TESTING FOR IMPROPRIETY OF MULTIVARIATE COMPLEX RANDOM PROCESSES

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ABSTRACT

We consider the problem of testing whether a complex-valued vector random sequence is proper. Past work on this problem is limited to a sequence of independent Gaussian random vectors whereas we allow an arbitrary stationary vector sequence that can be non-Gaussian. A binary hypothesis testing approach is formulated and a generalized likelihood ratio test (GLRT) is derived using the power spectral density estimator of an augmented sequence. An asymptotic analytical solution for calculating the test threshold is provided. The results are illustrated via simulations.

Index Terms— Improper complex random signals; generalized likelihood ratio test (GLRT); spectral analysis.

1. INTRODUCTION

A complex-valued random sequence is called improper if the cross-correlation function of the sequence with its complex conjugate is non-vanishing, else it is proper [1]. Quite often, algorithms for complex signal processing in communications and statistical signal processing have been derived assuming that the complex signals are proper [1, 2, 3]. However, this assumption of propriety is often not justified. For example, BPSK, offset QPSK and ASK modulation based signals are improper [1]. If the underlying signals are improper, much can be gained in performance if they are treated as improper [4]. If it is not known apriori whether a signal of interest is proper or improper, this information must be obtained from its noisy measurements. This problem has received considerable attention in the literature [1, 5, 6, 7, 8].

Relation to Prior Work: Past work on this problem of determination of propriety is limited to the case where the measurements consist of a sequence of independent Gaussian random vectors [1, 5, 6, 8], or independent possibly non-Gaussian random vectors [7]. In this paper we allow an arbitrary stationary (i.e., correlated) vector sequence that can be non-Gaussian.

Contributions: A binary hypothesis testing approach is formulated and a generalized likelihood ratio test (GLRT) is derived using the power spectral density estimator of an augmented sequence. An asymptotic analytical solution for calculating the test threshold is provided. The results are illustrated via simulations. The tests of [1, 5, 6, 8, 7] do not apply to the case of correlated sequences since their tests are not invariant under changes to the correlation structure of the null

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hypothesis signals which, in turn, precludes determination of the test threshold for a specified false-alarm rate either via simulations or analytically.

Notation: We use $\mathbf{S} \succeq 0$ and $\mathbf{S} \succ 0$ to denote that Hermitian \mathbf{S} is positive semi-definite and positive definite, respectively. For a square matrix \mathbf{A} , $|\mathbf{A}|$ and $\operatorname{etr}(\mathbf{A})$ denote the determinant and the exponential of the trace of \mathbf{A} , respectively, i.e., $\operatorname{etr}(\mathbf{A}) = \exp(\operatorname{tr}(\mathbf{A}))$, $\mathbf{B}_{k;i:l,j:m}$ denotes the submatrix of the matrix \mathbf{B}_k comprising its rows *i* through *l* and columns *j* through *m*, $\mathbf{B}_{k;ij}$ is its *ij*th element, and \mathbf{I} is the identity matrix. The superscripts *, *T* and *H* denote the complex conjugate, transpose and the Hermitian (conjugate transpose) operations, respectively.

2. SYSTEM MODEL

A stationary complex zero-mean process $\{\mathbf{x}(t)\}\$ of dimension p is said to be proper [1] if its matrix complementary correlation (covariance) function (called pseudo-correlation in [2]) $\tilde{\mathbf{R}}_{xx}(\tau)$ vanishes, i.e., if

$$\tilde{\mathbf{R}}_{xx}(\tau) = E\{\mathbf{x}(t+\tau)\mathbf{x}^{T}(t)\} = 0, \ \tau = 0, \pm 1, \cdots, \quad (1)$$

where $\mathbf{x}(t) = \mathbf{x}_r(t) + j\mathbf{x}_i(t)$, with $\mathbf{x}_r(t)$ and $\mathbf{x}_i(t)$ denoting its real and imaginary components, respectively.

Define $\mathbf{R}_{xx}(\tau) = \bar{E}\{\mathbf{x}(t + \tau)\mathbf{x}^{H}(t)\}\)$, the conventional matrix correlation function. Denote the power spectral density (PSD) of $\{\mathbf{x}(t)\}\)$ by $\mathbf{S}_{x}(f)$, where $\mathbf{S}_{x}(f) = \sum_{\tau=-\infty}^{\infty} \mathbf{R}_{xx}(\tau)e^{-j2\pi f\tau}\)$, the Fourier transform of $\mathbf{R}_{xx}(\tau)$. Denote the complementary PSD (C-PSD) of $\{\mathbf{x}(t)\}\)$ by $\mathbf{\tilde{S}}_{x}(f)$, with $\mathbf{\tilde{S}}_{x}(f) = \sum_{\tau=-\infty}^{\infty} \mathbf{\tilde{R}}_{xx}(\tau)e^{-j2\pi f\tau}$. Clearly, for a proper process, the C-PSD vanishes.

We observe $\mathbf{x}(t)$ for $t = 0, 1, \dots, N - 1$ (N samples). We employ multivariate spectral analysis to test if its C-PSD vanishes. Define the augmented complex process $\{\mathbf{y}(t)\}$ and the real-valued process $\{\mathbf{z}(t)\}$ as

$$\mathbf{y}(t) = \begin{bmatrix} \mathbf{x}(t) \\ \mathbf{x}^*(t) \end{bmatrix}, \quad \mathbf{z}(t) = \begin{bmatrix} \mathbf{x}_r(t) \\ \mathbf{x}_i(t) \end{bmatrix}.$$
(2)

We assume that $\{z(t)\}\$ satisfies Assumption 2.6.1 of [9] so that some asymptotic results from [9] regarding PSD estimators can be invoked; this assumption implies that the time series need not be Gaussian but its moments of all orders should be finite.

Consider the finite Fourier transform (FFT) $\mathbf{d}_z(f_n)$ of $\mathbf{z}(t), t = 1, 2, \cdots, N-1$, given by

$$\mathbf{d}_z(f_n) := \sum_{t=0}^{N-1} \mathbf{z}(t) e^{-j2\pi f_n t}$$
(3)

where $f_n = n/N$, $n = 0, 1, \dots, N-1$. Then the estimator of the PSD of $\mathbf{z}(t)$ at frequency f_n , based on the Daniell method, is given by

$$\hat{\mathbf{S}}_{z}(f_{n}) = \frac{1}{K} \sum_{l=-m_{t}}^{m_{t}} \left(N^{-1} \mathbf{d}_{z}(f_{n+l}) \mathbf{d}_{z}^{H}(f_{n+l}) \right)$$
(4)

where $K = 2m_t + 1$ is the smoothing window size. By [9, Theorem 7.3.3], $\hat{\mathbf{S}}_z(f_n)$ is asymptotically ("large" N) distributed as $W_C(2p, K, K^{-1}\mathbf{S}_z(f_n))$ so long as the smoothing window in (4) does not include the frequency at n = 0 or n = N/2, where $W_C(2p, K, K^{-1}\mathbf{S}_z(f_n))$ denotes the complex Wishart distribution of dimension 2p, degrees of freedom K, and mean value $\mathbf{S}_z(f_n)$. If a random matrix $\mathbf{X} \sim$ $W_C(p, K, \mathbf{S}(f))$, then by [9, Sec. 4.2], $E\{\mathbf{X}\} = K\mathbf{S}(f)$, $\operatorname{cov} \{\mathbf{X}_{jk}, \mathbf{X}_{lm}\} = K\mathbf{S}_{jl}(f)\mathbf{S}_{km}^*(f)$, and for $K \geq p$, the probability density function (pdf) of \mathbf{X} is given by

$$f_{\mathbf{X}}(\mathbf{X}) = \frac{1}{\Gamma_p(K)} \frac{1}{|\mathbf{S}(f)|^K} |\mathbf{X}|^{K-p} \operatorname{etr}\{-\mathbf{S}^{-1}(f)\mathbf{X}\} \quad (5)$$

where the pdf (5) is defined for $\mathbf{S}(f) \succ 0$ and $\mathbf{X} \succeq 0$, and is otherwise zero, and $\Gamma_p(K) := \pi^{p(p-1)/2} \prod_{j=1}^p \Gamma(K-j+1)$ where $\Gamma(n)$ denotes the (complete) Gamma function $\Gamma(z) := \int_0^\infty t^{z-1} e^{-t} dt$.

We will confine our attention to the frequency points over which the spectral estimators are (approximately) mutually independent, which for the Daniell method are given by

$$\tilde{f}_k = \frac{(k-1)K + m_t + 1}{N}, \ 1 \le k \le M = \left\lfloor \frac{\frac{N}{2} - m_t - 1}{K} \right\rfloor.$$
(6)

Let $\mathcal{M} := \{ \hat{f}_k : 1 \le k \le M \}$ denote the set of M frequency bins as in (6) of interest. From (2) we have

$$\mathbf{y}(t) = \mathcal{T}\mathbf{z}(t), \quad \mathcal{T} = \begin{bmatrix} \mathbf{I} & j\mathbf{I} \\ \mathbf{I} & -j\mathbf{I} \end{bmatrix}$$
(7)

where $(2p) \times (2p) \mathcal{T}$ is full-rank. Hence, $\mathbf{d}_y(f_n) = \mathcal{T}\mathbf{d}_z(f_n)$ and $\hat{\mathbf{S}}_y(f_n) = \mathcal{T}\hat{\mathbf{S}}_z(f_n)\mathcal{T}^H$ where $\mathbf{d}_y(f_n) = \sum_{t=0}^{N-1} \mathbf{y}(t)e^{-j2\pi f_n t}$ and

$$\hat{\mathbf{S}}_{y}(f_{n}) = \frac{1}{K} \sum_{l=-m_{t}}^{m_{t}} \left(N^{-1} \mathbf{d}_{y}(f_{n+l}) \mathbf{d}_{y}^{H}(f_{n+l}) \right).$$
(8)

By the complex-valued counterpart of [10, Thm. 3.2.5], for any $m \times p$ matrix **A** of rank m, if $\mathbf{X} \sim W_C(p, K, \mathbf{S}(f))$, then $\mathbf{AXA}^H \sim W_C(m, K, \mathbf{AS}(f)\mathbf{A}^H)$. Therefore $\hat{\mathbf{S}}_y(f_n)$ is (asymptotically) distributed as $W_C(2p, K, K^{-1}\mathbf{S}_y(f_n))$. Furthermore, $\hat{\mathbf{S}}_{y}(\hat{f}_{k})$'s for \hat{f}_{k} as in (6) are asymptotically mutually independent complex Wishart random matrices.

If $\mathbf{x}(t)$ is a proper random process, then

$$\mathbf{S}_{y}(f) = \begin{bmatrix} \mathbf{S}_{x}(f) & \mathbf{0} \\ \mathbf{0} & \mathbf{S}_{x}^{*}(-f) \end{bmatrix},$$
(9)

else $\mathbf{S}_y(f) \succeq 0$ with no specific structure. Testing for impropriety of $\mathbf{x}(t)$ is then cast as a binary hypothesis testing problem:

$$\begin{aligned} \mathcal{H}_0 : \quad & \dot{\mathbf{S}}_x(\tilde{f}_k) = \mathbf{0} \ \forall \tilde{f}_k \in \mathcal{M}, \ \mathbf{x}(t) \text{ is proper} \\ \mathcal{H}_1 : \quad & \dot{\mathbf{S}}_x(\tilde{f}_k) \not\equiv \mathbf{0}, \ \mathbf{x}(t) \text{ is improper} \end{aligned}$$
 (10)

where \mathcal{H}_0 is the null hypothesis and \mathcal{H}_1 is the alternative.

We assume that $\mathbf{S}_y(f) \succ 0$ for any f. Otherwise, one can add artificial proper white Gaussian noise to $\mathbf{x}(t)$ to achieve $\mathbf{S}_y(f) \succ 0$.

3. PSD-BASED GLRT FOR TESTING IMPROPRIETY

In this section we derive the GLRT. We will denote the spectral estimator at the k-th frequency bin \tilde{f}_k (see (6), acquired from $\{\mathbf{y}(t)\}_{t=0}^{N-1}$, via (8), as \mathbf{Y}_k . We have ($\stackrel{a}{\sim}$ denotes asymptotic distribution)

$$\mathbf{Y}_k \stackrel{a}{\sim} W_C\left(2p, K, K^{-1}\mathbf{S}_y(\tilde{f}_k)\right) \tag{11}$$

and \mathbf{Y}_k s are mutually independent for $k \in [1, M]$. The joint probability density function (pdf) of \mathbf{Y}_k for $\tilde{f}_k \in \mathcal{M}$ under \mathcal{H}_0 is maximized w.r.t. the Hermitian matrix $\mathbf{S}_x(\tilde{f}_k)$ for $\hat{\mathbf{S}}_x(\tilde{f}_k) = \mathbf{Y}_{k;1:p,1:p}$, and w.r.t. the Hermitian matrix $\mathbf{S}_x^*(-\tilde{f}_k)$ for $\hat{\mathbf{S}}_x^*(-\tilde{f}_k) = \mathbf{Y}_{k;1+p:2p,1+p:2p}$. Under \mathcal{H}_1 , the joint pdf of \mathbf{Y}_k for $k \in [1, M]$ is maximized w.r.t. the Hermitian matrix $\mathbf{S}_y(\tilde{f}_k)$ for $\hat{\mathbf{S}}_y(\tilde{f}_k) = \mathbf{Y}_k$. Then one gets the GLRT ($\mathcal{Y} = (\mathbf{Y}_k, k \in \mathcal{M})$)

$$\mathcal{L} := \frac{f_{\mathcal{Y}}(\mathbf{Y}_k, k \in \mathcal{M} | \mathcal{H}_1, \hat{\mathbf{S}}_y(\tilde{f}_k))}{f_{\mathcal{Y}}(\mathbf{Y}_k, k \in \mathcal{M} | \mathcal{H}_0, \hat{\mathbf{S}}_x(\tilde{f}_k), \hat{\mathbf{S}}_x^*(-\tilde{f}_k))} \stackrel{\mathcal{H}_1}{\underset{\mathcal{H}_0}{\overset{\geq}{\simeq}}} \tau_1 \quad (12)$$

where the threshold τ_1 is picked to achieve a pre-specified probability of false alarm $P_{fa} = P\{\mathcal{L} \ge \tau_1 | \mathcal{H}_0\}$. This requires pdf of \mathcal{L} under \mathcal{H}_0 which is discussed in Sec. 4. Simplifying, one obtains

$$\mathcal{L} = \prod_{k=1}^{M} \mathcal{L}_{k}, \ \mathcal{L}_{k} := \frac{|\mathbf{Y}_{k;1:p,1:p}|^{K} |\mathbf{Y}_{k;1+p:2p,1+p:2p}|^{K}}{|\mathbf{Y}_{k}|^{K}}$$
(13)

Invariance of GLRT: Note that \mathcal{L}_k is invariant to transformation $\mathbf{Y}_k \to \mathbf{A}_k \mathbf{Y}_k \mathbf{A}_k^H$ for any nonsingular $\mathbf{A}_k \in \mathbb{C}^{2p \times 2p}$ such that $\mathbf{A}_k = \text{block} - \text{diag}\{\mathbf{A}_k^{(1)}, \mathbf{A}_k^{(2)}\}, \mathbf{A}_k^{(i)} \in \mathbb{C}^{p \times p}$ for i = 1, 2. This observation allows us to transform any \mathbf{Y}_k to $\tilde{\mathbf{Y}}_k \sim W_C(2p, K, \mathbf{I})$ under \mathcal{H}_0 by choosing $\mathbf{A}_k^{(1)} = \sqrt{K}\mathbf{S}_x^{-1/2}(\tilde{f}_k)$ and $\mathbf{A}_k^{(2)} = \sqrt{K}(\mathbf{S}_x^{-1/2}(-\tilde{f}_k))^*$. Then \mathcal{L} is invariant and transformed $\tilde{\mathbf{Y}}_k$ s now correspond to proper i.i.d. (white) sequence $\mathbf{x}(t)$ which can be used to compute the test threshold via Monte Carlo simulations. This threshold is valid for any other PSD. However, in Sec. 4, we offer an analytical approach.

4. THRESHOLD SELECTION

We now turn to determination of an asymptotic expansion of the distribution of \mathcal{L} under \mathcal{H}_0 following [10, 11, 12], which allows us to calculate the test threshold analytically instead of via simulations. The main result is stated in Theorem 1.

First we need the following result: Lemma 1 : Under $\mathcal{H}_0, E\{\frac{1}{\mathcal{L}^h} | \mathcal{H}_0\}$

$$=\frac{\prod_{\ell=0}^{p-1}\Gamma^{2M}(K+\ell)}{\Gamma_{2p}^{M}(K)}\frac{\left[\prod_{k=1}^{2p}\Gamma(K(1+h)-k+1)\right]^{M}}{\left[\prod_{j=1}^{p}\Gamma(K(1+h)-j+1)\right]^{2M}}$$
(14)

Proof: Using the transformation specified in Sec. 3 to obtain $\tilde{\mathbf{Y}}_k \sim W_C(2p, K, \mathbf{I})$ under \mathcal{H}_0 and denoting $\tilde{\mathbf{Y}}_k^{(1)} = \tilde{\mathbf{Y}}_{k;1:p,1:p}, \tilde{\mathbf{Y}}_k^{(2)} = \tilde{\mathbf{Y}}_{k;1+p:2p,1+p:2p}$, we have

$$E\{1/\mathcal{L}_{k}^{h} | \mathcal{H}_{0}\} = \int \frac{|\tilde{\mathbf{Y}}_{k}|^{Kh+K-2p}}{|\tilde{\mathbf{Y}}_{k}^{(1)}|^{Kh}|\tilde{\mathbf{Y}}_{k}^{(2)}|^{Kh}} \\ \times \frac{1}{\Gamma_{2p}(K)} \operatorname{etr}\{-\tilde{\mathbf{Y}}_{k}\} d\tilde{\mathbf{Y}}_{k} \\ = \frac{\Gamma_{2p}(K+Kh)}{\Gamma_{2p}(K)} E\{|\tilde{\mathbf{Y}}_{k}^{\prime(1)}|^{-Kh}\} E\{|\tilde{\mathbf{Y}}_{k}^{\prime(2)}|^{-Kh}\}, \quad (15)$$

where $\tilde{\mathbf{Y}}'_k \sim W_C(2p, K(1+h), \mathbf{I}), \ \tilde{\mathbf{Y}}'^{(1)}_k = \tilde{\mathbf{Y}}'_{k;1:p,1:p} \sim W_C(p, K(1+h), \mathbf{I}), \ \tilde{\mathbf{Y}}'^{(2)}_k = \tilde{\mathbf{Y}}'_{k;1+p:2p,1+p:2p} \sim W_C(p, K(1+h), \mathbf{I}), \ \text{and} \ \tilde{\mathbf{Y}}'^{(1)}_k \ \text{is independent of} \ \tilde{\mathbf{Y}}'^{(2)}_k.$ Using [13, Theorem 3.8, p. 51], we have

$$E\{|\tilde{\mathbf{Y}}_{k}^{\prime(1)}|^{-Kh}\} = E\left\{\left(\prod_{l=1}^{p}(1/2)V_{l}\right)^{-Kh}\right\}$$
(16)

where $V_l \sim \chi^2_{2(K(1+h)-l+1}$ and V_l 's are mutually independent. Since (see [10, p. 101])

$$E\{W^r\} = \frac{2^r \Gamma((n/2) + r)}{\Gamma((n/2))} \quad \text{for} \quad W \sim \chi_n^2, \qquad (17)$$

we obtain

$$E\{|\tilde{\mathbf{Y}}_{k}^{\prime(1)}|^{-Kh}\} = \prod_{l=1}^{p} \frac{\Gamma(K-p+l)}{\Gamma(K(1+h)-p+l)}$$
(18)

 $= E\{|\tilde{\mathbf{Y}}_{k}^{\prime(2)}|^{-Kh}\}$. Now using (13), (15), (18) and $\Gamma_{p}(K) := \pi^{p(p-1)/2} \prod_{j=1}^{p} \Gamma(K-j+1)$, we get the desired result. \Box

In order to exploit Lemma 2 (stated next), we need to establish that $0 \le \mathcal{L}^{-1} \le 1$. Since \mathbf{Y}_k and its principal submatrices are $\succ 0$, clearly $\mathcal{L}^{-1} > 0$, and by Fischer's inequality [14, p. 478], we have $|\mathbf{Y}_k| \le |\mathbf{Y}_{k;1:p,1:p}||\mathbf{Y}_{k;1+p:2p,1+p:2p}|$ which implies $\mathcal{L}^{-1} \le 1$. The following result follows from [10, Sec. 8.2.4], [11, Sec. 8.5.1]:

Lemma 2 : Consider a random variable W ($0 \le W \le 1$)

with the *h*th moment $(h = 0, 1, 2, \cdots)$

$$E\{W^{h}\} = C\left(\frac{\prod_{j=1}^{b} y_{j}^{y_{j}}}{\prod_{k=1}^{a} x_{k}^{x_{k}}}\right)^{h} \frac{\prod_{k=1}^{a} \Gamma(x_{k}(1+h) + \xi_{k})}{\prod_{j=1}^{b} \Gamma(y_{j}(1+h) + \eta_{j})},$$
(19)

where a and b are integers, C is a constant such that $E\{W^{\hat{0}}\} = 1$ and $\sum_{k=1}^{a} x_k = \sum_{j=1}^{b} y_j$. Let $B_r(h)$ denote the Bernoulli polynomial of degree r and order unity. Define $\nu = -2\left[\sum_{k=1}^{a} \xi_k - \sum_{j=1}^{b} \eta_j - \frac{1}{2}(a-b)\right], \rho = 1 - \frac{1}{\nu}\left[\sum_{k=1}^{a} x_k^{-1}(\xi_k^2 - \xi_k + \frac{1}{6}) - \sum_{j=1}^{b} y_j^{-1}(\eta_j^2 - \eta_j + \frac{1}{6})\right], \beta_k = (1-\rho)x_k, \quad \epsilon_j = (1-\rho)y_j \text{ and}$ $\omega_r = \frac{(-1)^{r+1}}{r(r+1)}\left\{\sum_{k=1}^{a} \frac{B_{r+1}(\beta_k + \xi_k)}{(\rho x_k)^r} - \sum_{j=1}^{b} \frac{B_{r+1}(\epsilon_j + \eta_j)}{(\rho y_j)^r}\right\}.$ Then with χ_n^2 denoting a random variable with central chi-square distribution with n degrees of freedom (as well as the distribution itself),

$$P\{-2\rho \ln(W) \leq z\} = P\{\chi_{\nu}^{2} \leq z\} + \omega_{2} \left[P\{\chi_{\nu+4}^{2} \leq z\}\right]$$
$$-P\{\chi_{\nu}^{2} \leq z\} + \omega_{3} \left[P\{\chi_{\nu+6}^{2} \leq z\} - P\{\chi_{\nu}^{2} \leq z\}\right]$$
$$+\{\omega_{4} \left[P\{\chi_{\nu+8}^{2} \leq z\} - P\{\chi_{\nu}^{2} \leq z\}\right] + \frac{1}{2}\omega_{2}^{2} \left[P\{\chi_{\nu+8}^{2} \leq z\}\right]$$
$$-2P\{\chi_{\nu+4}^{2} \leq z\} + P\{\chi_{\nu}^{2} \leq z\}\right] \}$$
$$+\sum_{k=1}^{a} \mathcal{O}(x_{k}^{-5}) + \sum_{j=1}^{b} \mathcal{O}(y_{j}^{-5}).$$
(20)

Comparing (19) with (14), we find the correspondence

$$a = 2Mp, \ b = 2Mp, \ x_k = K,$$

$$\xi_k = 1 - k \mod(2p) \text{ for } k = 1, 2, \cdots, a,$$

$$= K \text{ and } \eta_j = 1 - j \mod(p) \text{ for } j = 1, 2, \cdots, b. \quad (21)$$

paring Lemmas 1 and 2, we further have

Comparing Lemmas 1 and 2, we further have

$$\beta_k = (1 - \rho) K \,\forall k, \quad \epsilon_j = (1 - \rho) K \,\forall j. \tag{22}$$

Furthermore, one has $E\{1/\mathcal{L}^0 | \mathcal{H}_0\} = 1$. Thus, Lemma 2 is applicable with $W = 1/\mathcal{L}$ and parameters specified in (21). Using these values in Lemma 2 and simplifying, one gets

$$\nu = 2p^2 M, \quad \rho = 1 - \frac{p}{K},$$
 (23)

$$\sum_{k=1}^{a} \frac{B_{r+1}(\beta_k + \xi_k)}{(\rho x_k)^r} = M \sum_{l=1}^{2p} \frac{B_{r+1}((1-\rho)K + 1-l)}{(\rho K)^r},$$
(24)
$$\sum_{j=1}^{b} \frac{B_{r+1}(\epsilon_j + \eta_j)}{(\rho y_j)^r} = 2M \sum_{l=1}^{p} \frac{B_{r+1}((1-\rho)K + 1-l)}{(2\rho K)^r}.$$
(25)

Therefore, we have

 y_j

$$\omega_r = \frac{(-1)^{r+1}M}{r(r+1)(\rho K)^r} \left\{ \left(\sum_{l=1}^{2p} B_{r+1}((1-\rho)K+1-l) \right) \right\}$$

$$-\left(\sum_{l=1}^{p} 2B_{r+1}((1-\rho)K+1-l)\right)\bigg\}.$$
 (26)

It then follows from Lemma 2 that

$$P\{2\rho \ln(\mathcal{L}) \leq z \mid \mathcal{H}_0\} = P\{\chi_{\nu}^2 \leq z\} + \omega_2 \left[P\{\chi_{\nu+4}^2 \leq z\} - P\{\chi_{\nu}^2 \leq z\}\right] + \omega_3 \left[P\{\chi_{\nu+6}^2 \leq z\} - P\{\chi_{\nu}^2 \leq z\}\right] + \left\{\omega_4 \left[P\{\chi_{\nu+8}^2 \leq z\} - P\{\chi_{\nu}^2 \leq z\}\right] + \frac{1}{2}\omega_2^2 \left[P\{\chi_{\nu+8}^2 \leq z\} - 2P\{\chi_{\nu+4}^2 \leq z\} + P\{\chi_{\nu}^2 \leq z\}\right] \right\} + \mathcal{O}(K^{-5})$$
(27)

where ω_r 's are given by (24)-(26), and

$$\ln(\mathcal{L}) = K \sum_{k=1}^{M} \left\{ \ln(|\mathbf{Y}_{k;1:p,1:p}|) + \ln(|\mathbf{Y}_{k;1+p:2p,1+p:2p}|) - \ln(|\mathbf{Y}_{k}|) \right\}.$$
(28)

We summarize the above in the following result.

Theorem 1. The GLRT for (10) is given by $2\rho \ln(\mathcal{L}) \overset{\mathcal{H}_1}{\underset{\mathcal{H}_0}{\geq}} \tau$ where ρ and $\ln(\mathcal{L})$ are given by (23) and (28), respectively. The threshold τ is picked to achieve a pre-specified $P_{fa} = 1 - P\{2\rho \ln(\mathcal{L}) \leq \tau \mid \mathcal{H}_0\}$ where $P\{2\rho \ln(\mathcal{L}) \leq \tau \mid \mathcal{H}_0\}$ is given by (27) and the various needed parameters are specified in (23)-(26) •

Theorem 1 allows us to calculate the test threshold analytically.



Fig. 1: Actual P_{fa} vs design P_{fa} , N = 256, K = 15, M = 8

5. NUMERICAL EXPERIMENTS

First we investigate the efficacy of Theorem 1 in computing the GLRT threshold for a given P_{fa} . We consider p antennas (p=1,2,3 or 4) with spatially uncorrelated, colored proper complex Gaussian noise { $\mathbf{n}(t)$ } generated by filtering p independent sequences through p separate linear filters each with impulse response {0.3, 1.0, 0.3}. To estimate the PSD of augmented $\mathbf{y}(t)$ for N = 256, we choose $m_t = 7$ leading to K = 15 and M = 8. In Fig. 1 we compare the actual P_{fa} and design P_{fa} based on 10,000 runs. It is seen that Theorem 1 is effective in accurately calculating the threshold value.



Next we show the receiver operating characteristic (ROC) curves. The $p \times 1$ random sequence $\mathbf{x}(t)$ was generated as $\mathbf{x}(t) = \mathbf{x}_s(t) + \mathbf{n}(t)$; the noise sequence $\mathbf{n}(t)$ is as in the previous example and $\mathbf{x}_s(t) = \sum_{l=0}^{4} \mathbf{h}(l)I(t-l)$ where I(t) is a scalar BPSK sequence and $p \times 1$ vector channel $\mathbf{h}(l)$ is Rayleigh fading with 5 taps, equal power delay profile, mutually independent components. Thus signal is improper and noise is proper. The probability of detection P_d versus false-alarm rate P_{fa} results for three different SNR values and p = 3, based on 10,000 runs, is shown in Fig. 2; SNR is defined as ratio of the sum of signal powers at the p antennas to the sum of noise powers. In all cases we have N=256, K=15 and M=8. It is seen that performance improves with increasing SNR, and our approach is able to detect improper random signals quite well at low SNRs. For the same set-up, Fig. 3 shows P_d vs SNR for p = 1,2,3 or 4, $P_{fa} = 0.005$.



Fig. 3: P_d vs. SNR, $P_{fa} = 0.005$, N = 256, K = 15, M = 8

6. CONCLUSIONS

In this paper we investigated a method based on analysis of the multivariate PSD of augmented received noisy complex signal to determine if the signal is proper or improper. Our proposed approach is based on GLRT. An analytical method for calculation of the test threshold was provided and illustrated via simulations. Past work on this problem is limited to a sequence of independent random vectors whereas we allow correlated signal sequences with unknown correlation.

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