ON SPARSE CONTROLLABILITY OF GRAPH SIGNALS

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ABSTRACT

Controlling the behavior of a signal defined over a graph by acting on a limited set of nodes is a problem that finds application in many fields. In this paper, we merge recently developed tools in graph signal processing with control theory of complex networks and consider the reconstruction of bandlimited graph signals from their samples through a diffusion process properly driven by a subset of control nodes. Then, we propose an optimization algorithm aimed at minimizing the control energy incorporating a regularization term whose goal is to promote sparsity across time and nodes jointly.

Index Terms— Graph signals, controllability, complex networks, sparse control.

1. INTRODUCTION

In many fields of current interest, the observed data can be represented as signals over a graph [1]. The signal is a vector whose entries are associated to the vertices of the graph, whereas the edges represent relations between nodes that interact directly with each other. Relevant examples are biological networks, smart grids or social networks. Motivated by the ubiquity and widespread applicability of the graph signal model, in recent years the signal processing research community has dedicated many efforts to devise specific tools for analyzing signals defined over a graph, or graph signal for short. In this paper we address the controllability of a graph signal, meaning the capability of driving a graph signal to assume a desired behavior, by acting on a limited number of nodes. This problem was recently addressed in [2]. This is indeed one of the basic problems in control theory, namely how to drive a dynamical system towards a desired state. More recently, the interest in controlling dynamical system has received a further push towards the control of complex networks. In spite of the efforts and progress, the control of a complex dynamical, typically nonlinear, network still remains an unsolved problem, because of the complicated interplay between network topologies and nonlinear dynamics. Even restricting the problem to linear systems, the problem of identifying the minimum set of nodes to be controlled in order to drive the whole network to a desired state is still open, especially for large size networks. Historically, the Kalman and later the equivalent Popov-Belevitch-Hautus (PBH) rank conditions have long been the basic tools to check controllability of a linear time-invariant system [3], [4]. A relatively recent ground-breaking contribution was given in [5], where the identification of the minimal set of nodes enabling full network control of complex networks based on graph-theoretic tools derived from structural control theory [6]. A very conceptually simple method to identify the minimum set of control nodes was recently expressed in terms of the geometric multiplicities of the eigenvalues of the state matrix in [7]. However, the practical implementation of this method over large size networks is critical because of numerical instabilities related to the identification of eigenvalues' multiplicities. Furthermore, in a practical setting, the search for the set of control nodes should be considered together with the energy to be spent to implement the control over a give time horizon. This issue was recently tackled in [8], where the authors specifically studied the trade-off between the number of control nodes and the energy to be spent to enforce the control. In this paper, we address this same problem and propose a few alternative algorithms aimed at striking an optimal trade-off between control energy and cardinality of the set of control nodes. More specifically, we formulate the minimum energy control problem by introducing alternative sparse regularization terms aimed at penalizing the use of large sets of control nodes.

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2. MINIMUM ENERGY CONTROLLABILITY

Let us consider a network represented by a graph $\mathcal{G} \triangleq (\mathcal{V}, \mathcal{E})$ where $\mathcal{V} \triangleq (1, \ldots, N)$ is the sets of N nodes and $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ is the set of edges. Let $\boldsymbol{W} = (w_{ij})_{i,j=1}^{N}$ denote the adjacency matrix associated with the graph $\mathcal{G}$ with entries $w_{ij} = 1$ if there is a link from node j to i and $w_{ij} = 0$ otherwise. The input degree of node i is defined as $d_i \triangleq \sum_{j=1}^{N} w_{ij}$ and $\boldsymbol{D} \triangleq \text{diag}(d_1, \ldots, d_N)$ is the associated diagonal degree matrix. The graph Laplacian matrix $\boldsymbol{L}$ is then $\boldsymbol{L} \triangleq \boldsymbol{D} - \boldsymbol{W}$. We associate to the network a state vector at time t defined as a mapping from the vertex set to real vectors of size N, i.e. $\boldsymbol{x}(t) : \mathcal{V} \rightarrow \mathbb{R}^N$. The evolution of the network state over time is described by the continuous, time-invariant, linear system

$$\dot{\boldsymbol{x}}(t) = -\boldsymbol{L}\boldsymbol{x}(t) \tag{1}$$

initialized with $\boldsymbol{x}(0) = \boldsymbol{x}_0$ where $\boldsymbol{x}_0$ is the vector at time $t = 0$. The equivalent discrete time evolution of (1) can be

expressed as

$$\boldsymbol{x}_{k+1} = (\boldsymbol{I} - \epsilon \boldsymbol{L})\boldsymbol{x}_k := \boldsymbol{A}\boldsymbol{x}_k \tag{2}$$

where we have introduced the so called state (or transition) matrix $\boldsymbol{A} = \boldsymbol{I} - \epsilon \boldsymbol{L}$. To guarantee the stability of the system, we assume $0 < \epsilon \leq 1/d_{max}$ with $d_{max} = \max_i d_i$. Steering the state vector from an initial state $\boldsymbol{x}_0$ to a desired final state $\boldsymbol{x}_f$ in a finite time horizon (finite number of steps) amounts to finding a set of M input vectors $\boldsymbol{u}_k$ acting on a subset S of nodes through the following dynamical system

$$\boldsymbol{x}_{k+1} = \boldsymbol{A}\boldsymbol{x}_k + \boldsymbol{B}\boldsymbol{u}_k \tag{3}$$

where

$$\boldsymbol{B} \triangleq [\boldsymbol{e}_{k_1}, \ldots, \boldsymbol{e}_{k_M}] \tag{4}$$

is the $N \times M$ control input matrix with $\boldsymbol{e}_{k_i}$ the N-dimensional canonical vector. Let us denote with S the set of $M \leq N$ control nodes, i.e. $S \triangleq \{s_1, \ldots, s_M\} \subseteq \mathcal{V}$. The dynamic of a network controlled at each time k with M external control inputs can be formulated as in (3) where $\boldsymbol{u}_k : \mathcal{S} \to \mathbb{R}^M$ is the control signal imposed on the set S of controlled nodes. The dynamic system in (3) is *controllable* in $K \in \mathbb{N}$ steps by the set of nodes S if and only if, for any desired final state $\boldsymbol{x}_f$, there exists an input $\boldsymbol{u}$ driving the network to $\boldsymbol{x}(K) = \boldsymbol{x}_f$ from the initial state $\boldsymbol{x}_0 = \boldsymbol{0}$. This is possible if and only if the $N \times KM$ controllability matrix

$$\boldsymbol{C} = [\boldsymbol{B}, \boldsymbol{A}\boldsymbol{B}, \ldots, \boldsymbol{A}^{K-1}\boldsymbol{B}] \tag{5}$$

has full row rank, that is rank$(\boldsymbol{C}) = N$. This represents the so called Kalman's controllability rank condition [4]. Equivalently, defining the K-steps controllability Gramian symmetric matrix as

$$\boldsymbol{G} \triangleq \sum_{m=0}^{K-1} \boldsymbol{A}^m \boldsymbol{B}\boldsymbol{B}^T (\boldsymbol{A}^T)^m = \boldsymbol{C}\boldsymbol{C}^T \tag{6}$$

the network is controllable in K steps by the nodes S if and only if $\boldsymbol{G}$ is positive definite. From a practical point of view, besides checking the rank condition, it is important to evaluate how much energy is necessary to drive the graph signal to the desired behavior within a finite time. To this end, let us define the energy to control the network in K steps as

$$\mathcal{E}_K = \| \boldsymbol{u} \|^2 = \sum_{k=0}^{K-1} \| \boldsymbol{u}_k \|^2 \tag{7}$$

where $\boldsymbol{u}$ is the $M \cdot K$-dimensional vector obtained by stacking the $M \times 1$ vectors $\boldsymbol{u}_k$, $k = 0, \ldots, K-1$. The optimal control signals $\boldsymbol{u}_k$ minimizing the energy $\mathcal{E}_K$ necessary to drive the overall graph signal from $\boldsymbol{x}_0$ to $\boldsymbol{x}_f$ is [4], [9]:

$$\boldsymbol{u}_k^\star = \boldsymbol{B}^T (\boldsymbol{A}^T)^{K-k-1} \boldsymbol{G}^{-1} (\boldsymbol{x}_f - \boldsymbol{A}^K \boldsymbol{x}_0), \quad k = 0, \ldots, K-1 \tag{8}$$

and the corresponding minimum energy is

$$\mathcal{E}_K = \| \boldsymbol{u} \|^2 = (\boldsymbol{x}_f - \boldsymbol{A}^K \boldsymbol{x}_0)^T \boldsymbol{G}^{-1} (\boldsymbol{x}_f - \boldsymbol{A}^K \boldsymbol{x}_0). \tag{9}$$

3. RECONSTRUCTION OF BANDLIMITED GRAPH SIGNALS FROM SPARSE SAMPLES THROUGH DIFFUSION PROCESS

In many cases of interest, graph signals exhibit a clustered behavior meaning that the signal is relatively smooth within each cluster whereas it can assume different values across different clusters. This class of signals typically admits a parsimonious representation based on the recently introduced Graph Fourier Transform (GFT) [1], [10]. Alternative definition exists, based either on the adjacency or on the Laplacian matrix. In the latter case, if the graph is undirected, the Laplacian matrix is symmetric and positive semi-definite. Hence, it may be diagonalized as

$$\mathbf{L} = \mathbf{V}\boldsymbol{\Lambda}\mathbf{V}^T = \sum_{i=1}^{N} \lambda_i \boldsymbol{v}_i \boldsymbol{v}_i^T, \tag{10}$$

where $\boldsymbol{\Lambda}$ is a diagonal matrix with non-negative real eigenvalues $\{\lambda_i\}$ on its diagonal, $\{\boldsymbol{v}_i\}$ is the set of real-valued orthonormal eigenvectors. From spectral graph theory, see e.g., [11], the Laplacian eigenvectors are well known to possess useful clustering properties. The GFT of a vector $\boldsymbol{x}$ is then defined as the projection of $\boldsymbol{x}$ onto the space spanned by the Laplacian eigenvectors, see e.g. [1]:

$$\hat{\boldsymbol{x}} = \mathbf{V}^T \boldsymbol{x} \tag{11}$$

with inverse Fourier transform $\boldsymbol{x} = \mathbf{V}\,\hat{\boldsymbol{x}}$. Given a subset of vertices $\mathcal{S} \subseteq \mathcal{V}$, we define a vertex-limiting operator as a diagonal matrix $\mathbf{D}_\mathcal{S}$ such that $\mathbf{D}_\mathcal{S} = \mathrm{Diag}\{\mathbf{1}_\mathcal{S}\}$ where $\mathbf{1}_\mathcal{S}$ is the set indicator vector, whose i-th entry is equal to one, if $i \in \mathcal{S}$, or zero otherwise. Similarly, the vertex-limiting operator over the complement set $\overline{\mathcal{S}}$, such that $\mathcal{V} = \mathcal{S} \cup \overline{\mathcal{S}}$ and $\mathcal{S} \cap \overline{\mathcal{S}} = \emptyset$, is $\overline{\mathbf{D}} = \boldsymbol{I} - \mathbf{D}_\mathcal{S}$. By duality, given a subset of frequency indices $\mathcal{F}$, we introduce the band-limiting operator

$$\mathbf{B}_\mathcal{F} = \mathbf{V}\boldsymbol{\Sigma}_\mathcal{F}\mathbf{V}^T, \tag{12}$$

where $\boldsymbol{\Sigma}_\mathcal{F}$ is a diagonal matrix defined as $\boldsymbol{\Sigma}_\mathcal{F} = \mathrm{Diag}\{\mathbf{1}_\mathcal{F}\}$. It is immediate to check that both matrices $\mathbf{D}_\mathcal{S}$ and $\mathbf{B}_\mathcal{F}$ represent orthogonal projectors. A signal $\boldsymbol{x}$ is perfectly bandlimited over a frequency set $\mathcal{F}$ if $\mathbf{B}_\mathcal{F}\boldsymbol{x} = \boldsymbol{x}$. Sampling a graph signal $\boldsymbol{x}$ over a set $\mathcal{S}$ of vertices gives rise to a vector $\boldsymbol{x}_s := \mathbf{D}_\mathcal{S}\boldsymbol{x}$. The basic question with sampling is how to reconstruct the overall signal from its samples. It was shown in [12] that it is possible to recover the overall vector $\boldsymbol{x}$ from its sampled counterpart $\boldsymbol{x}_s$ if and only if the matrix $\overline{\mathbf{D}}\mathbf{B}_\mathcal{F}$ has spectral norm strictly less than one. If such condition holds, the reconstruction formula is [12]:

$$\boldsymbol{x} = \left(\boldsymbol{I} - \overline{\mathbf{D}}\mathbf{B}_\mathcal{F}\right)^{-1} \boldsymbol{x}_s. \tag{13}$$

Within this framework, it is now of interest to check whether it is possible to enforce a bandlimited behavior $\boldsymbol{x}$ over the entire graph through a diffusion process properly driven through

a set of control nodes. This problem can be cast as in (3), where $\boldsymbol{x}_0 = \boldsymbol{x}_s$ and $\boldsymbol{x}_f = \boldsymbol{x}$. The optimal driving input is then as in (8). The controllability condition, or equivalently the invertibility of the Gramian matrix $\boldsymbol{G}$, requires now that there is no eigenvector of the graph Laplacian that is orthogonal to all columns of matrix $\boldsymbol{B}$ in (3). This means that, choosing $\boldsymbol{B}$ as in (4), there is no eigenvector $\boldsymbol{v}_i$ such that its samples corresponding to the position of the driving nodes (identified by the columns of $\boldsymbol{B}$) are all null. As a particular case, this implies that, if the graph is disconnected, there should be at least one driving node for every disconnected component, as expected. But also in case of a connected graph, these arguments suggest that the position of the driving nodes should avoid locations where all eigenvectors of the Laplacian assume very low values, to avoid ill-conditioning of the Gramian and, consequently, high control energies.

A numerical example, for a random geometric graph, is reported in Fig. 1, showing the energy spent for reconstructing a bandlimited signal whose spectrum is confined to the first 4 eigenvectors of the Laplacian ($|\mathcal{F}| = 4$) vs. the number of control nodes. For each number of control nodes, we picked up the optimal set as the one that minimizes the reconstruction energy. We can see that the signal can indeed be reconstructed through a diffusion process but that the smaller is the number of control nodes, the higher is the energy necessary for perfect reconstruction.

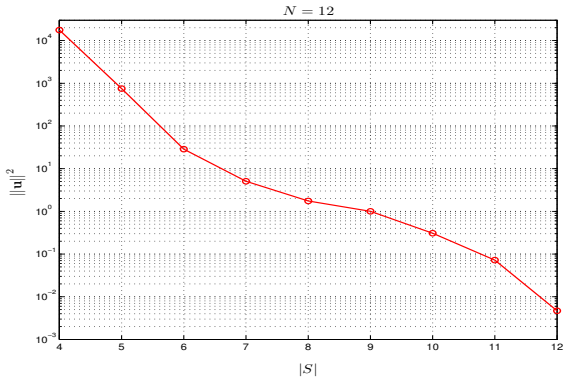

Fig. 1. Minimum control energy vs. number of control nodes.

4. OPTIMAL MINIMUM-ENERGY SPARSE CONTROL

The trade-offs between control energy and number of control nodes has been investigated in several papers [8], [5]. In particular in [8] it has been shown that some (theoretically) controllable networks are practically uncontrollable as the control energy grows exponentially with the ratio between the network size and the number of control nodes. Then identifying the optimal set of driver nodes to minimize energy consumption represents a fundamental topic. Sparse feedback control has been considered in several papers, see e.g. [13], [14]. In the following, we focus on the open loop (i.e., with no feedback) sparse control problem and our goal is to design the control inputs in order to strike the optimal balance between control energy and number of control nodes by acting, jointly, over time and network nodes, within a finite time horizon. We formulate our minimum energy control incorporating a sparsity constraint as

$$\begin{aligned} \min_{\substack{\boldsymbol{u}\in\mathbb{R}^{M\cdot K}\\ \boldsymbol{x}\in\mathbb{R}^{N\cdot(K+1)}}} \quad & J_0(\boldsymbol{u}) := \boldsymbol{u}^T\boldsymbol{u} + \beta\|\boldsymbol{u}\|_0, \qquad (\mathcal{P}_0) \\ \text{s.t.} \quad & \left.\begin{aligned} & \boldsymbol{x}_{k+1} = \boldsymbol{A}\boldsymbol{x}_k + \boldsymbol{B}\boldsymbol{u}_k,\ k = 0,\dots,K-1 \\ & \boldsymbol{x}_0 = \boldsymbol{0}, \\ & \boldsymbol{x}_K = \boldsymbol{x}_f \end{aligned}\right\} \triangleq \mathcal{X} \end{aligned} \tag{14}$$

defining $\boldsymbol{x} \doteq (\boldsymbol{x}_k)_{k=0}^K$. Note that in our model, sparsity of the input at each time step means that for each time k only a few columns of the matrix (basis) $\boldsymbol{B}$ are excited. To efficiently solve the non-convex L_0 norm problem in (14) we relax it by seeking the solution of a L_1 norm regularization term, i.e.

$$\begin{aligned} \min_{\boldsymbol{u}\in\mathbb{R}^{M\cdot K},\boldsymbol{x}\in\mathbb{R}^{N\cdot(K+1)}} \quad & J_1(\boldsymbol{u}) := \boldsymbol{u}^T\boldsymbol{u} + \beta\|\boldsymbol{u}\|_1, \qquad (\mathcal{P}_1) \\ \text{s.t.} \quad & (\boldsymbol{x},\boldsymbol{u}) \in \mathcal{X}. \end{aligned} \tag{15}$$

The latter formulation is amenable to an iterative solution strategy by using the alternating direction method of multipliers (ADMM), see [13],[14],[15] so that an iterative and efficient algorithm can be designed to solve it as discussed in details in the following section.

4.1. ADMM algorithm

By introducing the auxiliary variable $\boldsymbol{v} \in \mathbb{R}^{M\cdot K}$ we can write problem $\mathcal{P}_1$ in the equivalent form

$$\begin{aligned} \min_{\substack{\boldsymbol{u},\boldsymbol{v}\in\mathbb{R}^{M\cdot K}\\ \boldsymbol{x}\in\mathbb{R}^{N\cdot(K+1)}}} \quad & \boldsymbol{u}^T\boldsymbol{u} + \beta\|\boldsymbol{v}\|_1 + \frac{\rho}{2}\|\boldsymbol{u}-\boldsymbol{v}\|_2^2 \\ \text{s.t.} \quad & \text{a) } \boldsymbol{x}_{k+1} = \boldsymbol{A}\boldsymbol{x}_k + \boldsymbol{B}\boldsymbol{u}_k,\ k = 0,\dots,K-1 \\ & \text{b) } \boldsymbol{u}_k = \boldsymbol{v}_k, \qquad k = 0,\dots,K-1 \\ & \text{c) } \boldsymbol{x}_0 = \boldsymbol{0}, \\ & \text{d) } \boldsymbol{x}_K = \boldsymbol{x}_f \end{aligned} \tag{16}$$

where the introduction of the constraints $\boldsymbol{u}_k = \boldsymbol{v}_k$ leads to decouple the objective function with respect to $\boldsymbol{u}_k$ and $\boldsymbol{v}_k$ by ensuring, through the quadratic penalty, the strongly convexity of the problem for $\rho > 0$. The augmented Lagrangian associated with (16) is given by

$$\begin{aligned} \mathcal{L}_\rho(\boldsymbol{x},\boldsymbol{u},\boldsymbol{v},\boldsymbol{\lambda},\boldsymbol{z}) = {} & H_0(\boldsymbol{x}_0,\boldsymbol{u}_0,\boldsymbol{v}_0,\boldsymbol{\lambda}_1,\boldsymbol{z}_0) - \boldsymbol{\lambda}_K^T\boldsymbol{x}_K \\ & + \sum_{k=1}^{K-1}\left(H_k(\boldsymbol{x}_k,\boldsymbol{u}_k,\boldsymbol{v}_k,\boldsymbol{\lambda}_{k+1},\boldsymbol{z}_k) - \boldsymbol{\lambda}_k^T\boldsymbol{x}_k\right) \end{aligned}$$

where $\boldsymbol{\lambda} \triangleq (\boldsymbol{\lambda}_k)_{k=1}^K$ and $\boldsymbol{z} \triangleq (\boldsymbol{z}_k)_{k=0}^{K-1}$ are the Lagrangian multipliers associated respectively with the constraints a) and b) and the Hamiltonian function H_k is defined as

$$\begin{aligned} H_k(\boldsymbol{x}_k,\boldsymbol{u}_k,\boldsymbol{v}_k,\boldsymbol{\lambda}_{k+1},\boldsymbol{z}_k) = {} & \boldsymbol{u}_k^T\boldsymbol{u}_k + \beta\|\boldsymbol{v}_k\|_1 + \frac{\rho}{2}\|\boldsymbol{u}_k-\boldsymbol{v}_k\|_2^2 \\ & + \boldsymbol{\lambda}_{k+1}^T(\boldsymbol{A}\boldsymbol{x}_k + \boldsymbol{B}\boldsymbol{u}_k) + \boldsymbol{z}_k^T(\boldsymbol{u}_k - \boldsymbol{v}_k). \end{aligned}$$

According to the Lagrange-multiplier theory necessary conditions for a constrained minimum can be found by deriving $\mathcal{L}_\rho$ with respect to $(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{v}, \boldsymbol{\lambda}, \boldsymbol{z})$. Hence we get

$$\frac{\partial \mathcal{L}_\rho}{\partial \boldsymbol{x}_k} = \boldsymbol{\lambda}_{k+1}^T \boldsymbol{A} - \boldsymbol{\lambda}_k^T = 0, \qquad k = 1, \ldots, K-1$$
$$\frac{\partial \mathcal{L}_\rho}{\partial \boldsymbol{\lambda}_{k+1}} = \boldsymbol{A}\boldsymbol{x}_k + \boldsymbol{B}\boldsymbol{u}_k - \boldsymbol{x}_{k+1} = 0, \quad k = 0, \ldots, K-1$$

which leads to

$$\begin{aligned} \boldsymbol{\lambda}_k &= (\boldsymbol{A}^T)^{K-k} \boldsymbol{\lambda}_K, \qquad k = 1, \ldots, K-1 \\ \boldsymbol{x}_{k+1} &= \boldsymbol{A}\boldsymbol{x}_k + \boldsymbol{B}\boldsymbol{u}_k, \quad k = 0, \ldots, K-1. \end{aligned} \tag{17}$$

The ADMM algorithm is based on the iterative solution of a sequence of convex problems so that we will determine $\boldsymbol{u}$, $\boldsymbol{v}$ and $\boldsymbol{z}$ through the following steps:

$$\begin{aligned} \boldsymbol{u}^{n+1} &= \text{argmin}_{\boldsymbol{u}} \; \mathcal{L}_\rho(\boldsymbol{x}^n, \boldsymbol{u}, \boldsymbol{v}^n, \boldsymbol{\lambda}^n, \boldsymbol{z}^n) \\ \boldsymbol{v}^{n+1} &= \text{argmin}_{\boldsymbol{v}} \; \mathcal{L}_\rho(\boldsymbol{x}^n, \boldsymbol{u}^{n+1}, \boldsymbol{v}, \boldsymbol{\lambda}^n, \boldsymbol{z}^n) \\ \boldsymbol{z}^{n+1} &= \boldsymbol{z}^n + \rho(\boldsymbol{u}^{n+1} - \boldsymbol{v}^{n+1}) \end{aligned} \tag{18}$$

i.e. the optimal $\boldsymbol{u}_k$ and $\boldsymbol{v}_k$ are found in an alternating fashion and the dual update step adopts a step-size equal to ρ by ensuring the dual feasibility of $(\boldsymbol{u}^n, \boldsymbol{v}^n, \boldsymbol{z}^n)$ at each iteration. After some simple manipulations we yield from (18) for each k

$$\begin{aligned} \boldsymbol{u}_k^{n+1} &= (2+\rho)^{-1}(\rho \boldsymbol{v}_k^n - \boldsymbol{z}_k^n - \boldsymbol{B}^T \boldsymbol{\lambda}_{k+1}^n) \\ \boldsymbol{v}_k^{n+1} &= t_{\beta/\rho}(\boldsymbol{u}_k^{n+1} + \boldsymbol{z}_k^n/\rho) \\ \boldsymbol{z}_k^{n+1} &= \boldsymbol{z}_k^n + \rho(\boldsymbol{u}_k^{n+1} - \boldsymbol{v}_k^{n+1}) \end{aligned} \tag{19}$$

where $t_{\beta/\rho}(\boldsymbol{y})$ is an element-wise non-linear operator defined as

$$t_{\beta/\rho}(y_k) = \begin{cases} y_k - \beta/\rho & \text{if} \quad y_k \geq \beta/\rho \\ y_k + \beta/\rho & \text{if} \quad y_k \leq -\beta/\rho \\ 0 & \text{if} \quad -\beta/\rho < y_k < \beta/\rho. \end{cases} \tag{20}$$

To find a closed-form expression for $\boldsymbol{\lambda}_{k+1}^n$ we can replace in (17) the $\boldsymbol{u}_k^{n+1}$ expression in (19) by obtaining

$$\begin{aligned} \boldsymbol{\lambda}_K^n &= -(2+\rho)\boldsymbol{G}^{-1}\left[\boldsymbol{x}_f - \sum_{k=0}^{K-1} \frac{\boldsymbol{A}^{K-1-k}\boldsymbol{B}}{2+\rho}(\rho \boldsymbol{v}_k^n - \boldsymbol{z}_k^n)\right] \\ \boldsymbol{\lambda}_k^n &= (\boldsymbol{A}^T)^{K-k}\boldsymbol{\lambda}_K^n, \quad \forall k. \end{aligned}$$

The final ADMM iterative algorithm is described in Algorithm I. As a numerical example, in Fig. 2, we report the activation pattern of a graph across time (active nodes are in red, inactive nodes in blue). Rather interestingly, we can see that, initially, for most of the time only a few nodes are active. Only as time approach the horizon, the number of active nodes increases to meet the requirement on reaching the desired overall pattern within a finite time. Finally, in Fig. 3 we show the minimum energy consumption versus the percentage of control nodes obtained by varying the sparsity coefficient β for different time horizon K. It can be noted that, as

Algorithm 1 : ADMM iterative algorithm

Set $\boldsymbol{x}_0 = \boldsymbol{0}$, $\boldsymbol{x}_K = \boldsymbol{x}_f$, $0 < \epsilon \ll 1$, $n = 0$, initialize
$\boldsymbol{z}^n$, $\boldsymbol{v}^n$, $\boldsymbol{u}^n$ randomly;

Repeat

(a): $\boldsymbol{\lambda}_K^n = -\boldsymbol{G}^{-1}[(2+\rho)\boldsymbol{x}_f - \sum_{k=0}^{K-1} \boldsymbol{A}^{K-1-k}\boldsymbol{B}(\rho \boldsymbol{v}_k^n - \boldsymbol{z}_k^n)]$

(b): $\boldsymbol{\lambda}_k^n = (\boldsymbol{A}^T)^{K-k}\boldsymbol{\lambda}_K^n, \quad \forall k$

(c): $\boldsymbol{u}_k^{n+1} = (2+\rho)^{-1}(\rho \boldsymbol{v}_k^n - \boldsymbol{z}_k^n - \boldsymbol{B}^T \boldsymbol{\lambda}_{k+1}^n), \quad \forall k$

(d): $\boldsymbol{v}_k^{n+1} = t_{\beta/\rho}(\boldsymbol{u}_k^{n+1} + \boldsymbol{z}_k^n/\rho), \quad \forall k$

(e): $\boldsymbol{z}_k^{n+1} = \boldsymbol{z}_k^n + \rho(\boldsymbol{u}_k^{n+1} - \boldsymbol{v}_k^{n+1}), \quad \forall k$

(f): $n = n+1$

until $\| \boldsymbol{u}^n - \boldsymbol{u}^{n-1} \|^2 + \| \boldsymbol{v}^n - \boldsymbol{v}^{n-1} \|^2 \leq \epsilon$.

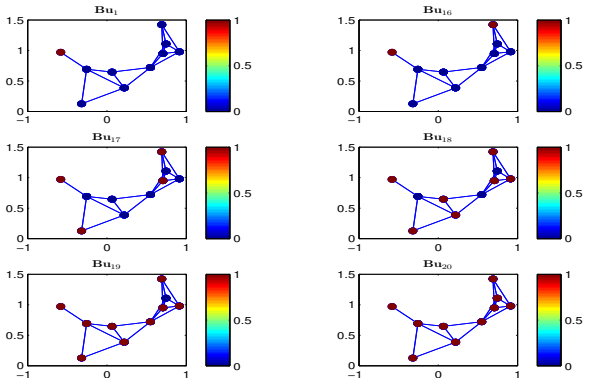

Fig. 2. Activation pattern across time.

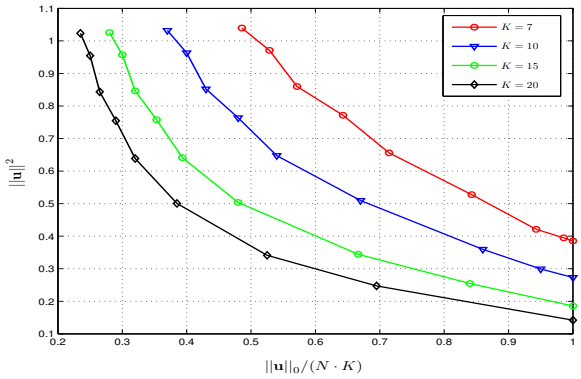

Fig. 3. Minimum energy vs. the percentage of control nodes, for different time horizons.

K increases, a given control energy can be achieved with a lower number of control nodes.

In conclusion, in this paper, we have proposed algorithms aimed to strike the best trade-off between the control energy and the number of control nodes by promoting sparsity across time and nodes jointly. As a particular case, we considered the reconstruction of band-limited graph signals through a diffusion process properly driven by a subset of control nodes.

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