A ROBUST GAUSSIAN APPROXIMATE FILTER FOR NONLINEAR SYSTEMS WITH HEAVY TAILED MEASUREMENT NOISES

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ABSTRACT

The scale matrix and degrees of freedom (dof) parameter of a Student's t distribution are important for nonlinear robust inference, and it is difficult to determine exact values in practical application due to complex environments. To solve this problem, an improved robust Gaussian approximate (GA) filter is derived based on the variational Bayesian approach, where the state together with unknown scale matrix and dof parameter are inferred. The proposed filter is applied to a target tracking problem with measurement outliers, and its performance is compared with an existing robust GA filter with fixed scale matrix and dof parameter. The results show the efficiency and superiority of the proposed filter as compared with the existing filter.

Index Terms— Gaussian approximate filter, Student's t distribution, variational Bayesian, heavy tailed measurement noise, outliers

1. INTRODUCTION

Gaussian approximate (GA) filters have been gaining more attention because they can provide tradeoffs between computational complexity and estimation accuracy in many practical applications [1]–[4]. In the framework of the GA filter, the posterior probability density function (PDF) is approximated as Gaussian, and the technical challenge is how to compute Gaussian weighted integrals [4]. So far, several forms of GA filter have been developed based on different numerical integral methods [3]–[8]. However, these standard GA filters are sensitive to heavy tailed measurement noises induced by measurement outliers from unreliable sensors [9].

To solve the filtering problem of linear systems with heavy tailed measurement noises, many linear robust filters have been derived by modelling measurement noises as a multivariate Student's t distribution and using the variational Bayesian (VB) approach to infer the state [10]–[13]. Piché et al. derived a robust GA filter by extending the above linear robust filters to nonlinear state space models [9]. In the design of the existing robust GA filter, the scale matrix and degrees of freedom (dof) parameter of the Student's t distribution for measurement noises are assumed to be known exactly. However, in practical application, it is difficult to determine exact values due to complex environments, and the scale matrix and dof parameter may be time varying. Thus, the existing robust GA filter may show poor estimation performance when the choices of scale matrix and dof parameter are inaccurate.

In this paper, to solve this problem, an improved robust GA filter for nonlinear systems with heavy tailed measurement noises is proposed, where the state together with unknown scale matrix and dof parameter are inferred. The posterior PDFs of state, scale matrix and dof parameter are updated as Gaussian, inverse Wishart and Gamma respectively by assuming appropriate conjugate prior and using the VB approach. Simulation results show the proposed filter outperforms the existing robust GA filter for the case of heavy tailed measurement noise.

2. IMPROVED ROBUST GA FILTER

Consider the following nonlinear state-space model

$$\begin{cases} \mathbf{x}_k = \mathbf{f}_{k-1}(\mathbf{x}_{k-1}) + \mathbf{w}_{k-1} \\ \mathbf{z}_k = \mathbf{h}_k(\mathbf{x}_k) + \mathbf{v}_k \end{cases} \tag{1}$$

where $\mathbf{x}_k \in \mathbb{R}^n$ is the state vector, and $\mathbf{z}_k \in \mathbb{R}^m$ is the measurements vector, and $\mathbf{f}_{k-1}(\cdot)$ and $\mathbf{h}_k(\cdot)$ are nonlinear process and measurement functions respectively. $\mathbf{w}_k \in \mathbb{R}^n$ is the zero-mean Gaussian white process noise satisfying $E[\mathbf{w}_k\mathbf{w}_l^T] = \mathbf{\Sigma}_k^w \delta_{kl}$, where $\mathbf{\Sigma}_k^w$ is the process noise covariance matrix and δ_{kl} is the Kronecker delta function. $\mathbf{v}_k \in \mathbb{R}^m$ is the heavy tailed measurement noise, and it is modelled as a Student's t distribution

$$p(\mathbf{v}_k) = \mathrm{St}(\mathbf{v}_k; \mathbf{0}, \mathbf{R}_k, \nu_k) = \int_0^{+\infty} \mathrm{N}(\mathbf{v}_k; \mathbf{0}, \mathbf{R}_k/\lambda_k) \times \mathrm{G}(\lambda_k; \frac{\nu_k}{2}, \frac{\nu_k}{2}) d\lambda_k \tag{2}$$

where $\mathrm{St}(\cdot; \mu, \mathbf{\Sigma}, \nu)$ denotes the Student's t PDF with mean vector μ, scale matrix $\mathbf{\Sigma}$, and dof parameter ν, $\mathrm{N}(\cdot; \mu, \mathbf{\Sigma})$ de-

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notes the Gaussian PDF with mean vector μ and covariance matrix $\mathbf{\Sigma}$, and $\mathrm{G}(\cdot;\alpha,\beta)$ denotes the Gamma PDF with shape parameter α and rate parameter β. The initial state $\mathbf{x}_0$ is a Gaussian random vector with mean $\hat{\mathbf{x}}_{0|0}$ and covariance matrix $\mathbf{P}_{0|0}$, and $\mathbf{x}_0$, $\mathbf{w}_k$ and $\mathbf{v}_k$ are assumed to be mutually independent.

In the paper, the state $\mathbf{x}_k$, the scale matrix $\mathbf{R}_k$ and the dof parameter ν_k are assumed to have Gaussian, inverse Wishart and Gamma prior respectively, i.e.

$$p(\mathbf{x}_k,\lambda_k,\mathbf{R}_k,\nu_k|\mathbf{z}_{1:k-1}) = \mathrm{N}(\mathbf{x}_k;\hat{x}_{k|k-1},\mathbf{P}_{k|k-1}) \\ \mathrm{G}(\lambda_k;\frac{\nu_k}{2},\frac{\nu_k}{2})\mathrm{IW}(\mathbf{R}_k;\hat{u}_{k|k-1},\mathbf{U}_{k|k-1}) \\ \mathrm{G}(\nu_k;\hat{a}_{k|k-1},\hat{b}_{k|k-1}) \quad (3)$$

where $\mathrm{IW}(\cdot;u,\mathbf{U})$ denotes the inverse Wishart PDF with dof parameter u and inverse scale matrix $\mathbf{U}$ [14].

To estimate the state $\mathbf{x}_k$, an auxiliary parameter λ_k, scale matrix $\mathbf{R}_k$ and dof parameter ν_k, we need to compute the joint posterior PDF $p(\mathbf{x}_k,\lambda_k,\mathbf{R}_k,\nu_k|\mathbf{z}_{1:k})$. For a general nonlinear system, there is not an analytical solution for the posterior PDF $p(\mathbf{x}_k,\lambda_k,\mathbf{R}_k,\nu_k|\mathbf{z}_{1:k})$. Thus, to obtain an approximate solution, the standard VB approach is used to look for a free form factored approximate PDF for $p(\mathbf{x}_k,\lambda_k,\mathbf{R}_k,\nu_k|\mathbf{z}_{1:k})$, i.e.

$$p(\mathbf{x}_k,\lambda_k,\mathbf{R}_k,\nu_k|\mathbf{z}_{1:k}) \approx q(\mathbf{x}_k)q(\lambda_k)q(\mathbf{R}_k)q(\nu_k) \quad (4)$$

where $q(\mathbf{x}_k)$, $q(\lambda_k)$, $q(\mathbf{R}_k)$ and $q(\nu_k)$ are the approximate posterior PDFs of $\mathbf{x}_k$, λ_k, $\mathbf{R}_k$ and ν_k, respectively. According to the standard VB approach, the optimal solution satisfies the following equations [15, 16]

$$\log q(\mathbf{x}_k) = E_{\lambda,\mathbf{R},\nu}[\log p(\mathbf{x}_k,\lambda_k,\mathbf{R}_k,\nu_k,\mathbf{z}_{1:k})] + c_{\mathbf{x}} \quad (5)$$

$$\log q(\lambda_k) = E_{\mathbf{x},\mathbf{R},\nu}[\log p(\mathbf{x}_k,\lambda_k,\mathbf{R}_k,\nu_k,\mathbf{z}_{1:k})] + c_{\lambda} \quad (6)$$

$$\log q(\mathbf{R}_k) = E_{\mathbf{x},\lambda,\nu}[\log p(\mathbf{x}_k,\lambda_k,\mathbf{R}_k,\nu_k,\mathbf{z}_{1:k})] + c_{\mathbf{R}} \quad (7)$$

$$\log q(\nu_k) = E_{\mathbf{x},\lambda,\mathbf{R}}[\log p(\mathbf{x}_k,\lambda_k,\mathbf{R}_k,\nu_k,\mathbf{z}_{1:k})] + c_{\nu} \quad (8)$$

where $E[\cdot]$ denotes the expectation operation, and $c_{\mathbf{x}}$, c_λ, $c_{\mathbf{R}}$ and c_ν are constants with respect to the variables $\mathbf{x}_k$, λ_k, $\mathbf{R}_k$ and ν_k respectively. Since the variational parameters of $q(\mathbf{x}_k)$, $q(\lambda_k)$, $q(\mathbf{R}_k)$ and $q(\nu_k)$ are coupled, we need to utilize fixed-point iterations to solve equations (5)-(8), where only one factor in (4) is updated while keeping the other factors fixed [14, 15].

2.1. Measurement update

According to the Bayesian theorem and using (1)-(3), the joint PDF $p(\mathbf{x}_k,\lambda_k,\mathbf{R}_k,\nu_k,\mathbf{z}_{1:k})$ can be computed as

$$p(\mathbf{x}_k,\lambda_k,\mathbf{R}_k,\nu_k,\mathbf{z}_{1:k}) = \mathrm{N}(\mathbf{z}_k;\mathbf{h}_k(\mathbf{x}_k),\mathbf{R}_k/\lambda_k) \\ \mathrm{N}(\mathbf{x}_k;\hat{x}_{k|k-1},\mathbf{P}_{k|k-1})\mathrm{G}(\lambda_k;\frac{\nu_k}{2},\frac{\nu_k}{2})\mathrm{IW}(\mathbf{R}_k; \\ \hat{u}_{k|k-1},\mathbf{U}_{k|k-1})\mathrm{G}(\nu_k;\hat{a}_{k|k-1},\hat{b}_{k|k-1})p(\mathbf{z}_{1:k-1}) \quad (9)$$

Using (9) in (5), $\log q^{(i+1)}(\mathbf{x}_k)$ can be computed as

$$\log q^{(i+1)}(\mathbf{x}_k) = -0.5E^{(i)}[\lambda_k][\mathbf{z}_k-\mathbf{h}_k(\mathbf{x}_k)]^T E^{(i)}[\mathbf{R}_k^{-1}] \\ [\mathbf{z}_k-\mathbf{h}_k(\mathbf{x}_k)] + \log\mathrm{N}(\mathbf{x}_k;\hat{x}_{k|k-1},\mathbf{P}_{k|k-1}) + c_{\mathbf{x}} \quad (10)$$

where $q^{(i+1)}(\cdot)$ is the approximation of PDF $q(\cdot)$ at the $i+1$th iteration, and $E^{(i)}[\xi]$ is the expectation of variable ξ at the ith iteration. Define the modified measurement noise covariance matrix $\tilde{\mathbf{R}}_k^{(i)}$ as follows

$$\tilde{\mathbf{R}}_k^{(i)} = (E^{(i)}[\lambda_k]E^{(i)}[\mathbf{R}_k^{-1}])^{-1} = (E^{(i)}[\mathbf{R}_k^{-1}])^{-1}/E^{(i)}[\lambda_k] \quad (11)$$

Using (10)-(11), the approximate posterior PDF $q^{(i+1)}(\mathbf{x}_k)$ can be computed as

$$q^{(i+1)}(\mathbf{x}_k) = \frac{\mathrm{N}(\mathbf{z}_k;\mathbf{h}_k(\mathbf{x}_k),\tilde{\mathbf{R}}_k^{(i)})\mathrm{N}(\mathbf{x}_k;\hat{x}_{k|k-1},\mathbf{P}_{k|k-1})}{\int \mathrm{N}(\mathbf{z}_k;\mathbf{h}_k(\mathbf{x}_k),\tilde{\mathbf{R}}_k^{(i)})\mathrm{N}(\mathbf{x}_k;\hat{x}_{k|k-1},\mathbf{P}_{k|k-1})d\mathbf{x}_k} \quad (12)$$

It can be seen from (12) that $q^{(i+1)}(\mathbf{x}_k)$ has the same form as the posterior PDF of state in a standard nonlinear system with modified likelihood PDF $\mathrm{N}(\mathbf{z}_k;\mathbf{h}_k(\mathbf{x}_k),\tilde{\mathbf{R}}_k^{(i)})$. Thus, $q^{(i+1)}(\mathbf{x}_k)$ can be approximated as a Gaussian PDF by using the measurement update of the standard GA filter [4] with modified likelihood PDF $\mathrm{N}(\mathbf{z}_k;\mathbf{h}_k(\mathbf{x}_k),\tilde{\mathbf{R}}_k^{(i)})$, i.e.

$$q^{(i+1)}(\mathbf{x}_k) = \mathrm{N}(\mathbf{x}_k;\hat{x}_{k|k}^{(i+1)},\mathbf{P}_{k|k}^{(i+1)}) \quad (13)$$

where $\hat{x}_{k|k}^{(i+1)}$ and $\mathbf{P}_{k|k}^{(i+1)}$ are the filtering estimate and corresponding estimate error covariance matrix at the $i+1$th iteration.

Using (9) in (6), the variational form for $q^{(i+1)}(\lambda_k)$ obeys

$$\log q^{(i+1)}(\lambda_k) = (\frac{m+E^{(i)}[\nu_k]}{2}-1)\log\lambda_k - \\ \frac{1}{2}\{\mathrm{tr}(\mathbf{D}_k^{(i+1)}E^{(i)}[\mathbf{R}_k^{-1}]) + E^{(i)}[\nu_k]\}\lambda_k + c_\lambda \quad (14)$$

where $\mathrm{tr}(\cdot)$ denotes the trace operation of a matrix and $\mathbf{D}_k^{(i+1)}$ is given by

$$\mathbf{D}_k^{(i+1)} = E^{(i+1)}[(\mathbf{z}_k-\mathbf{h}_k(\mathbf{x}_k))(\mathbf{z}_k-\mathbf{h}_k(\mathbf{x}_k))^T] \quad (15)$$

According to (14), $q^{(i+1)}(\lambda_k)$ can be updated as

$$q^{(i+1)}(\lambda_k) = \mathrm{G}(\lambda_k;\alpha_k^{(i+1)},\beta_k^{(i+1)}) \quad (16)$$

where the shape parameter $\alpha_k^{(i+1)}$ and rate parameter $\beta_k^{(i+1)}$ are given by

$$\alpha_k^{(i+1)} = 0.5(m+E^{(i)}[\nu_k]) \quad (17)$$

$$\beta_k^{(i+1)} = 0.5\{\mathrm{tr}(\mathbf{D}_k^{(i+1)}E^{(i)}[\mathbf{R}_k^{-1}]) + E^{(i)}[\nu_k]\} \quad (18)$$

Using (9) in (7), the variational form for $q^{(i+1)}(\mathbf{R}_k)$ obeys

$$\log q^{(i+1)}(\mathbf{R}_k) = -0.5(\hat{u}_{k|k-1}+1+m+1)\log|\mathbf{R}_k| \\ -0.5\mathrm{tr}((E^{(i+1)}[\lambda_k]\mathbf{D}_k^{(i+1)}+\mathbf{U}_{k|k-1})\mathbf{R}_k^{-1}) + c_{\mathbf{R}} \quad (19)$$

which yields

$$q^{(i+1)}(\mathbf{R}_k) = \mathrm{IW}(\mathbf{R}_k; \hat{u}_{k|k}^{(i+1)}, \mathbf{U}_{k|k}^{(i+1)}) \tag{20}$$

where the dof parameter $\hat{u}_{k|k}^{(i+1)}$ and inverse scale matrix $\mathbf{U}_{k|k}^{(i+1)}$ are given by

$$\hat{u}_{k|k}^{(i+1)} = \hat{u}_{k|k-1} + 1 \tag{21}$$

$$\mathbf{U}_{k|k}^{(i+1)} = \mathbf{U}_{k|k-1} + E^{(i+1)}[\lambda_k]\mathbf{D}_k^{(i+1)} \tag{22}$$

Substituting (9) in (8), $\log q^{(i+1)}(\nu_k)$ is given by

$$\begin{aligned}\log q^{(i+1)}(\nu_k) &= \frac{\nu_k}{2}\log\frac{\nu_k}{2} - \log\Gamma(\frac{\nu_k}{2}) + (\frac{\nu_k}{2} - 1) \\ E^{(i+1)}&[\log\lambda_k] - \frac{\nu_k}{2}E^{(i+1)}[\lambda_k] + (\hat{a}_{k|k-1} - 1)\log\nu_k \\ -\hat{b}_{k|k-1}&\nu_k + c_\nu \end{aligned} \tag{23}$$

Using Stirling's approximation: $\log\Gamma(\frac{\nu_k}{2}) \approx \frac{\nu_k - 1}{2}\log\frac{\nu_k}{2} - \frac{\nu_k}{2}$ in (23) [12], $\log q^{(i+1)}(\nu_k)$ can be recomputed as

$$\begin{aligned}\log q^{(i+1)}(\nu_k) &= (\hat{a}_{k|k-1} + 0.5 - 1)\log\nu_k - (\hat{b}_{k|k-1} \\ -0.5 - 0.5E^{(i+1)}&[\log\lambda_k] + 0.5E^{(i+1)}[\lambda_k])\nu_k + c_\nu\end{aligned} \tag{24}$$

which yields

$$q^{(i+1)}(\nu_k) = \mathrm{G}(\nu_k; \hat{a}_{k|k}^{(i+1)}, \hat{b}_{k|k}^{(i+1)}) \tag{25}$$

where shape parameter $\hat{a}_{k|k}^{(i+1)}$ and rate parameter $\hat{b}_{k|k}^{(i+1)}$ are given by

$$\hat{a}_{k|k}^{(i+1)} = \hat{a}_{k|k-1} + 0.5 \tag{26}$$

$$\hat{b}_{k|k}^{(i+1)} = \hat{b}_{k|k-1} - 0.5 - 0.5E^{(i+1)}[\log\lambda_k] + 0.5E^{(i+1)}[\lambda_k] \tag{27}$$

Next, we compute the expectations which are needed for the computations of the approximate posterior PDFs $q^{(i+1)}(\mathbf{x}_k)$, $q^{(i+1)}(\lambda_k)$, $q^{(i+1)}(\mathbf{R}_k)$ and $q^{(i+1)}(\nu_k)$. Using (13), (16), (20), (25), the expectations $E^{(i)}[\mathbf{R}_k^{-1}]$, $E^{(i)}[\lambda_k]$, $E^{(i)}[\nu_k]$, $\mathbf{D}_k^{(i)}$ and $E^{(i)}[\log\lambda_k]$ can be computed as

$$E^{(i)}[\mathbf{R}_k^{-1}] = (\hat{u}_{k|k}^{(i)} - m - 1)(\mathbf{U}_{k|k}^{(i)})^{-1} \tag{28}$$

$$E^{(i)}[\lambda_k] = \alpha_k^{(i)}/\beta_k^{(i)} \tag{29}$$

$$E^{(i)}[\nu_k] = \hat{a}_{k|k}^{(i)}/\hat{b}_{k|k}^{(i)} \tag{30}$$

$$\mathbf{D}_k^{(i)} = \int(\mathbf{z}_k - \mathbf{h}_k(\mathbf{x}_k))(\mathbf{z}_k - \mathbf{h}_k(\mathbf{x}_k))^T \mathrm{N}(\mathbf{x}_k; \hat{x}_{k|k}^{(i)}, \mathbf{P}_{k|k}^{(i)})d\mathbf{x}_k \tag{31}$$

$$E^{(i)}[\log\lambda_k] = \psi(\alpha_k^{(i)}) - \log\beta_k^{(i)} \tag{32}$$

where $\psi(\cdot)$ denotes the digamma function [12].

After fixed point iteration N, the approximate posterior PDFs $q(\mathbf{x}_k)$, $q(\mathbf{R}_k)$ and $q(\nu_k)$ can be updated as

$$q(\mathbf{x}_k) = \mathrm{N}(\mathbf{x}_k; \hat{x}_{k|k}^{(N)}, \mathbf{P}_{k|k}^{(N)}) = \mathrm{N}(\mathbf{x}_k; \hat{x}_{k|k}, \mathbf{P}_{k|k}) \tag{33}$$

$$q(\mathbf{R}_k) = \mathrm{IW}(\mathbf{R}_k; \hat{u}_{k|k}^{(N)}, \mathbf{U}_{k|k}^{(N)}) = \mathrm{IW}(\mathbf{R}_k; \hat{u}_{k|k}, \mathbf{U}_{k|k}) \tag{34}$$

$$q(\nu_k) = \mathrm{G}(\nu_k; \hat{a}_{k|k}^{(N)}, \hat{b}_{k|k}^{(N)}) = \mathrm{G}(\nu_k; \hat{a}_{k|k}, \hat{b}_{k|k}) \tag{35}$$

2.2. Time update

In the time update, the dynamic models for the scale matrix $\mathbf{R}_k$ and dof parameter ν_k need to be chosen so that $p(\mathbf{R}_k|\mathbf{z}_{1:k-1})$ and $p(\nu_k|\mathbf{z}_{1:k-1})$ are inverse Wishart and Gamma PDFs respectively. Using similar heuristics as in [12] and [17], the dynamic models for scale matrix $\mathbf{R}_k$ and dof parameter ν_k can be obtained as follows

$$\begin{cases} \hat{u}_{k|k-1} = \rho(\hat{u}_{k-1|k-1} - m - 1) + m + 1 \\ \mathbf{U}_{k|k-1} = \rho\mathbf{U}_{k-1|k-1} \\ \hat{a}_{k|k-1} = \rho\hat{a}_{k-1|k-1} \quad \hat{b}_{k|k-1} = \rho\hat{b}_{k-1|k-1} \end{cases} \tag{36}$$

where $\rho \in (0\ 1]$ is a forgetting factor which indicates the extent of time-fluctuations. The proposed improved robust GA filter is shown in Algorithm 1.

3. SIMULATION

In this simulation, the superior performance of the proposed filter as compared with existing robust GA filter [9] is shown in the problem of tracking a target in two dimensional space executing a maneuvering turn with unknown and time-varying turn rate. The process and measurement models and simulation parameters are the same as [4]. Similar to [18], outlier corrupted measurement noise is generated according to

$$\mathbf{v}_k \sim \begin{cases} N(\mathbf{0}, \boldsymbol{\Sigma}^v) & \text{w.p. } 1 - p_c \\ N(\mathbf{0}, 100\boldsymbol{\Sigma}^v) & \text{w.p. } p_c \end{cases} \tag{37}$$

where $\boldsymbol{\Sigma}^v = \mathrm{diag}[\sigma_r^2\ \sigma_\theta^2]$, $\sigma_r = 10$m and $\sigma_\theta = \sqrt{10}$mrad. Equation (37) means that $\mathbf{v}_k$ is drawn from $N(\mathbf{0}, \boldsymbol{\Sigma}^v)$ with probability $1 - p_c$ and $N(\mathbf{0}, 100\boldsymbol{\Sigma}^v)$ with probability p_c. That is to say, the probability of the measurement outlier is p_c. Measurement noise, which is generated in terms of (37), has heavy tails.

In this simulation, the existing robust GA filter and the proposed robust GA filter are tested. The parameters of the proposed filter are set as: $\hat{a}_{0|0} = 5$, $\hat{b}_{0|0} = 1$, $\hat{u}_{0|0} = 4$, $\mathbf{U}_{0|0} = \boldsymbol{\Sigma}^v$, $\rho = 1 - \exp(-5)$, $N = 5$. The scale matrix of the existing robust GA filter is set as $\boldsymbol{\Sigma}^v$. The third-degree spherical radial cubature rule [4] is used to implement the proposed filter and the existing robust GA filter, and corresponding improved robust cubature Kalman filter (CKF) and existing robust CKF can be obtained. The root-mean square errors

Algorithm 1: One time step of the proposed filter

Inputs: $\mathbf{f}_{k-1}(\cdot)$, $\mathbf{h}_k(\cdot)$, $\mathbf{z}_k$, $\mathbf{\Sigma}_k^w$, $\hat{x}_{k-1|k-1}$, $\mathbf{P}_{k-1|k-1}$, $\hat{u}_{k-1|k-1}$, $\mathbf{U}_{k-1|k-1}$, $\hat{a}_{k-1|k-1}$, $\hat{b}_{k-1|k-1}$, ρ, N

Time update

1. Compute $\hat{\mathbf{x}}_{k|k-1}$ and $\mathbf{P}_{k|k-1}$ by the time update of standard GA filter [4]

2. Compute $\hat{u}_{k|k-1}$, $\mathbf{U}_{k|k-1}$, $\hat{a}_{k|k-1}$, $\hat{b}_{k|k-1}$ using (36)

Measurement update

3. Initialization: $\hat{u}_{k|k}^{(0)} \leftarrow \hat{u}_{k|k-1} + 1$, $\mathbf{U}_{k|k}^{(0)} \leftarrow \mathbf{U}_{k|k-1}$, $\hat{a}_{k|k}^{(0)} \leftarrow \hat{a}_{k|k-1} + 0.5$, $\hat{b}_{k|k}^{(0)} \leftarrow \hat{b}_{k|k-1}$, $\alpha_k^{(0)} \leftarrow 0.5\hat{a}_{k|k-1}/\hat{b}_{k|k-1}$, $\beta_k^{(0)} \leftarrow 0.5\hat{a}_{k|k-1}/\hat{b}_{k|k-1}$

4. Compute initial expectations using (28)-(32)

For $i = 0 : N-1$

5. Compute $\tilde{\mathbf{R}}_k^{(i)}$ using (11)

6. Compute $\hat{\mathbf{x}}_{k|k}^{(i+1)}$ and $\mathbf{P}_{k|k}^{(i+1)}$ by the measurement update of standard GA filter with $\tilde{\mathbf{R}}_k^{(i)}$ [4]

7. Compute $\mathbf{D}_k^{(i+1)}$ using (31)

8. Compute $\alpha_k^{(i+1)}$ and $\beta_k^{(i+1)}$ using (17)-(18)

9. Compute $E^{(i+1)}[\lambda_k]$ and $E^{(i+1)}[\log\lambda_k]$ using (29) and (32)

10. Compute $\hat{u}_{k|k}^{(i+1)}$, $\mathbf{U}_{k|k}^{(i+1)}$, $\hat{a}_{k|k}^{(i+1)}$ and $\hat{b}_{k|k}^{(i+1)}$ using (21)-(22) and (26)-(27)

11. Compute $E^{(i+1)}[\mathbf{R}_k^{-1}]$ and $E^{(i+1)}[\nu_k]$ using (28) and (30)

End For

12. $\hat{x}_{k|k} \leftarrow \hat{x}_{k|k}^{(N)}$, $\mathbf{P}_{k|k} \leftarrow \mathbf{P}_{k|k}^{(N)}$, $\hat{u}_{k|k} \leftarrow \hat{u}_{k|k}^{(N)}$, $\mathbf{U}_{k|k} \leftarrow \mathbf{U}_{k|k}^{(N)}$, $\hat{a}_{k|k} \leftarrow \hat{a}_{k|k}^{(N)}$, $\hat{b}_{k|k} \leftarrow \hat{b}_{k|k}^{(N)}$

Outputs: $\hat{x}_{k|k}$, $\mathbf{P}_{k|k}$, $\hat{u}_{k|k}$, $\mathbf{U}_{k|k}$, $\hat{a}_{k|k}$, $\hat{b}_{k|k}$

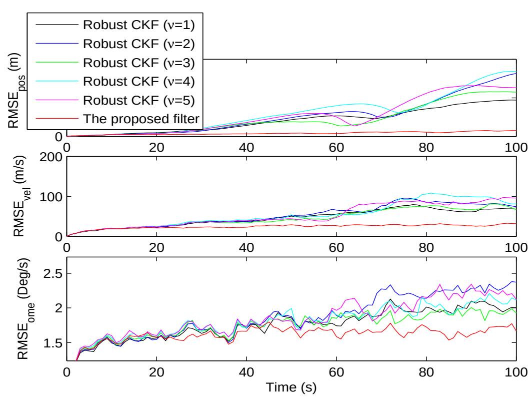

Fig. 1: RMSEs of existing robust CKF and the proposed filter

(RMSEs) of position, velocity and turn rate, which were defined in [4], are chosen as performance metrics.

Fig.1 shows the RMSEs of the proposed filter and existing robust CKF with fixed dof parameters $\nu = 1, 2, 3, 4, 5$ when $p_c = 0.3$. It can be seen from Fig. 1 that the proposed

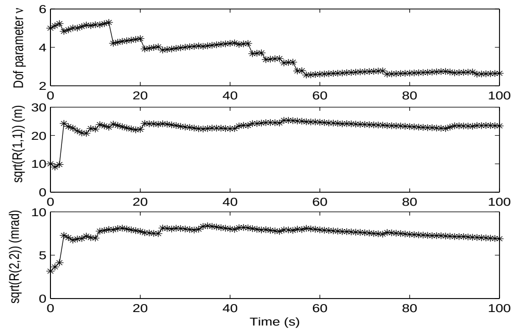

Fig. 2: Estimated dof parameter, $\sqrt{\mathbf{R}(1,1)}$ and $\sqrt{\mathbf{R}(2,2)}$

Table 1: Averaged estimate of dof parameter and diagonal elements of scale matrix

p_c	ν	$\sqrt{\mathbf{R}(1,1)}$(m)	$\sqrt{\mathbf{R}(2,2)}$(mrad)
0	6.702	10.486	3.432
0.1	3.733	15.722	5.003
0.2	2.961	21.152	6.732
0.3	2.724	27.276	8.682
0.4	2.705	34.223	10.972

filter has higher estimation accuracy than existing robust GA filter. Fig. 2 shows the estimated dof parameter, $\sqrt{\mathbf{R}(1,1)}$ and $\sqrt{\mathbf{R}(2,2)}$ from the proposed filter when $p_c = 0.3$. It can be seen from Fig. 2 that the estimated dof parameter is time varying, and the estimated $\sqrt{\mathbf{R}(1,1)}$ and $\sqrt{\mathbf{R}(2,2)}$ are double σ_r and σ_θ respectively. This justifies from another aspect the proposed filter benefits over the existing robust GA filter with fixed scale matrix $\mathbf{\Sigma}^v$ and fixed dof parameters. Table 1 shows the averaged estimate of dof parameter, $\sqrt{\mathbf{R}(1,1)}$ and $\sqrt{\mathbf{R}(2,2)}$ from the proposed filter when $p_c = 0, 0.1, 0.2, 0.3, 0.4$. We can see from Table 1 that ν decreases and $\sqrt{\mathbf{R}(1,1)}$ and $\sqrt{\mathbf{R}(2,2)}$ are far away from σ_r and σ_θ respectively as p_c increases.

4. CONCLUSION

In this paper, the authors focused on solving the problem of the choices of scale matrix and dof parameter. An improved robust GA filter was proposed based on the VB approach to estimate the state together with unknown scale matrix and dof parameter. The performance of the proposed filter was tested in the simulation of target tracking with measurement outliers. In our simulation results, the proposed filter showed higher estimation accuracy than the existing robust GA filter with fixed scale matrix and dof parameter, which is induced by the fact that the proposed filter can iteratively find better estimate of scale matrix and dof parameter.

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