\mathcal{V}}_s \in \mathbb{C}^{M_2\times F\times M_3}$ contain the corresponding singular vectors. The $\boldsymbol{\mathcal{S}}_n, \boldsymbol{\mathcal{U}}_s$ and $\boldsymbol{\mathcal{V}}_s$ are the corresponding residual terms. This can be regarded as a truncated t-SVD. Therefore, we can define

$$\boldsymbol{\mathcal{P}} = \boldsymbol{\mathcal{U}}_s, \quad \boldsymbol{\mathcal{Q}} = \boldsymbol{\mathcal{S}}_s * \boldsymbol{\mathcal{V}}_s^H \tag{16}$$

and solve the problem (13). Due to the reason that the weighting tensor $\boldsymbol{\mathcal{D}}$ contains $\boldsymbol{\mathcal{P}}$ and $\boldsymbol{\mathcal{Q}}$, we cannot obtain the final solution immediately. An iterative procedure, named as iteratively reweighted t-SVD (IR-t-SVD), is adopted and is shown in Table 2, where the superscript $^{(k)}$ denotes the kth iteration. Notice that we can also use the idea of SCM [14] for the initialization step in Table 2, that is, compute $\boldsymbol{\mathcal{P}}^{(0)}$ and $\boldsymbol{\mathcal{Q}}^{(0)}$ from the truncated t-SVD of $\boldsymbol{\mathcal{Z}}$ where $z_{m_1,m_2,n} = y_{m_1,m_2,n}/|y_{m_1,m_2,n}|$.

After obtaining $\boldsymbol{\mathcal{P}}$ and $\boldsymbol{\mathcal{Q}}$ through the IR-t-SVD, we can get the recovered signal

$$\boldsymbol{\mathcal{Y}}_{\text{recover}} = \boldsymbol{\mathcal{P}} * \boldsymbol{\mathcal{Q}} \tag{17}$$

and then apply the conventional multi-dimensional HR algorithms, such as UE [7], UTE [9] and TEV [11] to compute the final estimates of the harmonics. Note that automatic parameter pairing is achieved in these algorithms. Furthermore, when there are identical frequencies in at least one dimension, these algorithms can still operate well. However, when there are no identical frequencies along any dimension, we can follow the idea of [17] and use the MUSIC algorithm [16] to solve the harmonics for all the dimensions one by one individually.

According to (3), (16) and (17), we have

$$\text{MatVec}(\boldsymbol{\mathcal{Y}}_{\text{recover}}) = \text{circ}(\boldsymbol{\mathcal{U}}_s^{(K)}) * \text{MatVec}(\boldsymbol{\mathcal{Q}}) \tag{18}$$

Since $(\boldsymbol{\mathcal{U}}^{(K)})^H * \boldsymbol{\mathcal{U}}^{(K)} = \boldsymbol{\mathcal{I}}_{FFM_3}$, we reach to the conclusion that $\text{circ}(\boldsymbol{\mathcal{U}}_s^{(K)})^H\text{circ}(\boldsymbol{\mathcal{U}}_s^{(K)}) = \mathbf{I}_{FN}$ and have

$$\begin{aligned}&\text{MatVec}(\boldsymbol{\mathcal{Y}}_{\text{recover}})^H\text{MatVec}(\boldsymbol{\mathcal{Y}}_{\text{recover}})\\ =&\text{MatVec}(\boldsymbol{\mathcal{Q}})^H\text{circ}(\boldsymbol{\mathcal{U}}_s^{(K)})^H\text{circ}(\boldsymbol{\mathcal{U}}_s^{(K)})\text{MatVec}(\boldsymbol{\mathcal{Q}})\\ =&\text{MatVec}(\boldsymbol{\mathcal{Q}})^H\text{MatVec}(\boldsymbol{\mathcal{Q}})\end{aligned} \tag{19}$$

which is the conjugate sample covariance matrix of the second dimension of the recovered signal, indicating that $\boldsymbol{\mathcal{Q}}$ contains the parameter information of the second dimension, therefore the root-MUSIC technique [22] can be used to do the estimation. Similarly, harmonics of the first dimension can be retrieved, and the final step is to associate them via the pairing step [23].

(i) Initialize $\boldsymbol{\mathcal{P}}_F$ and $\boldsymbol{\mathcal{Q}}_F$ with all frontal slices being random matrices of full column rank and full row rank, and then assign $\boldsymbol{\mathcal{P}}^{(0)} = \text{ifft}(\boldsymbol{\mathcal{P}}_F, [\,], 3))$ and $\boldsymbol{\mathcal{Q}}^{(0)} = \text{ifft}(\boldsymbol{\mathcal{Q}}_F, [\,], 3))$.
for $k = 0, 1, \cdots, K-1$, **do**
(ii) Compute $\boldsymbol{\mathcal{R}}^{(k)}$ and then $\boldsymbol{\mathcal{D}}^{(k)}$.
(iii) Perform truncated t-SVD on $\boldsymbol{\mathcal{D}}^{(k)} \odot \boldsymbol{\mathcal{Y}} = \boldsymbol{\mathcal{U}}_s * \boldsymbol{\mathcal{S}}_s * \boldsymbol{\mathcal{V}}_s^H$.
(iv) Assign $\boldsymbol{\mathcal{P}}^{(k+1)} = \boldsymbol{\mathcal{U}}_s$ and $\boldsymbol{\mathcal{Q}}^{(k+1)} = \boldsymbol{\mathcal{S}}_s * \boldsymbol{\mathcal{V}}_s^H$.
end

Table 2: IR-t-SVD Algorithm

3.2. Extension to Higher Dimension

For the general multi-dimensional case, we suggest to use the idea of general unfolding [24] to reshape the received data first. For a tensor $\boldsymbol{\mathcal{Y}} \in \mathbb{C}^{M_1\times M_2\times M_3\times N}$, we define $\boldsymbol{\mathcal{Z}} = \text{stack}(\boldsymbol{\mathcal{Y}}) \in \mathbb{C}^{M_1M_3\times M_2\times N}$ by stacking the samples in the 3rd dimension of $\boldsymbol{\mathcal{Y}}$ to the end of the 1st dimension. Similarly, for a tensor $\boldsymbol{\mathcal{Y}} \in \mathbb{C}^{M_1\times M_2\times\cdots\times M_R\times N}$, we can reshape it as

$$\boldsymbol{\mathcal{Z}} = \text{stack}(\boldsymbol{\mathcal{Y}}) \in \mathbb{C}^{M_{r1}\times M_{r2}\times N} \tag{20}$$

where $M_{r1}M_{r2} = M = \prod_{r=1}^{R} M_r$. As the operation of t-SVD contains a set of SVDs to the frontal slices of $\text{fft}(\boldsymbol{\mathcal{Z}}, [\,], 3)$, we propose to make $M_{r1} \approx M_{r2}$ to save time. Then we can substitute $\boldsymbol{\mathcal{Y}}$ with $\boldsymbol{\mathcal{Z}}$ for the retrieval process.

4. SIMULATION RESULTS

Computer simulations have been carried out to evaluate the performance of the proposed R-D HR approach by comparing with state-of-the-art algorithms, the IR-SVD-UE [16] and IR-HOSVD-MUSIC [17]. As [16] is not designed for multi-dimensional HR, we modify it as follows: firstly, unfold the mutli-dimensional data into a 2-D matrix, and use the IR-SVD approach in [16] to recover the signal subspace; Secondly, apply the recovered signal into UE [7] to get the final estimates. The IR-HOSVD-MUSIC [17] method, on the other hand, recover the signal subspace for each dimension individually, and cannot be applied with the UE type algorithms. Furthermore, the direct UE approach is also considered to show the advantages of robust methods against the non-robust one. Similarly, we combine the proposed IR-t-SVD method with both UE and MUSIC, named as IR-t-SVD-UE and IR-t-SVD-MUSIC, in all the testes. All the results are obtained based on 1000 independent runs. The statistical performance is evaluated in terms of average mean square frequency error (AMSFE), which is computed by averaging over all the number of sources and dimensions. The impulsive noise is modeled using the SαS process. The characteristic function of the SαS distribution with zero location is determined by the characteristic exponent $0 < \alpha < 2$ and dispersion $\gamma > 0$ [25]. In our study, we set $\gamma = 1$. Furthermore, generalized signal-to-noise ratio (GSNR) is adopted to quantify the relative strength between signal and noise [16] since S-NR is not applicable. Unless stated otherwise, $p = 1.3$ is employed in all estimators.

In the first test, we assume that there are three frequencies in a 3-D space. The power vector of the uncorrelated sources is $\boldsymbol{\sigma}^2 =$

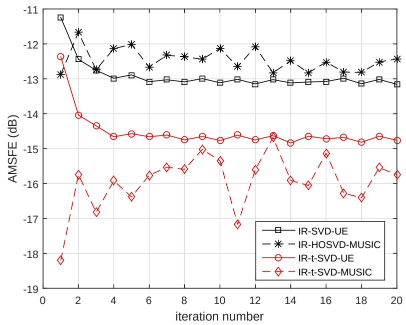

Fig. 1. AMSFE versus iteration number

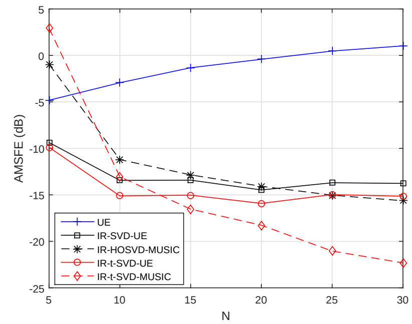

Fig. 3. AMSFE versus N

[1 1 1], while the frequencies are $[\omega_{1,1}\ \omega_{1,2}\ \omega_{1,3}] = [0.1\ 0.4\ 0.6]\pi$ and $[\omega_{1,1}\ \omega_{1,2}\ \omega_{1,3}] = [0.1\ 0.3\ 0.8]\pi$. The size of the data set is $M_1 \times M_2 \times N = 15 \times 15 \times 15$. Figure 1 shows the ARMSE results versus iteration number in SαS stable noise with $\alpha = 1.3$ and GSNR $= 20$dB. It shows that the UE-type algorithms converge in a few iterations, while the MUSIC-type ones fail to converge to a stable optimum point. This is due to the reason that the MUSIC based algorithms recover the subspace of each dimension and do the estimation individually, therefore some useful information of the data set is missed in the estimation process. However, in most cases, the IR-t-SVD-MUSIC is superior to IR-t-SVD-UE, and the proposed IR-t-SVD based approaches outperforms the others by 2-3dB.

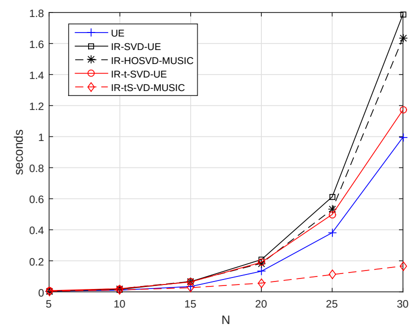

Fig. 4. Computational time versus N

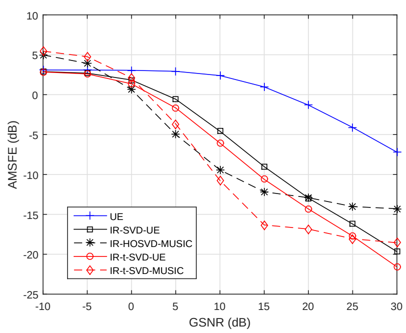

Fig. 2. AMSFE versus GSNR

In the second test, we assume the same parameter setting as the previous one, and shows the ARMSE results versus GSNR in Figure 2. It is observed that the MUSIC based methods suffer from the convergence problem, while the UE based ones do not. When the GSNR is sufficiently large, the proposed methods give the best performance among all the methods, and there is a gap between the robust methods and the non-robust one. The average computation times of the UE, IR-SVD-UE, IR-HOSVD-MUSIC, IR-t-SVD-UE and IR-t-SVD-MUSIC algorithms in a single run are measured as 0.0580s, 0.1172s, 0.1400s, 0.1101s and 0.0341s, respectively, showing the computational attractiveness of the IR-t-SVD-MUSIC approach.

We test the estimation accuracy and computational complexity against the size of the data in the third simulation and the results are shown in Figures 3 and 4. The parameter settings are the same as that in the second test with $M_1 = M_2 = N$ varying from 5 to 30 and GSNR $= 20$dB. This time, we find out that the IR-t-SVD-MUSIC is the most attractive one when the data size is large. It runs the fastest and gives the most accurate result among all the estimators, while the IR-t-SVD-UE performs not as good as the IR-t-SVD-MUSIC in large data size situation comparing to small data size case. As the UE-type algorithms use UE in the estimation step, when $M_1 = M_2 = N$, the UE method unfolds the recovered tensor into a matrix of size $M_1 M_2 \times N$ and compute the subspace using SVD. Since this is a tall matrix, the performance might be downgraded because of the inaccurate computation of the signal subspace [9], leading to the observed result. Furthermore, the time consumed by the proposed IR-t-SVD-UE, which uses UE in the estimation step, becomes close to the time consumed by UE method when data size is large, indicating that the estimation process instead of recovery process becomes dominant in large sample case.

5. CONCLUSION

A tensor structure based approach for R-D HR in the presence of impulsive noise is devised. The main idea is to minimize the ℓ_p-norm of the residual error of the data samples. With the use of t-SVD and the iteratively reweighted technique, the tensor subspace can be estimated, leading to a recovery of the signal or signal subspace for subspace based HR methods. Computer simulations show that the proposed algorithms have an outstanding performance in terms of computational complexity and/or estimation accuracy.

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