MEASURE-TRANSFORMED QUASI LIKELIHOOD RATIO TEST

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ABSTRACT

In this paper, a generalization of the Gaussian quasi likelihood ratio test (GQLRT) for simple hypotheses is developed. The proposed generalization, called measure-transformed GQLRT (MT-GQLRT), selects a Gaussian probability model that best empirically fits a transformed probability measure of the data. By judicious choice of the transform we show that, unlike the GQLRT, the proposed test can gain sensitivity to higher-order statistical moments and resilience to outliers leading to significant mitigation of the model mismatch effect on the decision performance. Under some mild regularity conditions we show that the proposed test statistic is asymptotically normal. A data driven procedure for optimal selection of the measure transformation parameters is developed that maximizes an empirical estimate of the asymptotic power given a fixed empirical asymptotic size. The MT-GQLRT is applied to signal classification in a simulation example that illustrates its sensitivity to higher-order statistical moments and resilience to outliers.

Index Terms— Higher-order statistics, hypothesis testing, probability measure transform, robust statistics, signal classification.

1. INTRODUCTION

Classical simple binary hypothesis testing deals with deciding between two hypotheses based on a sequence of multivariate samples from an underlying probability distribution that is equal to one of two probability measures [1]. When the probability distributions under each hypothesis are correctly specified the likelihood ratio test (LRT), which is the most powerful test for a given size [2], can be implemented. In many practical scenarios the probability distributions are unknown, and therefore, one must resort to suboptimal tests.

A popular suboptimal test of this kind is the Gaussian quasi LRT (GQLRT) [3]-[9] which assumes that the samples obey Gaussian distributions under each hypothesis. The GQLRT operates by selecting the Gaussian probability model that best fits the data. When the observations are i.i.d. this selection is carried out by comparing the empirical Kullback-Leibler divergences [10] between the underlying probability distribution and the assumed normal probability measures. The GQLRT has gained popularity due to its implementation simplicity, ease of performance analysis, and its geometrical interpretations. Despite the model mismatch, introduced by the normality assumption, the GQLRT has the appealing property of consistency when the mean vectors and covariance matrices are correctly specified and identifiable over the considered hypotheses [6]. However, in some circumstances, such as for certain types of non-Gaussian data, deviation from normality can inflict poor decision performance. This can occur when the first and second-order statistical moments are

weakly identifiable over the considered hypotheses, or in the case of heavy-tailed data when the non-robust sample mean and covariance provide poor estimates in the presence of outliers.

In this paper, a generalization of the GQLRT is proposed that operates by selecting a Gaussian probability model that has the best empirical fit to a transformed probability distribution of the data. Under the proposed generalization new tests can be obtained that can gain sensitivity to higher-order statistical moments, resilience to outliers, and yet have the computational and implementation advantages of the GQLRT. This generalization, called measure-transformed GQLRT (MT-GQLRT), is inspired by the measure transformation approach that was recently applied to canonical correlation analysis [11], [12], multiple signal classification (MUSIC) [13], [14] and parameter estimation [15].

The proposed transform is structured by a non-negative function, called the MT-function, and maps the probability distribution into a set of new probability measures on the observation space. By modifying the MT-function, classes of measure transformations can be obtained that have different useful properties. Under the proposed transform we define the measure-transformed (MT) mean vector and covariance matrix, derive their strongly consistent estimates, and study their sensitivity to higher-order statistical moments and resilience to outliers.

Similarly to the GQLRT, the proposed MT-GQLRT compares the empirical Kullback-Leibler divergences between the transformed probability distribution of the data and two normal probability measures that are characterized by the MT-mean vector and MT-covariance matrix under each hypothesis. Under some mild regularity conditions we show that the proposed test statistic is asymptotically normal. Furthermore, given two training sequences from the probability distribution under each hypothesis, a data-driven procedure for optimal selection of the MT-function within some parametric class of functions is developed that maximizes an empirical estimate of the asymptotic power given a fixed empirical asymptotic size.

We illustrate the MT-GQLRT for the problem of signal classification in the presence of heavy-tailed spherically contoured noise [16] that produces outliers. By specifying the MT-function within the family of zero-centered Gaussian functions parameterized by a scale parameter, we show that the MT-GQLRT outperforms the non-robust GQLRT and attains classification performance that are significantly closer to those obtained by the omniscient LRT that, unlike the MT-GQLRT, requires complete knowledge of the likelihood function under each hypothesis.

The paper is organized as follows. In Section 2, the principles of our proposed probability measure transform are reviewed. In Section 3, we use this transformation to construct the MT-GQLRT. The proposed test is applied to a signal classification problem in Section 4. In Section 5, the main points of this contribution are summarized. Proofs for the theorem and propositions stated throughout the paper will be provided in the full length journal version.

2. PROBABILITY MEASURE TRANSFORM

In this section, we develop a transform on the probability measure of a random vector. Under the proposed transform, we define the measure-transformed mean vector and covariance matrix, derive their strongly consistent estimates, and establish their sensitivity to higher-order statistical moments and resilience to outliers. These quantities will be used in the following section to construct the proposed measure-transformed GQLRT.

2.1. Probability measure transformation

We define the measure space $(\mathcal{X}, \mathcal{S}_\mathcal{X}, P_{\mathbf{X};\boldsymbol{\theta}})$, where $\mathcal{X} \subseteq \mathbb{C}^p$ is the observation space of a random vector $\mathbf{X}$, $\mathcal{S}_\mathcal{X}$ is a σ-algebra over $\mathcal{X}$ and $P_{\mathbf{X};\boldsymbol{\theta}}$ is an unknown probability measure on $\mathcal{S}_\mathcal{X}$ parameterized by a vector parameter $\boldsymbol{\theta}$ that belongs to a pair set $\boldsymbol{\Theta} \triangleq \{\boldsymbol{\theta}_0, \boldsymbol{\theta}_1\}$.

Definition 1. *Given a non-negative function $u : \mathbb{C}^p \rightarrow \mathbb{R}_+$ satisfying*

$$0 < \mathrm{E}\left[u\left(\mathbf{X}\right); P_{\mathbf{X};\boldsymbol{\theta}}\right] < \infty, \tag{1}$$

where $\mathrm{E}\left[u\left(\mathbf{X}\right); P_{\mathbf{X};\boldsymbol{\theta}}\right] \triangleq \int_\mathcal{X} u\left(\mathbf{x}\right) dP_{\mathbf{X};\boldsymbol{\theta}}\left(\mathbf{x}\right)$ and $\mathbf{x} \in \mathcal{X}$, a transform on $P_{\mathbf{X};\boldsymbol{\theta}}$ is defined via the relation:

$$Q_{\mathbf{X};\boldsymbol{\theta}}^{(u)}\left(A\right) \triangleq \mathrm{T}_u\left[P_{\mathbf{X};\boldsymbol{\theta}}\right]\left(A\right) = \int_A \varphi_u\left(\mathbf{x};\boldsymbol{\theta}\right) dP_{\mathbf{X};\boldsymbol{\theta}}\left(\mathbf{x}\right), \tag{2}$$

where $A \in \mathcal{S}_\mathcal{X}$ and $\varphi_u\left(\mathbf{x};\boldsymbol{\theta}\right) \triangleq u\left(\mathbf{x}\right)/\mathrm{E}\left[u\left(\mathbf{X}\right); P_{\mathbf{X};\boldsymbol{\theta}}\right]$. The function $u\left(\cdot\right)$ is called the MT-function.

Proposition 1 (Properties of the transform). *Let $Q_{\mathbf{X};\boldsymbol{\theta}}^{(u)}$ be defined by relation (2). Then 1) $Q_{\mathbf{X};\boldsymbol{\theta}}^{(u)}$ is a probability measure on $\mathcal{S}_\mathcal{X}$. 2) $Q_{\mathbf{X};\boldsymbol{\theta}}^{(u)}$ is absolutely continuous w.r.t. $P_{\mathbf{X};\boldsymbol{\theta}}$, with Radon-Nikodym derivative [17]:*

$$dQ_{\mathbf{X};\boldsymbol{\theta}}^{(u)}\left(\mathbf{x}\right)/dP_{\mathbf{X};\boldsymbol{\theta}}\left(\mathbf{x}\right) = \varphi_u\left(\mathbf{x};\boldsymbol{\theta}\right). \tag{3}$$

The probability measure $Q_{\mathbf{X};\boldsymbol{\theta}}^{(u)}$ is said to be generated by the MT-function $u\left(\cdot\right)$. By modifying $u\left(\cdot\right)$, such that the condition (1) is satisfied, virtually any probability measure on $\mathcal{S}_\mathcal{X}$ can be obtained.

2.2. The MT-mean and MT-covariance

According to (3) the mean vector and covariance matrix of $\mathbf{X}$ under $Q_{\mathbf{X};\boldsymbol{\theta}}^{(u)}$ are given by:

$$\boldsymbol{\mu}_{\mathbf{X};\boldsymbol{\theta}}^{(u)} \triangleq \mathrm{E}\left[\mathbf{X}\varphi_u\left(\mathbf{X};\boldsymbol{\theta}\right); P_{\mathbf{X};\boldsymbol{\theta}}\right] \tag{4}$$

and

$$\boldsymbol{\Sigma}_{\mathbf{X};\boldsymbol{\theta}}^{(u)} \triangleq \mathrm{E}\left[\mathbf{X}\mathbf{X}^H \varphi_u\left(\mathbf{X};\boldsymbol{\theta}\right); P_{\mathbf{X};\boldsymbol{\theta}}\right] - \boldsymbol{\mu}_{\mathbf{X};\boldsymbol{\theta}}^{(u)}\boldsymbol{\mu}_{\mathbf{X};\boldsymbol{\theta}}^{(u)H}, \tag{5}$$

respectively. Equations (4) and (5) imply that $\boldsymbol{\mu}_{\mathbf{X};\boldsymbol{\theta}}^{(u)}$ and $\boldsymbol{\Sigma}_{\mathbf{X};\boldsymbol{\theta}}^{(u)}$ are weighted mean and covariance of $\mathbf{X}$ under $P_{\mathbf{X};\boldsymbol{\theta}}$, with the weighting function $\varphi_u\left(\cdot;\cdot\right)$ defined below (2). By modifying the MT-function $u\left(\cdot\right)$, such that the condition (1) is satisfied, the MT-mean and MT-covariance under $Q_{\mathbf{X};\boldsymbol{\theta}}^{(u)}$ are modified. In particular, by choosing $u\left(\cdot\right)$ to be any non-zero constant valued function we have $Q_{\mathbf{X};\boldsymbol{\theta}}^{(u)} = P_{\mathbf{X};\boldsymbol{\theta}}$, for which the standard mean vector $\boldsymbol{\mu}_{\mathbf{X};\boldsymbol{\theta}}$ and covariance matrix $\boldsymbol{\Sigma}_{\mathbf{X};\boldsymbol{\theta}}$ are obtained. Alternatively, when $u\left(\cdot\right)$ is non-constant analytic function, which has a convergent Taylor series expansion, the resulting MT-mean and MT-covariance involve higher-order statistical moments of $P_{\mathbf{X};\boldsymbol{\theta}}$.

2.3. The empirical MT-mean and MT-covariance

Given a sequence of N i.i.d. samples from $P_{\mathbf{X};\boldsymbol{\theta}}$ the empirical estimators of $\boldsymbol{\mu}_{\mathbf{X};\boldsymbol{\theta}}^{(u)}$ and $\boldsymbol{\Sigma}_{\mathbf{X};\boldsymbol{\theta}}^{(u)}$ are defined as:

$$\hat{\boldsymbol{\mu}}_{\mathbf{X}}^{(u)} \triangleq \sum_{n=1}^{N} \mathbf{X}_n \hat{\varphi}_u\left(\mathbf{X}_n\right) \tag{6}$$

and

$$\hat{\boldsymbol{\Sigma}}_{\mathbf{X}}^{(u)} \triangleq \sum_{n=1}^{N} \mathbf{X}_n \mathbf{X}_n^H \hat{\varphi}_u\left(\mathbf{X}_n\right) - \hat{\boldsymbol{\mu}}_{\mathbf{X}}^{(u)}\hat{\boldsymbol{\mu}}_{\mathbf{X}}^{(u)H}, \tag{7}$$

respectively, where $\hat{\varphi}_u\left(\mathbf{X}_n\right) \triangleq u\left(\mathbf{X}_n\right)/\sum_{n=1}^{N} u\left(\mathbf{X}_n\right)$. According to Proposition 2 in [13], if $\mathrm{E}\left[\|\mathbf{X}\|^2 u\left(\mathbf{X}\right); P_{\mathbf{X};\boldsymbol{\theta}}\right] < \infty$ then $\hat{\boldsymbol{\mu}}_{\mathbf{X}}^{(u)} \xrightarrow[N\rightarrow\infty]{\text{w.p. }1} \boldsymbol{\mu}_{\mathbf{X};\boldsymbol{\theta}}^{(u)}$ and $\hat{\boldsymbol{\Sigma}}_{\mathbf{X}}^{(u)} \xrightarrow[N\rightarrow\infty]{\text{w.p. }1} \boldsymbol{\Sigma}_{\mathbf{X};\boldsymbol{\theta}}^{(u)}$, where "$\xrightarrow{\text{w.p. }1}$" denotes convergence with probability (w.p.) 1 [18].

2.4. Robustness to outliers

Robustness of the empirical MT-covariance (7) to outliers was studied in [13] using its influence function [19] which describes the effect on the estimator of an infinitesimal contamination at some point $\mathbf{y} \in \mathbb{C}^p$. An estimator is said to be B-robust if its influence function is bounded [19]. In [13] we have shown that if the MT-function $u(\mathbf{y})$ and the product $u(\mathbf{y})\|\mathbf{y}\|^2$ are bounded over $\mathbb{C}^p$ then the influence function of the empirical MT-covariance is bounded. Similarly, it can be shown that under the same conditions the influence function of the empirical MT-mean (6) is bounded.

3. MEASURE-TRANSFORMED GAUSSIAN QUASI LIKELIHOOD RATIO TEST

In this section we use the measure transformation (2) to construct a test between the null and alternative simple hypotheses $H_0 : \boldsymbol{\theta} = \boldsymbol{\theta}_0$ and $H_1 : \boldsymbol{\theta} = \boldsymbol{\theta}_1$ based on a sequence of samples $\mathbf{X}_n$, $n = 1, \ldots, N$ from $P_{\mathbf{X};\boldsymbol{\theta}}$. Regularity conditions for asymptotic normality of the corresponding test statistic are derived. Under these conditions, consistent estimates of the asymptotic size and power are derived. Optimal selection of the MT-function $u\left(\cdot\right)$ out of some parametric class of functions is also discussed.

3.1. The MT-GQLRT

Let $\Phi_{\mathbf{X};\boldsymbol{\theta}}^{(u)}$ denote a complex circular Gaussian probability distribution [20] characterized by the MT-mean $\boldsymbol{\mu}_{\mathbf{X};\boldsymbol{\theta}}^{(u)}$ and MT-covariance $\boldsymbol{\Sigma}_{\mathbf{X};\boldsymbol{\theta}}^{(u)}$. The KLD between $Q_{\mathbf{X};\boldsymbol{\theta}}^{(u)}$ and $\Phi_{\mathbf{X};\boldsymbol{\theta}_k}^{(u)}$, $k \in \{0, 1\}$ is defined as [10]:

$$D_{\mathrm{KL}}\left[Q_{\mathbf{X};\boldsymbol{\theta}}^{(u)}||\Phi_{\mathbf{X};\boldsymbol{\theta}_k}^{(u)}\right] \triangleq \mathrm{E}\left[\log \frac{q^{(u)}\left(\mathbf{X};\boldsymbol{\theta}\right)}{\phi^{(u)}\left(\mathbf{X};\boldsymbol{\theta}_k\right)}; Q_{\mathbf{X};\boldsymbol{\theta}}^{(u)}\right], \tag{8}$$

where $q^{(u)}\left(\mathbf{x};\boldsymbol{\theta}\right)$ and $\phi^{(u)}\left(\mathbf{x};\boldsymbol{\theta}_k\right)$ are the density functions of $Q_{\mathbf{X};\boldsymbol{\theta}}^{(u)}$ and $\Phi_{\mathbf{X};\boldsymbol{\theta}_k}^{(u)}$, respectively. When $\Phi_{\mathbf{X};\boldsymbol{\theta}_0}^{(u)} \neq \Phi_{\mathbf{X};\boldsymbol{\theta}_1}^{(u)}$, the difference $D_{\mathrm{KL}}\left[Q_{\mathbf{X};\boldsymbol{\theta}}^{(u)}||\Phi_{\mathbf{X};\boldsymbol{\theta}_0}^{(u)}\right] - D_{\mathrm{KL}}\left[Q_{\mathbf{X};\boldsymbol{\theta}}^{(u)}||\Phi_{\mathbf{X};\boldsymbol{\theta}_1}^{(u)}\right]$ will be negative under H_0 and positive under H_1. This motivates an empirical estimate of this difference as a test statistic for testing H_0 versus H_1. According to (3), $D_{\mathrm{KL}}\left[Q_{\mathbf{X};\boldsymbol{\theta}}^{(u)}||\Phi_{\mathbf{X};\boldsymbol{\theta}_k}^{(u)}\right]$ can be estimated using only samples from $P_{\mathbf{X};\boldsymbol{\theta}}$. Therefore, similarly to (6) and (7), an empirical estimate of (8) given a sequence of N samples from $P_{\mathbf{X};\boldsymbol{\theta}}$ is defined as:

$$\hat{D}_{\mathrm{KL}}\left[Q_{\mathbf{X};\boldsymbol{\theta}}^{(u)}||\Phi_{\mathbf{X};\boldsymbol{\theta}_k}^{(u)}\right] \triangleq \sum_{n=1}^{N} \hat{\varphi}_u\left(\mathbf{X}_n\right) \log \frac{q^{(u)}\left(\mathbf{X}_n;\boldsymbol{\theta}\right)}{\phi^{(u)}\left(\mathbf{X}_n;\boldsymbol{\theta}_k\right)}, \tag{9}$$

where $\hat{\varphi}_u(\cdot)$ is defined below (7). Hence, we propose the following test statistic, which is independent of the unknown density function $q^{(u)}(\mathbf{x};\boldsymbol{\theta})$:

$$
\begin{aligned}
T_u &\triangleq \hat{D}_{\mathrm{KL}}\left[Q_{\mathbf{X};\boldsymbol{\theta}}^{(u)}||\Phi_{\mathbf{X};\boldsymbol{\theta}_0}^{(u)}\right] - \hat{D}_{\mathrm{KL}}\left[Q_{\mathbf{X};\boldsymbol{\theta}}^{(u)}||\Phi_{\mathbf{X};\boldsymbol{\theta}_1}^{(u)}\right] \quad (10)\\
&= \sum_{n=1}^{N}\hat{\varphi}_u(\mathbf{X}_n)\,\psi_u(\mathbf{X}_n;\boldsymbol{\theta}_0,\boldsymbol{\theta}_1)\\
&= \left(D_{\mathrm{LD}}\left[\hat{\boldsymbol{\Sigma}}_{\mathbf{X}}^{(u)}||\boldsymbol{\Sigma}_{\mathbf{X};\boldsymbol{\theta}_0}^{(u)}\right]+\left\|\hat{\boldsymbol{\mu}}_{\mathbf{X}}^{(u)}-\boldsymbol{\mu}_{\mathbf{X};\boldsymbol{\theta}_0}^{(u)}\right\|_{(\boldsymbol{\Sigma}_{\mathbf{X};\boldsymbol{\theta}_0}^{(u)})^{-1}}^{2}\right)\\
&\quad - \left(D_{\mathrm{LD}}\left[\hat{\boldsymbol{\Sigma}}_{\mathbf{X}}^{(u)}||\boldsymbol{\Sigma}_{\mathbf{X};\boldsymbol{\theta}_1}^{(u)}\right]+\left\|\hat{\boldsymbol{\mu}}_{\mathbf{X}}^{(u)}-\boldsymbol{\mu}_{\mathbf{X};\boldsymbol{\theta}_1}^{(u)}\right\|_{(\boldsymbol{\Sigma}_{\mathbf{X};\boldsymbol{\theta}_1}^{(u)})^{-1}}^{2}\right),
\end{aligned}
$$

where

$$
\psi_u(\mathbf{X};\boldsymbol{\theta}_0,\boldsymbol{\theta}_1) \triangleq \log\frac{\phi^{(u)}(\mathbf{X}_n;\boldsymbol{\theta}_1)}{\phi^{(u)}(\mathbf{X}_n;\boldsymbol{\theta}_0)},
$$

$D_{\mathrm{LD}}[\mathbf{A}||\mathbf{B}] \triangleq \mathrm{tr}[\mathbf{A}\mathbf{B}^{-1}] - \log\det[\mathbf{A}\mathbf{B}^{-1}] - p$ is the log-determinant divergence [21] between positive definite matrices $\mathbf{A}, \mathbf{B} \in \mathbb{C}^{p \times p}$ and $\|\mathbf{a}\|_{\mathbf{C}} \triangleq \sqrt{\mathbf{a}^H \mathbf{C} \mathbf{a}}$ denotes the weighted Euclidian norm of a vector $\mathbf{a} \in \mathbb{C}^p$ with positive-definite weighting matrix $\mathbf{C} \in \mathbb{C}^{p \times p}$. The decision rule based on the test statistic (10) is:

$$
T_u \underset{H_0}{\overset{H_1}{\gtrless}} t, \quad (11)
$$

where $t \in \mathbb{R}$ denotes a threshold value. Notice that for any non-zero constant MT-function, $u(\cdot)$, $Q_{\mathbf{X};\boldsymbol{\theta}}^{(u)} = P_{\mathbf{X};\boldsymbol{\theta}}$ and the standard GQLRT is obtained which only involves first and second-order moments.

3.2. Asymptotic performance analysis

Here, we study the asymptotic performance of the proposed test (11). For simplicity, we assume that a sequence of i.i.d. samples $\mathbf{X}_n$, $n = 1, \ldots, N$ from $P_{\mathbf{X};\boldsymbol{\theta}}$ is available.

Theorem 1 (Asymptotic normality). *Assume that the following conditions are satisfied: 1) $\boldsymbol{\mu}_{\mathbf{X};\boldsymbol{\theta}_0}^{(u)} \neq \boldsymbol{\mu}_{\mathbf{X};\boldsymbol{\theta}_1}^{(u)}$ or $\boldsymbol{\Sigma}_{\mathbf{X};\boldsymbol{\theta}_0}^{(u)} \neq \boldsymbol{\Sigma}_{\mathbf{X};\boldsymbol{\theta}_1}^{(u)}$. 2) $\boldsymbol{\Sigma}_{\mathbf{X};\boldsymbol{\theta}_0}^{(u)}$ and $\boldsymbol{\Sigma}_{\mathbf{X};\boldsymbol{\theta}_1}^{(u)}$ are non-singular matrices. 3) $\mathrm{E}\left[u^2(\mathbf{X});P_{\mathbf{X};\boldsymbol{\theta}}\right]$ and $\mathrm{E}\left[\|\mathbf{X}\|_2^4 u^2(\mathbf{X});P_{\mathbf{X};\boldsymbol{\theta}}\right]$ are finite for $\boldsymbol{\theta} = \boldsymbol{\theta}_0$ and $\boldsymbol{\theta} = \boldsymbol{\theta}_1$. Then,*

$$
\sqrt{N}\left(T_u - \eta_{\boldsymbol{\theta}}^{(u)}\right) \xrightarrow[N\to\infty]{D} \mathcal{N}\left(0, \lambda_{\boldsymbol{\theta}}^{(u)}\right) \quad \forall \boldsymbol{\theta} \in \boldsymbol{\Theta},
$$

where "$\xrightarrow{D}$" denotes convergence in distribution [18], the mean $\eta_{\boldsymbol{\theta}}^{(u)} \triangleq \mathrm{E}\left[\varphi_u(\mathbf{X};\boldsymbol{\theta})\psi_u(\mathbf{X};\boldsymbol{\theta}_0,\boldsymbol{\theta}_1);P_{\mathbf{X};\boldsymbol{\theta}}\right]$, and the variance $\lambda_{\boldsymbol{\theta}}^{(u)} \triangleq \mathrm{Var}\left[\varphi_u(\mathbf{X};\boldsymbol{\theta})\left(\psi_u(\mathbf{X};\boldsymbol{\theta}_0,\boldsymbol{\theta}_1)-\eta_{\boldsymbol{\theta}}^{(u)}\right);P_{\mathbf{X};\boldsymbol{\theta}}\right].$

Corollary 1 (Asymptotic size and power). *Assume that the conditions stated in Proposition 1 are satisfied. The asymptotic size and power of the test (11) are given by:*

$$
\alpha_u \triangleq Q\left(\frac{t-\eta_{\boldsymbol{\theta}_0}^{(u)}}{\sqrt{N^{-1}\lambda_{\boldsymbol{\theta}_0}^{(u)}}}\right) \quad and \quad \beta_u \triangleq Q\left(\frac{t-\eta_{\boldsymbol{\theta}_1}^{(u)}}{\sqrt{N^{-1}\lambda_{\boldsymbol{\theta}_1}^{(u)}}}\right), \quad (12)
$$

respectively, where $Q(\cdot)$ denotes the tail probability of the standard normal distribution.

In the following Proposition, strongly consistent estimates of the asymptotic size and power (12) are constructed based on two i.i.d. training sequences from $P_{\mathbf{X};\boldsymbol{\theta}_0}$ and $P_{\mathbf{X};\boldsymbol{\theta}_1}$. These quantities will be used in the sequel for optimal selection of the MT-function.

Proposition 2 (Empirical asymptotic size and power). *Let $\mathbf{X}_n^{(k)}$, $n = 1, \ldots, N_k$, $k = 0, 1$ denote sequences of i.i.d. samples from $P_{\mathbf{X};\boldsymbol{\theta}_0}$ and $P_{\mathbf{X};\boldsymbol{\theta}_1}$, respectively. Define the empirical asymptotic size and power:*

$$
\hat{\alpha}_u \triangleq Q\left(\frac{t-\hat{\eta}_{\boldsymbol{\theta}_0}^{(u)}}{\sqrt{N^{-1}\hat{\lambda}_{\boldsymbol{\theta}_0}^{(u)}}}\right) \quad and \quad \hat{\beta}_u \triangleq Q\left(\frac{t-\hat{\eta}_{\boldsymbol{\theta}_1}^{(u)}}{\sqrt{N^{-1}\hat{\lambda}_{\boldsymbol{\theta}_1}^{(u)}}}\right), \quad (13)
$$

respectively, where $\hat{\eta}_{\boldsymbol{\theta}_k}^{(u)} \triangleq \sum_{n=1}^{N_k}\hat{\varphi}_u\left(\mathbf{X}_n^{(k)}\right)\psi_u\left(\mathbf{X}_n^{(k)};\boldsymbol{\theta}_0,\boldsymbol{\theta}_1\right)$ and $\hat{\lambda}_{\boldsymbol{\theta}_k}^{(u)} \triangleq N_k\sum_{n=1}^{N_k}\hat{\varphi}_u^2\left(\mathbf{X}_n^{(k)}\right)\left(\psi_u\left(\mathbf{X}_n^{(k)};\boldsymbol{\theta}_0,\boldsymbol{\theta}_1\right)-\hat{\eta}_{\boldsymbol{\theta}_k}^{(u)}\right)^2$. Assume that $\mathrm{E}\left[u^2(\mathbf{X});P_{\mathbf{X};\boldsymbol{\theta}}\right]$ and $\mathrm{E}\left[\|\mathbf{X}\|^4 u^2(\mathbf{X});P_{\mathbf{X};\boldsymbol{\theta}}\right]$ are finite for any $\boldsymbol{\theta} \in \boldsymbol{\Theta}$. Then, $\hat{\alpha}_u \xrightarrow[N_0\to\infty]{w.p.1} \alpha_u$ and $\hat{\beta}_u \xrightarrow[N_1\to\infty]{w.p.1} \beta_u$.

3.3. Selection of the MT-function

We propose to specify the MT-function within some parametric family $\{u(\mathbf{X};\boldsymbol{\omega}), \boldsymbol{\omega} \in \boldsymbol{\Omega} \subseteq \mathbb{C}^r\}$ that satisfies the conditions stated in Definition 1 and Theorem 1. An optimal choice of the MT-function parameter $\boldsymbol{\omega}$ would be the one that maximizes the empirical asymptotic power (13) at a fixed empirical asymptotic size $\hat{\alpha}_u = \alpha$, i.e., we maximize the following objective function:

$$
\hat{\beta}_u^{(\alpha)}(\boldsymbol{\omega}) = Q\left(\frac{\hat{\eta}_{\boldsymbol{\theta}_0}^{(u)}(\boldsymbol{\omega})-\hat{\eta}_{\boldsymbol{\theta}_1}^{(u)}(\boldsymbol{\omega})+\sqrt{N^{-1}\hat{\lambda}_{\boldsymbol{\theta}_0}^{(u)}(\boldsymbol{\omega})}Q^{-1}(\alpha)}{\sqrt{N^{-1}\hat{\lambda}_{\boldsymbol{\theta}_1}^{(u)}(\boldsymbol{\omega})}}\right)
$$
$$(14)$$

4. EXAMPLE

We consider a signal classification problem that is stated as the problem of testing the following simple hypotheses:

$$
\begin{aligned}
H_0 &: \quad \mathbf{X}_n = \mathbf{S}_n(\boldsymbol{\theta}_0) + \mathbf{W}_n, \quad n = 1, \ldots, N, \quad (15)\\
H_1 &: \quad \mathbf{X}_n = \mathbf{S}_n(\boldsymbol{\theta}_1) + \mathbf{W}_n, \quad n = 1, \ldots, N,
\end{aligned}
$$

where $\mathbf{X}_n \in \mathbb{C}^p$ is an observation vector, $\mathbf{S}_n(\boldsymbol{\theta}) \triangleq S_n\boldsymbol{\theta}$ is a latent random vector, $S_n \in \mathbb{C}$ is a first-order stationary random signal that is symmetrically distributed about the origin, $\boldsymbol{\theta} \in \mathbb{C}^p$ is a deterministic unit norm vector and $\mathbf{W}_n \in \mathbb{C}^p$ is a first-order stationary additive noise that is statistically independent of S_n. We assume that the noise component is spherically contoured with stochastic representation [16]:

$$
\mathbf{W}_n = \nu_n \mathbf{Z}_n, \quad (16)
$$

where $\nu_n \in \mathbb{R}_{++}$ is a first-order stationary process and $\mathbf{Z}_n \in \mathbb{C}^p$ is a proper-complex wide-sense stationary Gaussian process with zero-mean and scaled unit covariance $\sigma_{\mathbf{Z}}^2\mathbf{I}$. The processes ν_n and $\mathbf{Z}_n$ are assumed to be statistically independent.

In order to gain robustness against outliers, as well as sensitivity to higher-order moments, we specify the MT-function in the zero-centred Gaussian family of functions parametrized by a width parameter ω, i.e.,

$$
u(\mathbf{x};\omega) = \exp\left(-\|\mathbf{x}\|^2/\omega^2\right), \quad \omega \in \mathbb{R}_{++}. \quad (17)
$$

Notice that the MT-function (17) satisfies the B-robustness conditions stated in Subsection 2.4. Using (4), (5) and (15)-(17) it can be shown that the MT-mean and MT-covariance under $Q_{\mathbf{X};\boldsymbol{\theta}}^{(u)}$ are:

$$\boldsymbol{\mu}_{\mathbf{X};\boldsymbol{\theta}}^{(u)}(\omega) = \mathbf{0} \tag{18}$$

and

$$\boldsymbol{\Sigma}_{\mathbf{X};\boldsymbol{\theta}}^{(u)}(\omega) = r_S(\omega)\boldsymbol{\theta}\boldsymbol{\theta}^H + r_W(\omega)\mathbf{I}, \tag{19}$$

respectively, where $r_S(\omega)$ and $r_W(\omega)$ are some strictly positive functions of ω. Hence, by substituting (17)-(19) into (10) followed by normalization by the observation-independent factor $c(\omega) \triangleq \frac{r_S(\omega)}{r_W(\omega)(r_S(\omega)+r_W(\omega))}$, the MT-GQLRT (11) simplifies to

$$T_u' \triangleq T_u/c(\omega) = \boldsymbol{\theta}_1^H \hat{\mathbf{C}}_{\mathbf{X}}^{(u)}(\omega)\boldsymbol{\theta}_1 - \boldsymbol{\theta}_0^H \hat{\mathbf{C}}_{\mathbf{X}}^{(u)}(\omega)\boldsymbol{\theta}_0 \overset{H_1}{\underset{H_0}{\gtrless}} t',$$

where $\hat{\mathbf{C}}_{\mathbf{X}}^{(u)}(\omega) \triangleq \hat{\boldsymbol{\Sigma}}_{\mathbf{X}}^{(u)}(\omega) + \hat{\boldsymbol{\mu}}_{\mathbf{X}}^{(u)}(\omega)\hat{\boldsymbol{\mu}}_{\mathbf{X}}^{(u)H}(\omega)$ and $t' \triangleq t/c(\omega)$.

Under the considered settings, it can be shown that the conditions stated in Proposition 1 are satisfied. The resulting asymptotic power (12) at a given asymptotic size $\alpha_u = \alpha$ takes the form:

$$\beta_u^{(\alpha)}(\omega) = Q\left(-\sqrt{\frac{2N}{V(\omega,\boldsymbol{\theta}_0,\boldsymbol{\theta}_1)}} + Q^{-1}(\alpha)\right), \tag{20}$$

where $V(\omega,\boldsymbol{\theta}_0,\boldsymbol{\theta}_1) \triangleq G(\omega)\left(1 - \left|\boldsymbol{\theta}_0^H\boldsymbol{\theta}_1\right|^2\right)^{-1} - U(\omega) + \frac{1}{2}D(\omega)$,

- $G(\omega) \triangleq \frac{\mathrm{E}\left[g(S,\sqrt{2}\bar{\nu},\omega)h_2(\sqrt{2}S,\sqrt{2}\bar{\nu},\omega);P_{S,\nu}\right]}{\mathrm{E}^2\left[|S|^2 h_2(S,\bar{\nu},\omega);P_{S,\nu}\right]}$,

- $U(\omega) \triangleq \frac{\mathrm{E}\left[|S|^2 h_2(\sqrt{2}S,\sqrt{2}\bar{\nu},\omega);P_{S,\nu}\right]}{\mathrm{E}\left[h_0(S,\bar{\nu},\omega);P_{S,\nu}\right]\mathrm{E}\left[|S|^2 h_2(S,\bar{\nu},\omega);P_{S,\nu}\right]}$,

- $D(\omega) \triangleq \frac{\mathrm{E}\left[h_0(\sqrt{2}S,\sqrt{2}\bar{\nu},\omega);P_{S,\nu}\right]}{\mathrm{E}^2\left[|S|^2 h_0(S,\bar{\nu},\omega);P_{S,\nu}\right]}$,

$\bar{\nu} \triangleq \nu\sigma_{\mathbf{z}}$, $g(S,\nu,\omega) \triangleq \left(\frac{\omega^2|S|^2}{\omega^2+\nu^2}\right)^2 + 3\nu^2\left(\frac{\omega^2|S|^2}{\omega^2+\nu^2}\right) + \nu^4$, and $h_k(S,\nu,\omega) \triangleq \left(\frac{\omega^2}{\nu^2+\omega^2}\right)^{p+k}\exp\left(-\frac{|S|^2}{\nu^2+\omega^2}\right)$. Furthermore, its empirical estimate (14) is given by

$$\hat{\beta}_u^{(\alpha)}(\omega) = Q\left(\frac{\tilde{\eta}_{\boldsymbol{\theta}_0}^{(u)}(\omega) - \tilde{\eta}_{\boldsymbol{\theta}_1}^{(u)}(\omega) + \sqrt{N^{-1}\tilde{\lambda}_{\boldsymbol{\theta}_0}^{(u)}(\omega)}Q^{-1}(\alpha)}{\sqrt{N^{-1}\tilde{\lambda}_{\boldsymbol{\theta}_1}^{(u)}(\omega)}}\right), \tag{21}$$

where $\tilde{\eta}_{\boldsymbol{\theta}_k}^{(u)}(\omega) \triangleq \frac{\hat{\eta}_{\boldsymbol{\theta}_k}^{(u)}(\omega)}{c(\omega)} = \sum_{n=1}^{N_k}\hat{\varphi}_u\left(\mathbf{X}_n^{(k)};\omega\right)\xi\left(\mathbf{X}_n^{(k)};\boldsymbol{\theta}_0,\boldsymbol{\theta}_1\right)$, $\tilde{\lambda}_{\boldsymbol{\theta}_k}^{(u)}(\omega) \triangleq \frac{\hat{\lambda}_{\boldsymbol{\theta}_k}^{(u)}(\omega)}{c^2(\omega)} = N_k\sum_{n=1}^{N_k}\hat{\varphi}_u^2\left(\mathbf{X}_n^{(k)};\omega\right)\left(\xi\left(\mathbf{X}_n^{(k)};\boldsymbol{\theta}_0,\boldsymbol{\theta}_1\right) - \tilde{\eta}_{\boldsymbol{\theta}_k}^{(u)}\right)^2$, $k = 0,1$, and $\xi(\mathbf{X};\boldsymbol{\theta}_0,\boldsymbol{\theta}_1) \triangleq \left|\boldsymbol{\theta}_1^H\mathbf{X}\right|^2 - \left|\boldsymbol{\theta}_0^H\mathbf{X}\right|^2$.

In the following simulation examples, the vectors $\boldsymbol{\theta}_0$ and $\boldsymbol{\theta}_1$ were set to $\boldsymbol{\theta}_k \triangleq \frac{1}{\sqrt{p}}\left[1, e^{-i\pi\sin(\vartheta_k)}, \ldots, e^{-i\pi(p-1)\sin(\vartheta_k)}\right]^T$, $k = 0, 1$, were $\vartheta_0 = 0$, $\vartheta_1 = \pi/3$ and $p = 4$. We consider a BPSK signal with variance σ_S^2 and an ϵ-contaminated Gaussian noise model [16] under which the texture component ν in (16) is a binary random variable satisfying $\nu = 1$ w.p. $1 - \epsilon$ and $\nu = \delta$ w.p. ϵ. The parameters ϵ and δ that control the heaviness of the noise tails were set to 0.2 and 10, respectively. We define the signal-to-noise-ratio (SNR) as $\mathrm{SNR} \triangleq 10\log_{10}\sigma_S^2/\left[\sigma_{\mathbf{z}}^2\left((1-\epsilon)+\epsilon\delta^2\right)\right]$. In all simulation examples the sample size was set to $N = 10^3$. The empirical asymptotic power (21) was obtained using two i.i.d. training sequences from $P_{\mathbf{X};\boldsymbol{\theta}_0}$ and $P_{\mathbf{X};\boldsymbol{\theta}_1}$ containing $N_0 = N_1 = 10^5$ samples.

In the first example, we compared the asymptotic power (20) to its empirical estimate (21) at test size equal to 0.05 as a function of ω for $\mathrm{SNR} = -10$ [dB]. Observing Fig. 1, one sees that due to the consistency of (21) the compared quantities are very close.

In the second example, we compared the empirical, asymptotic (20) and empirical asymptotic (21) power of the MT-GQLRT to the empirical powers of the GQLRT and the omniscient LRT for test size equal to 0.05. The optimal Gaussian MT-function parameter ω_{opt} was obtained by maximizing (21) over $\Omega = [1, 100]$. The empirical power curves were obtained using 10^4 Monte-Carlo simulations. The SNR is used to index the performances as depicted in Fig. 2. One sees that the MT-GQLRT outperforms the non-robust GQLRT and for most examined SNR values performs similarly to the LRT that, unlike the MT-GQLRT, requires complete knowledge of the likelihood function under each hypothesis.
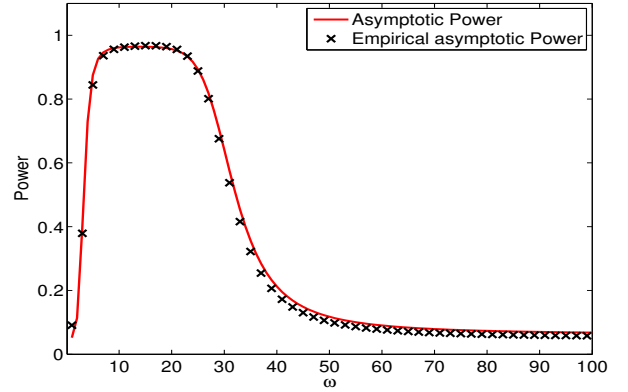

Fig. 1. Asymptotic power (20) and its empirical estimate (21) at test size $\alpha = 0.05$ versus the width parameter ω of the MT-function (17).
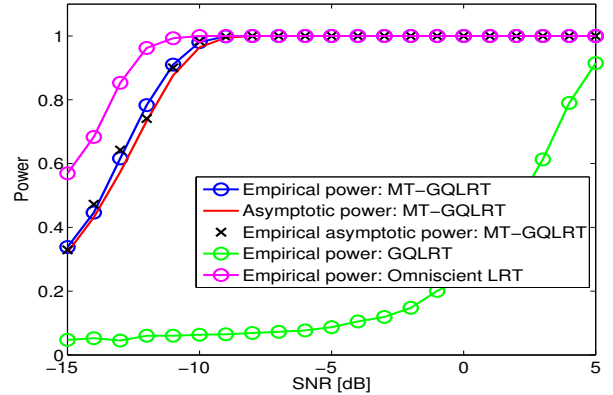

Fig. 2. The empirical, asymptotic (20) and empirical asymptotic (21) powers of the proposed MT-GQLRT as compared to the empirical powers of the GQLRT and the omniscient LRT at test size $\alpha = 0.05$.

5. CONCLUSION

In this paper a new test, called MT-GQLRT, was derived by applying a transform to the probability distribution of the data. By specifying the MT-function in the Gaussian family, the proposed test was applied to signal classification in non-Gaussian noise. Exploration of other MT-functions may result in additional tests in this class that have different useful properties.

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