ESTIMATING PARAMETERS IN NOISY LOW FREQUENCY EXPONENTIALLY DAMPED SINUSOIDS AND EXPONENTIALS

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ABSTRACT

There has been much recent interest in damped sinusoidal models, probably as a result of their relevance to magnetic resonance imaging. In [1], a model which allowed the sinusoid to decay to 0 was examined, and a Fourier coefficient estimation procedure was proposed. [2] noted that in order for any asymptotic theory to be available, the decay should not be allowed to complete, and examined the asymptotic behavior of a Fourier coefficient procedure based on this assumption, for which the asymptotic behavior of nonlinear least squares estimators had already been derived in [3]. In this paper, we consider the problem of estimating the frequency and damping factor when the frequency is so low that only a finite number of periods appear in the data. Additionally, we consider a Fourier technique for estimating the damping factor in a noisy real exponential.

Index Terms**—** exponentially damped sinsuoid estimation, Fourier coefficient method

1. INTRODUCTION

In [2] the model considered initially was

$$X_t = \mu + Ae^{-\gamma t}\cos(\omega t+\phi)+\varepsilon_t,\ t=0,1,\ldots,T-1 \quad (1)$$

where $\mu, A>0, \gamma>0, \omega$ and ϕ are unknown parameters, and $\{\varepsilon_t\}$ is some general 'noise' process, not necessarily Gaussian or white. Interest was in the estimation of these unknown parameters, and their asymptotic properties as $T\to\infty$. However, as the amplitude $Ae^{-\gamma t}$ converges to 0 as $T\to\infty$, the Cramér-Rao lower bound does not converge to 0 as $T\to\infty$ and so the estimators are inconsistent. The model was reparametrized as

$$X_t = \mu + Ae^{-\gamma t/T}\cos(\omega t+\phi)+\varepsilon_t,\ t=0,1,\ldots,T-1, \quad (2)$$

as in [3] in order to avoid this problem. A review of estimation techniques was conducted and a generalization of [4] produced. Of note in (2) is that although the amplitude of the sinusoid does not converge to 0 as $T\to\infty$, the number of periods of the sinusoid is linear in T, and therefore diverges to ∞. In [5], a similar idea is used with model given by (1), but at the times $t=0,1/(T-1),2/(T-1),\ldots,1$, the number of periods of the sinusoid is fixed, and the stochastic properties of the noise process $\{\varepsilon_t\}$ thus become problematic.

In this paper, we propose the following model for the case of a damped sinusoid

$$X_t = \mu + Ae^{-\gamma t/T}\cos(at/T+\phi)+\varepsilon_t,\ t=0,1,\ldots,T-1 \quad (3)$$

for which there is a fixed number of sinusoidal periods. The same idea was used in [6], where limit theory was established for the least squares estimator of the frequency of a sinusoid, when the frequency was 'low'. We derive the asymptotic theory for the least squares estimators of the parameters. We then propose Fourier transform estimators of γ and a. A special case is that of $a=0$, i.e. a purely exponential signal. The Fourier transform technique outperforms least squares from the computational point of view, and has very similar asymptotics. The technique is generalized to a broad class of nonlinear functions, using a more general class of transforms. Simulations are performed to evaluate the accuracy of the asymptotics in relatively small samples.

2. LEAST SQUARES AND THE GAUSSIAN CRLB

[6] examined (3) when $\gamma=0$. The least squares procedure was defined and analyzed imposing only weak conditions on $\{\varepsilon_t\}$. In particular, Gaussianity and whiteness are not needed for the parameter estimators to satisfy a central limit theorem, which depends on $\{\varepsilon_t\}$ only through its spectral density $f(\omega)$ at 0 frequency. The derivation of the central limit theorem is complicated by the fact that (3) has *three* sinusoidal terms that 'interfere' with each other, at frequencies $-a/T, 0$ and a/T. In [7] it is shown that $T^{1/2}(\widehat{a}_T-a)$ is asymptotically normal with mean 0 and variance of the form

$$\frac{48\pi f(0)}{A^2}\left\{\xi\cos^2\psi+\zeta\sin^2\psi\right\},$$

where ξ and ζ depend only on a and $\psi=\phi+a/2$. Here we rewrite the model as

$$X_t = \nu+\alpha\left\{e^{-\gamma t/T}\cos(at/T)-c\right\}+\beta\left\{e^{-\gamma t/T}\sin(at/T)-s\right\}+\varepsilon_t,$$

where $\nu=\mu-\alpha c-\beta s$ and

$$c=T^{-1}\sum_{t=0}^{T-1}e^{-\gamma t/T}\cos(at/T),\ s=T^{-1}\sum_{t=0}^{T-1}e^{-\gamma t/T}\sin(at/T).$$

We thus minimize with respect to ν, α, β and a,

$$S(\nu,\alpha,\beta,a,\gamma) = \sum_{t=0}^{T-1}\Big[X_t - \nu - \alpha\left\{e^{-\gamma t/T}\cos(at/T) - c\right\} \\ -\beta\left\{e^{-\gamma t/T}\sin(at/T) - s\right\}\Big]^2. \tag{4}$$

Under Gaussian assumptions on $\{\varepsilon_t\}$, the log-likelihood is

$$l = -\frac{T}{2}\log\left(2\pi\sigma^2\right) - \frac{1}{2\sigma^2}S(\nu,\alpha,\beta,a,\gamma),$$

and so the least squares estimators of ν, α, β, a and γ are also the Gaussian maximum likelihood estimators. Now for fixed a and γ, S is minimized with respect to ν, α and β when $\nu = \overline{X} = T^{-1}\sum_{t=0}^{T-1} X_t$ and

$$\begin{bmatrix}\alpha\\ \beta\end{bmatrix} = \begin{bmatrix} D_{11} & D_{12}\\ D_{12} & D_{22}\end{bmatrix}^{-1}\begin{bmatrix}C_1\\ C_2\end{bmatrix}$$

where

$$\begin{bmatrix}C_1\\ C_2\end{bmatrix} = \begin{bmatrix}\sum_{t=0}^{T-1}\left(X_t - \overline{X}\right)e^{-\gamma t/T}\cos(at/T)\\ \sum_{t=0}^{T-1}\left(X_t - \overline{X}\right)e^{-\gamma t/T}\sin(at/T)\end{bmatrix},$$

$$\begin{bmatrix}D_{11}\\ D_{12}\\ D_{22}\end{bmatrix} = \begin{bmatrix}\sum_{t=0}^{T-1} e^{-2\gamma t/T}\cos^2(at/T) - Tc^2\\ \sum_{t=0}^{T-1} e^{-2\gamma t/T}\cos(at/T)\sin(at/T) - Tsc\\ \sum_{t=0}^{T-1} e^{-2\gamma t/T}\sin^2(at/T) - Ts^2\end{bmatrix}$$

The least squares procedure is then the same as maximizing

$$P(a,\gamma) = \begin{bmatrix} C_1 & C_2\end{bmatrix}\begin{bmatrix} D_{11} & D_{12}\\ D_{12} & D_{22}\end{bmatrix}^{-1}\begin{bmatrix}C_1\\ C_2\end{bmatrix}.$$

Note that the elements of D may be asymptotically approximated by applying (8) to $z = -2\gamma$ and $z = -2\gamma + 2ja$. $P(a,\gamma)$ can also be made to look more like a periodogram by using a Lomb-Scargle [8, 9] type trick, replacing $e^{-\gamma t/T}\cos(at/T+\phi)$ in (3) by

$$e^{-\gamma(t-\tau)/T}\cos\left(a(t-\tau)/T + \phi\right),$$

recalculating the above and finding a τ that forces D_{12} to be 0. The details are fairly simple and will not be given here. Indeed, in light of comments made in [10], computing τ and using the modified formulae might cause numerical instability.

Gaussian (asymptotic) Cramér-Rao bounds and the asymptotic distribution of the estimators of a and γ are quite complicated, but simplified by using the original model (3). The log-likelihood l is then given by

$$l = -\frac{T}{2}\log\left(2\pi\sigma^2\right) - \frac{1}{2\sigma^2}V_T(\Theta), \tag{5}$$

where $\Theta = \begin{bmatrix}\mu & A & \phi & a & \gamma\end{bmatrix}'$ and

$$V_T(\Theta) = \sum_{t=0}^{T-1}\left\{X_t - \mu - Ae^{-\gamma t/T}\cos(at/T+\phi)\right\}^2 \tag{6}$$

The (asymptotic) Cramér-Rao bounds are computed in the appendix, and are given by the diagonal elements of $T^{-1}\sigma^2\Omega^{-1}$. In fact, these are also the asymptotic variances in the central limit theorem even under non-Gaussian and colored noise assumptions: Let $\widehat{\Theta}_T$ be the minimizer of $V_T(\Theta)$ with respect to Θ. Then under the same assumptions as in [6], $\widehat{\Theta}_T$ converges almost surely to 0 as $T \to \infty$, and the distribution of $T^{1/2}\left(\widehat{\Theta}_T - \Theta\right)$ converges as $T \to \infty$ to the normal with mean 0 and variance $2\pi f(0)\,\Omega^{-1}$. The fixed-frequency case has been discussed in [3, 11, 2].

3. FOURIER COEFFICIENT ESTIMATION TECHNIQUE

Let

$$Y_k = \sum_{t=0}^{T-1} X_t e^{-j2\pi kt/T}, U_k = \sum_{t=0}^{T-1}\varepsilon_t e^{-j2\pi kt/T}.$$

Then, with δ_{ij} denoting Kronecker's delta, Y_k is given by

$$T\mu\delta_{0k} + D\frac{1-e^{-\gamma+ja}}{1-e^{-(\gamma-ja+2\pi jk)/T}} + D^*\frac{1-e^{-\gamma-ja}}{1-e^{-(\gamma+ja+2\pi jk)/T}} + U_k,$$

where $D = Ae^{j\phi}/2$. The major complication in implementing a Fourier coefficient method, which does not occur in the fixed-frequency case, is that the term above involving D^* is not insignificant compared with the term involving D. As in [4], suppose that $a = 2\pi(n+\delta)$, where $\delta \in (-1/2, 1/2)$. Then, although n is unknown, it may be shown that, if $n > 0$,

$$\operatorname{argmax}_{1\le k\le\lfloor(T-1)/2\rfloor}|Y_k|^2 \to n,$$

almost surely as $T \to \infty$, and may thus be used to estimate n. If $|\delta| = 1/2$, the limit points are the set $\{n-1, n, n+1\}$, but this will not matter, for the same reason as in [7]. Assume first that $a > 3\pi$. Then for $k = -1, 0, 1$ and $n \ge 2$,

$$Y_{n+k} = D\frac{1-e^{-\gamma+2\pi j\delta}}{1-e^{-(\gamma-2\pi j\delta+2\pi jk)/T}} + D^*\frac{1-e^{-\gamma-2\pi j\delta}}{1-e^{-(\gamma+2\pi j\delta+4\pi jk)/T}} + U_{n+k}.$$

As in [4], solving the equations

$$Y_{n+1} = D\frac{1-e^{-\gamma+2\pi j\delta}}{1-e^{-(\gamma-2\pi j\delta+2\pi j)/T}} + D^*\frac{1-e^{-\gamma-2\pi j\delta}}{1-e^{-(\gamma+2\pi j\delta+4\pi j)/T}}$$

$$Y_n = D\frac{1-e^{-\gamma+2\pi j\delta}}{1-e^{-(\gamma-2\pi j\delta)/T}} + D^*\frac{1-e^{-\gamma-2\pi j\delta}}{1-e^{-(\gamma+2\pi j\delta)/T}}$$

yields one set of estimators of D, γ and δ, since the equations above represent four (real) equations in four (real) unknowns. Solving

$$Y_{n-1} = D\frac{1-e^{-\gamma+2\pi j\delta}}{1-e^{-(\gamma-2\pi j\delta-2\pi j)/T}} + D^*\frac{1-e^{-\gamma-2\pi j\delta}}{1-e^{-(\gamma+2\pi j\delta-4\pi j)/T}}$$

$$Y_n = D\frac{1-e^{-\gamma+2\pi j\delta}}{1-e^{-(\gamma-2\pi j\delta-2\pi j)/T}} + D^*\frac{1-e^{-\gamma-2\pi j\delta}}{1-e^{-(\gamma+2\pi j\delta-4\pi j)/T}}$$

gives another. There appear to be no closed-form formulae for solving the equations, or choosing between the two sets of solutions, even if asymptotic versions of the equations are used. Moreover, when $a \le 3\pi$, Y_0 cannot be used, as it involves μ, and is also real. Thus Y_1 and Y_2 need to be used when $a < 5\pi$.

3.1. A special case: $a = 0$

When $a = \phi = 0$, we have

$$Y_k = T\mu\delta_{0k} + A\frac{1-e^{-\gamma}}{1-e^{-(\gamma+2\pi jk)/T}} + U_k.$$

We may thus estimate γ by solving

$$Y_1 = A\frac{1-e^{-\gamma}}{1-e^{-(\gamma+2\pi j)/T}},$$

which reduces to

$$\frac{\operatorname{Re}(Y_1)}{\operatorname{Im}(Y_1)} = \frac{\operatorname{Re}\left(\frac{1-e^{-(\gamma-j2\pi)/T}}{\left|1-e^{-(\gamma+j2\pi)/T}\right|^2}\right)}{\operatorname{Im}\left(\frac{1-e^{-(\gamma-j2\pi)/T}}{\left|1-e^{-(\gamma+j2\pi)/T}\right|^2}\right)} = \frac{1-e^{-\gamma/T}\cos(2\pi/T)}{e^{-\gamma/T}\sin(2\pi/T)},$$

for which the solution is

$$\gamma = \widehat{\gamma}_T = T \log \left(\cos(2\pi/T) - \frac{\operatorname{Re}(Y_1)}{\operatorname{Im}(Y_1)} \sin(2\pi/T) \right)$$
$$\sim -2\pi \operatorname{Re}(Y_1) / \operatorname{Im}(Y_1).$$

The estimator $\widehat{\gamma}_T$ is remarkably simple, and certainly much faster to compute than the nonlinear least squares estimator, found by minimizing with respect to μ, A and γ,

$$\sum_{t=0}^{T-1} \left\{ X_t - \mu - A e^{-\gamma t/T} \right\}^2,$$

or equivalently by maximizing with respect to γ

$$\frac{\left\{ \sum_{t=0}^{T-1} \left(X_t - \overline{X} \right) e^{-\gamma t/T} \right\}^2}{\sum_{t=0}^{T-1} e^{-2\gamma t/T} - T^{-1} \left(\sum_{t=0}^{T-1} e^{-\gamma t/T} \right)^2}$$

4. GENERALIZATION

The above suggests an estimation procedure for 'low frequency' nonlinear regression problems. Suppose we wish to fit

$$X_t = \mu + \beta f(\gamma t/T) + \varepsilon_t, t = 0, 1, \ldots, T-1$$

where $\{\varepsilon_t\}$ is 'noise' and f is known. Let $\{g_k(x)\}$ be a family of functions whose domains are $[0, 1]$, and put

$$\begin{aligned} Y_k &= \sum_{t=0}^{T-1} X_t g_k(t/T) \\ &= \mu \sum_{t=0}^{T-1} g_k(t/T) + \beta \sum_{t=0}^{T-1} g_k(t/T) f(\gamma t/T) + \sum_{t=0}^{T-1} \varepsilon_t g_k(t/T). \end{aligned}$$

As long as $\{g_k(x)\}$ is suitably well-behaved,

$$\operatorname{var} \left\{ T^{-1/2} \sum_{t=0}^{T-1} \varepsilon_t g_k(t/T) \right\} \to 2\pi f(0) \int_0^1 g_k^2(x)\, dx.$$

Thus, at least in probability as $T \to \infty$,

$$\begin{aligned} T^{-1} Y_k &\to \mu \int_0^1 g_k(x)\, dx + \beta \int_0^1 g_k(x) f(\gamma x)\, dx \\ &= \mu G_k + \beta H_k(\gamma), \end{aligned}$$

say. For fixed γ, we might thus estimate μ and β by solving the above equation for $k = 0, 1$, viz.

$$\begin{bmatrix} \mu \\ \beta \end{bmatrix} = \begin{bmatrix} G_0 & H_0(\gamma) \\ G_1 & H_1(\gamma) \end{bmatrix}^{-1} \begin{bmatrix} T^{-1} Y_0 \\ T^{-1} Y_1 \end{bmatrix},$$

and thus estimate γ by solving for γ

$$\begin{aligned} T^{-1} Y_2 &= \begin{bmatrix} G_2 & H_2(\gamma) \end{bmatrix} \begin{bmatrix} G_0 & H_0(\gamma) \\ G_1 & H_1(\gamma) \end{bmatrix}^{-1} \begin{bmatrix} T^{-1} Y_0 \\ T^{-1} Y_1 \end{bmatrix} \\ &= T^{-1} \frac{G_2 \{ H_1(\gamma) Y_0 - H_0(\gamma) Y_1 \} + H_2(\gamma)(G_0 Y_1 - G_1 Y_0)}{G_0 H_1(\gamma) - G_1 H_0(\gamma)}, \end{aligned}$$

i.e. by finding zeros of

$$\begin{aligned} \kappa(\gamma) = {} & (G_0 Y_1 - G_1 Y_0) H_2(\gamma) + (G_2 Y_0 - G_0 Y_2) H_1(\gamma) \quad (7) \\ & + (G_1 Y_2 - G_2 Y_1) H_0(\gamma). \end{aligned}$$

For example, suppose $f(x) = e^{-x}$, and

$$g_k(x) = \begin{cases} 1 & ; \quad k = 0 \\ \cos(ax) & ; \quad k = 1 \\ \sin(ax) & ; \quad k = 2. \end{cases}$$

Then

$$G_k = \begin{cases} 1 & ; \quad k = 0 \\ \sin a / a & ; \quad k = 1 \\ (1 - \cos a)/a & ; \quad k = 2 \end{cases}$$

and

$$\begin{aligned} H_0(\gamma) &= \left(1 - e^{-\gamma}\right)/\gamma \\ H_1(\gamma) &= \left(\gamma - \gamma \cos a\, e^{-\gamma} + a \sin a\, e^{-\gamma}\right) / \left(a^2 + \gamma^2\right) \\ H_2(\gamma) &= \left(a - a \cos a\, e^{-\gamma} - \gamma \sin a\, e^{-\gamma}\right) / \left(a^2 + \gamma^2\right) \end{aligned}$$

In the special case where $a = 2n\pi$, n an integer, $G_k = \delta_{0k}$,

$$\begin{aligned} H_0(\gamma) &= \left(1 - e^{-\gamma}\right)/\gamma \\ H_1(\gamma) &= \gamma \left(1 - e^{-\gamma}\right) / \left(4n^2\pi^2 + \gamma^2\right) \\ H_2(\gamma) &= 2n\pi \left(1 - e^{-\gamma}\right) / \left(4n^2\pi^2 + \gamma^2\right) \end{aligned}$$

and so

$$\kappa(\gamma) = (\gamma Y_1 - 2n\pi Y_2)\left(1 - e^{-\gamma}\right) / \left(4n^2\pi^2 + \gamma^2\right).$$

The estimator $\widehat{\gamma}_T$ of γ is thus $2n\pi Y_2/Y_1$, agreeing with the formula in section 3.1 for the case $n = 1$. Unless a is of the form given, zeros of $\kappa(\gamma)$ have to be found by search or some iterative procedure. In the general case, $\widehat{\gamma}_T$ converges almost surely to γ, and $T^{1/2}(\widehat{\gamma}_T - \gamma)$ is asymptotically normal with mean 0 and variance

$$2\pi f(0) \frac{c'(\gamma_0)\, \Omega c(\gamma_0)}{A^2 \lambda^2(\gamma_0)},$$

where

$$\begin{aligned} \lambda(\gamma) = {} & -\frac{d}{d\gamma} H_2(\gamma) \\ & + \begin{bmatrix} G_2 & H_2(\gamma) \end{bmatrix} \begin{bmatrix} G_0 & H_0(\gamma) \\ G_1 & H_1(\gamma) \end{bmatrix}^{-1} \begin{bmatrix} \frac{d}{d\gamma} H_0(\gamma) \\ \frac{d}{d\gamma} H_1(\gamma) \end{bmatrix} \end{aligned}$$

$$c'(\gamma_0) = \begin{bmatrix} -\begin{bmatrix} G_2 & H_2(\gamma) \end{bmatrix} \begin{bmatrix} G_0 & H_0(\gamma) \\ G_1 & H_1(\gamma) \end{bmatrix}^{-1} & 1 \end{bmatrix}$$

and Ω is the 3×3 matrix with (j,k)th entry $\int_0^1 g_j(x) g_k(x)\, dx$. When $a = 2n\pi$, n an integer,

$$\begin{aligned} \lambda(\gamma) &= -\frac{d}{d\gamma} H_2(\gamma) + \frac{H_2(\gamma)}{H_1(\gamma)} \frac{d}{d\gamma} H_1(\gamma) \\ \Omega &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/2 \end{bmatrix} \\ c'(\gamma_0) &= \begin{bmatrix} 0 & -\frac{H_2(\gamma)}{H_1(\gamma)} & 1 \end{bmatrix}, \end{aligned}$$

and so the asymptotic variance is

$$\pi f(0) \frac{1 + \left\{ \frac{H_2(\gamma)}{H_1(\gamma)} \right\}^2}{\left\{ -\frac{d}{d\gamma} H_2(\gamma) + \frac{H_2(\gamma)}{H_1(\gamma)} \frac{d}{d\gamma} H_1(\gamma) \right\}^2}.$$

5. SIMULATIONS

Only a few results for the $a = 0$ case are reported. Figure 1 shows that the theoretical and simulated, least squares and Fourier estimates are all in close agreement. Of interest is the fact that the mean square errors initially decrease as γ increases, but then increase, the least squares estimates showing superiority at low and high values of γ. Figures 2 and 3 show that there is a threshold effect for fixed γ with decreasing SNR. Below threshold, the theoretical and simulated mean square errors agree, while the least squares estimates again eventually show superiority with decreasing SNR. There were 5000 replications for each combination of parameters, and the noise was simulated Gaussian and white.

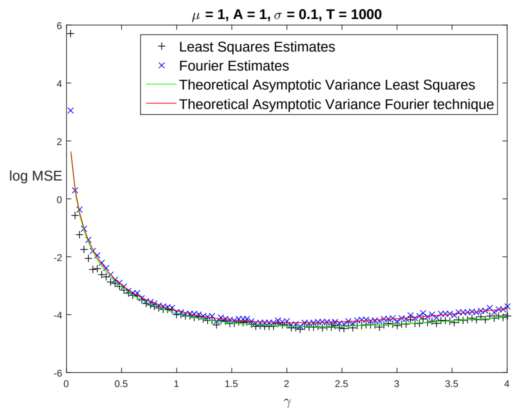

Fig 1. MSE for fixed σ as function of γ

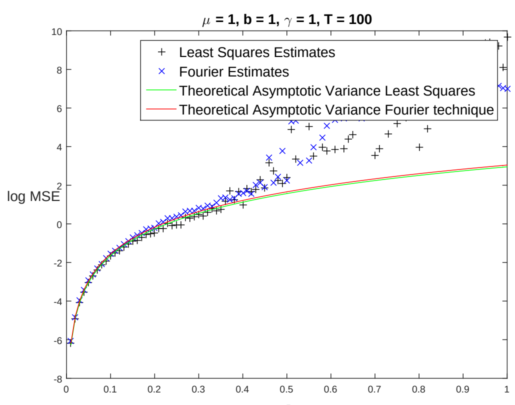

Fig 2. MSE for fixed γ as a function of σ

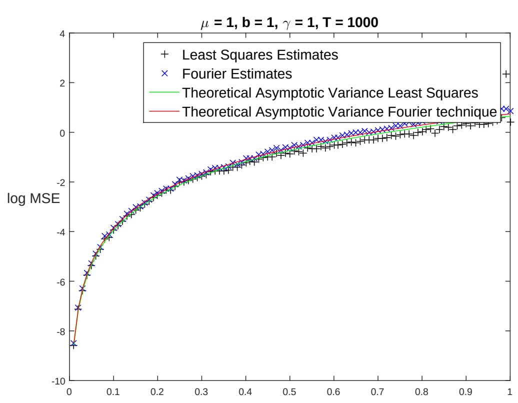

Fig 3. MSE for fixed γ as a function of σ

6. APPENDIX

With the log-likelihood l defined by (5), and $V_T(\Theta)$ by (6), the information matrix $\mathcal{I}_T$ may be shown to satisfy

$$T\mathcal{I}_T^{-1} \to \begin{bmatrix} 2\left(\sigma^2\right)^2 & 0 \\ 0 & \sigma^2\Omega^{-1} \end{bmatrix},$$

where Ω is the symmetric 5×5 matrix with entries given by

$$\Omega_{11} = 1, \Omega_{12} = \operatorname{Re} I_{0,1,1}, \Omega_{13} = -A \operatorname{Im} I_{0,1,1},$$
$$\Omega_{14} = -A \operatorname{Im} I_{1,1,1}, \Omega_{15} = -A \operatorname{Re} I_{1,1,1},$$
$$\Omega_{22} = \frac{1}{2}\left(I_{0,2,0} + \operatorname{Re} I_{0,2,2}\right), \Omega_{23} = -\frac{A}{2} \operatorname{Im} I_{0,2,1},$$
$$\Omega_{24} = -\frac{A}{2} \operatorname{Im} I_{1,2,1}, \Omega_{25} = -\frac{A}{2}\left(I_{1,2,0} + \operatorname{Re} I_{1,2,2}\right),$$
$$\Omega_{33} = \frac{A^2}{2}\left(I_{0,2,0} - \operatorname{Re} I_{0,2,2}\right), \Omega_{34} = \frac{A^2}{2}\left(I_{1,2,0} - \operatorname{Re} I_{1,2,2}\right),$$
$$\Omega_{35} = \frac{A^2}{2} \operatorname{Im} I_{1,2,2}, \Omega_{44} = \frac{A^2}{2}\left(I_{2,2,0} - \operatorname{Re} I_{2,2,2}\right),$$
$$\Omega_{45} = \frac{A^2}{2} \operatorname{Im} I_{2,2,2}, \Omega_{55} = \frac{A^2}{2}\left(I_{2,2,0} - \operatorname{Re} I_{2,2,2}\right)$$

where

$$I_{k,n,m} = \int_0^1 x^k e^{-n\gamma x + jm(ax+\phi)} dx.$$

To see this, note, for example, that as $T \to \infty$,

$$T^{-1}\frac{\partial^2 l}{\partial\mu\partial A} = -\frac{1}{\sigma^2}\frac{1}{T}\sum_{t=0}^{T-1} e^{-\gamma t/T}\cos\left(at/T + \phi\right)$$
$$= -\frac{1}{\sigma^2}\int_0^1 e^{-\gamma x}\cos\left(ax+\phi\right)dx + O\left(T^{-1}\right).$$

Ω may be computed exactly using the result that for $k = 0, 1, \ldots$ and complex z,

$$\int_0^1 x^k e^{zx} dx = \frac{d^k}{dz^k}\int_0^1 e^{zx} dx = \frac{d^k}{dz^k}\frac{e^z - 1}{z}. \qquad (8)$$

7. CONCLUSIONS

Techniques have been proposed for estimating the parameters in exponentially damped sinusoids when the damping has not been completed by the end of the time period, and the number of periods of the sinusoid is fixed as the sample size increases. This enables the asymptotic behavior of the estimation procedures to be evaluated without restrictive distributional assumptions being made. Generalisation has been suggested, and simulations reported for the case of a noisy pure real exponential.

Excluded from this paper, for reasons of space, are the asymptotic properties of the low-frequency Fourier coefficient method, and a computationally efficient implementation. Moreover, many of the ideas are applicable to the complex signal case, with some modification due to, for example, the fact that interference is only between the zero and positive frequency components. As well, the purely exponential case in not as straightforward as there are more parameters. Extension to the multicomponent case is also of interest, and will be the subject of further research.

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