GRAPH FILTER BANKS WITH M-CHANNELS, MAXIMAL DECIMATION, AND PERFECT RECONSTRUCTION

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ABSTRACT

Signal processing on graphs finds applications in many areas. Motivated by recent developments, this paper studies the concept of spectrum folding (aliasing) for graph signals under the downsample-then-upsample operation. In this development, we use a special eigenvector structure that is unique to the adjacency matrix of M-block cyclic matrices. We then introduce M-channel maximally decimated filter banks. Manipulating the characteristics of the aliasing effect, we construct polynomial filter banks with perfect reconstruction property. Later we describe how we can remove the eigenvector condition by using a generalized decimator. In this study graphs are assumed to be general with a possibly non-symmetric and complex adjacency matrix.

Index Terms— Multirate processing of graph signals, aliasing on graphs, bandlimited graph signals, PR filterbanks on graphs.

1. INTRODUCTION

In many circumstances, signals cannot be represented via time series and often have irregular patterns. Such signals can be expressed using a graph which models the underlying dependency structure between the data sources. This type of signal structure can be found in a variety of different contexts such as sensor networks [1], social and economic networks [2], biological networks [3] and others [4–8]. A detailed introduction to processing of such graph signals can be found in [5], and in the tutorial articles [6, 7].

There are two different approaches to graph signal processing. The first set of papers [7–12] showed how two-channel filter banks with perfect reconstruction (PR) property can be constructed on bipartite graphs. These results were developed for graphs which have a real, symmetric adjacency matrix, and all results were based on the graph Laplacian. In the other approach [5,6,13], the graph adjacency matrix $\boldsymbol{A}$ was allowed to be arbitrary – possibly non-symmetric (and complex). By proposing that the adjacency matrix can be regarded as a graph-shift operator, an extension of the basic concepts of linear shift invariant systems on graphs was developed in [5, 13]. Further extensions of these results were also developed in [14–17], and many other important aspects of graph filter banks are studied in [18–20].

Inspired by the pioneering contributions of [5] and [8], this paper studies the concept of aliasing in graph signals and develops the construction of M-channel maximally decimated filter banks (FB) with perfect reconstruction property on graphs. A complete discussion about multirate graph signal processing can be found in [21–23].

In the following, we first provide a brief review of DSP on graphs, define the decimator and expander operators and introduce M-block cyclic graphs. These graphs have a unique eigenvector structure which allows us to extend a number of multirate results to such graphs. Using this special eigenvector structure, we then consider the downsample-then-upsample operation and observe the aliasing effect on graph signals in Section 3. In Section 4 we introduce maximally decimated M-channel filter banks and explain

how we can construct ideal perfect reconstruction filter banks for graphs with the said eigenvector structure. In Section 5, we consider a similarity-transformation on the graph and remove the eigenvector constraint by using generalized decimators and expanders.

2. PRELIMINARIES

In the following, $\mathcal{C}^N$ denotes the set of complex column vectors of size N; $\mathcal{C}^{N \times M}$ denotes the set of $N \times M$ complex matrices; $\mathcal{M}^N$ denotes the set of square matrices with size N. $\boldsymbol{A}^T$ is the transpose of $\boldsymbol{A}$. We use $\otimes$ to denote the Kronecker product.

2.1. Digital Signal Processing on Graphs

We will follow the construction presented in [5, 13]. Let $\boldsymbol{x} \in \mathcal{C}^N$ be a graph signal on a graph of size N (i.e., N nodes or vertices) whose adjacency matrix is denoted as $\boldsymbol{A} \in \mathcal{M}^N$. We will call a graph with the adjacency matrix $\boldsymbol{A}$ simply as graph $\boldsymbol{A}$. i^{th} node of the graph is supposed to produce the ith element of the signal $\boldsymbol{x}$. $a_{i,j}$ denotes the $(i, j)^{th}$ element of $\boldsymbol{A}$ and is the weight of the edge from the jth node to the ith node. When $a_{i,j} = 0$, it means that there is no edge. We consider the general case and allow $a_{i,j}$ to be different from $a_{j,i}$. Moreover, edge weights can be complex valued, i.e. $a_{i,j} \in \mathcal{C}$.

In DSP on graphs, the adjacency matrix of the graph of interest is considered to be *the unit shift* operator for a signal on the graph [5]. Namely, let $\boldsymbol{x}$ be a signal on a graph $\boldsymbol{A}$. Then the signal $\boldsymbol{y}$ computed as $\boldsymbol{y} = \boldsymbol{Ax}$ is called as the unit shifted version of $\boldsymbol{x}$.

In general, any square matrix of size N, $\boldsymbol{H} \in \mathcal{M}^N$, is considered as a *linear graph filter* on the graph. In this study, we are interested in a special type of linear filters, namely polynomial filters. A linear filter on a graph $\boldsymbol{A}$ is said to polynomial if it can be written as

$$\boldsymbol{H} = H(\boldsymbol{A}) = \sum_{k=0}^{N-1} h_k \, \boldsymbol{A}^k, \qquad (1)$$

for some set of $h_k \in \mathcal{C}$. For a graph with the adjacency matrix $\boldsymbol{A}$, let the following denote the eigenvalue decomposition (Jordan form if $\boldsymbol{A}$ is not diagonalizable [5, 24]) of the adjacency matrix

$$\boldsymbol{A} = \boldsymbol{V} \, \boldsymbol{\Lambda} \, \boldsymbol{V}^{-1}, \qquad (2)$$

where $\boldsymbol{V}$ is composed of the (generalized) eigenvectors of the adjacency matrix and $\boldsymbol{\Lambda}$ is diagonal matrix consisting of the eigenvalues of $\boldsymbol{A}$ (or the Jordan normal form of $\boldsymbol{A}$). Using the decomposition in (2), the *graph Fourier transform* of a graph signal $\boldsymbol{x}$ is given as $\widehat{\boldsymbol{x}} = \boldsymbol{V}^{-1}\boldsymbol{x}$; the *Fourier domain operation* of a linear filter $\boldsymbol{H}$ is given as $\widehat{\boldsymbol{H}} = \boldsymbol{V}^{-1}\boldsymbol{H}\boldsymbol{V}$. Notice that for a graph signal $\boldsymbol{x}$ and a linear filter $\boldsymbol{H}$, we have $\boldsymbol{y} = \boldsymbol{H}\boldsymbol{x}$ if and only if $\widehat{\boldsymbol{y}} = \widehat{\boldsymbol{H}}\widehat{\boldsymbol{x}}$.

2.2. Decimator and Expander

One of the most essential building blocks for multirate signal processing is the *decimation operation* [25]. In the graph signal pro-

cessing, we will assume that this operator retains N/M samples of the original graph signal $\boldsymbol{x}$ which has N samples. It will be assumed that M is a divisor of N. Since the numbering of the graph vertices is flexible [5, 26], we will assume without loss of generality that the first N/M samples of $\boldsymbol{x}$ are retained by the decimator:

Definition 1 (Canonical Graph Decimator). *The M-fold graph decimation operator is defined by the matrix*

$$\boldsymbol{D} = \left[\begin{array}{cccc} \boldsymbol{I}_{N/M} & \boldsymbol{0}_{N/M} & \cdots & \boldsymbol{0}_{N/M} \end{array} \right] \in \mathcal{C}^{(N/M) \times N}. \quad (3)$$

Thus, given a graph signal $\boldsymbol{x}$, the decimated graph signal *is $\boldsymbol{D}\boldsymbol{x}$.*

We refer to $\boldsymbol{D}$ as *canonical decimator* with decimation ratio M. This is a mapping from $\mathcal{C}^N$ to $\mathcal{C}^{N/M}$. Similar definitions for the decimator operator have been introduced in [8, 9, 16, 27].

Next, we take the *upsampling operation* as $\boldsymbol{D}^T \in \mathcal{C}^{N \times (N/M)}$ which is a mapping from $\mathcal{C}^{N/M}$ to $\mathcal{C}^N$. This operation merely inserts blocks of zeros, analogous to conventional expanders [25, 28]. This selection of expander is justified in [22]. With these definitions, we have the following equalities for upsample-then-downsample (UD) and downsample-then-upsample (DU) operations:

$$\boldsymbol{D}\boldsymbol{D}^T = \boldsymbol{I}_{N/M}, \qquad \boldsymbol{D}^T\boldsymbol{D} = \left[\begin{array}{cc} \boldsymbol{I}_{N/M} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} \end{array} \right] \in \mathcal{M}^N, \quad (4)$$

where zero blocks have appropriate sizes, respectively.

2.3. M-Block Cyclic Graphs

In this section, we will introduce M-block cyclic graphs [22]. Adjacency matrices of this type of graphs have a unique eigenvector structure which will play a key role in the development of the concept of aliasing. We call a graph (balanced) M-block cyclic when its adjacency matrix has the form

$$\boldsymbol{A} = \left[\begin{array}{ccccc} \boldsymbol{0} & \cdots & \boldsymbol{0} & \boldsymbol{A}_M \\ \boldsymbol{A}_1 & \cdots & \boldsymbol{0} & \boldsymbol{0} \\ \vdots & \ddots & \boldsymbol{0} & \vdots \\ \boldsymbol{0} & \cdots & \boldsymbol{A}_{M\text{-}1} & \boldsymbol{0} \end{array} \right] \in \mathcal{M}^N, \quad (5)$$

where each $\boldsymbol{A}_j \in \mathcal{M}^{N/M}$. See Fig. 1(a) for a 5-block cyclic graph of size 20. Assume that $\boldsymbol{A}$ is diagonalizable. For the eigenvalue decomposition of the adjacency matrix $\boldsymbol{A} = \boldsymbol{V}\boldsymbol{\Lambda}\boldsymbol{V}^{-1}$, if we have the following double indexing scheme

$$\boldsymbol{\Lambda} = diag\Big(\big[\lambda_{1,1} \cdots \lambda_{1,M} \;\cdots\; \lambda_{N/M,1} \cdots \lambda_{N/M,M}\big]\Big), \quad (6)$$

$$\boldsymbol{V} = \Big[\boldsymbol{v}_{1,1} \cdots \boldsymbol{v}_{1,M} \;\cdots\; \boldsymbol{v}_{N/M,1} \cdots \boldsymbol{v}_{N/M,M} \Big], \quad (7)$$

it can be shown that eigenvalues and eigenvectors of M-block cyclic graphs have the following relation [22]

$$\lambda_{i,j+k} = w^k \lambda_{i,j}, \quad (8)$$

$$\boldsymbol{v}_{i,j+k} = \boldsymbol{\Omega}^k \boldsymbol{v}_{i,j}, \quad (9)$$

where

$$\boldsymbol{\Omega} = diag\Big(\big[1 \; w^{\text{-}1} \; \cdots \; w^{\text{-}(M\text{-}1)}\big]\Big) \otimes \boldsymbol{I}_{N/M}, \qquad w = e^{-j2\pi/M}. \quad (10)$$

Notice that the eigenvalues are located on concentric circles, with M equispaced eigenvalues on each circle (See Fig. 1(b)).

3. CONCEPT OF SPECTRUM FOLDING AND ALIASING

Before getting into maximally decimated perfect reconstruction graph filter banks, we need to first define what aliasing is in graph signals. For this purpose we now revisit the downsample-then-upsample (DU) operation. As given in (4), DU replaces samples with zeros, hence it is a lossy operation and the erased samples cannot be reconstructed back from the remaining data in general. We now analyze the effect of the DU operator from the frequency
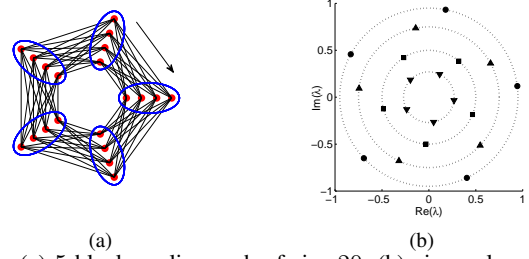

(a) (b)

Fig. 1. (a) 5-block cyclic graph of size 20, (b) eigenvalues of the graph. Notice that all the edges of the graph are directed along with the clock-wise direction.

domain viewpoint, and explain the *spectrum folding* or *aliasing* effect. A similar approach is presented for two-channel systems in [8], where graph signal processing is based on the graph Laplacian. In our development the graph $\boldsymbol{A}$ is allowed to have complex edge weights and can be directed.

Using the canonical definition of the decimator in (3) and eigenvector-shift operator $\boldsymbol{\Omega}$ in (10), the DU operator can be written as a sum of powers of $\boldsymbol{\Omega}$. That is,

$$\boldsymbol{D}^T\boldsymbol{D} = \frac{1}{M} \sum_{k=0}^{M\text{-}1} \boldsymbol{\Omega}^k. \quad (11)$$

Now consider the DU version of a graph signal $\boldsymbol{x}$, namely $\boldsymbol{y} = \boldsymbol{D}^T\boldsymbol{D}\boldsymbol{x}$. Let $\widehat{\boldsymbol{x}}$ and $\widehat{\boldsymbol{y}}$ denote the graph Fourier transform of the input and output signal of the DU system. Let $\boldsymbol{G}$ denote the Fourier domain operation of the DU operator. That is, $\widehat{\boldsymbol{y}} = \boldsymbol{G}\,\widehat{\boldsymbol{x}}$. Therefore we have

$$\boldsymbol{G} = \boldsymbol{V}^{\text{-}1}\boldsymbol{D}^T\boldsymbol{D}\boldsymbol{V}. \quad (12)$$

Using (11), we can write $\boldsymbol{G}$ as follows:

$$\boldsymbol{G} = \frac{1}{M} \sum_{k=0}^{M\text{-}1} \boldsymbol{V}^{\text{-}1}\boldsymbol{\Omega}^k\boldsymbol{V}. \quad (13)$$

In the following, we will not constrain ourselves to M-block cyclic graphs. We will only assume that $\boldsymbol{A}$ is diagonalizable with eigenvectors constrained as in (9), and let the eigenvalues be arbitrary. In Section 5 we will discuss how this assumption on the eigenvectors can be removed by appropriately generalizing the definition of the decimator $\boldsymbol{D}$.

Now notice from (9) that $\boldsymbol{\Omega}^k\boldsymbol{V}$ is the column permuted version of $\boldsymbol{V}$. $\boldsymbol{\Omega}^k$ circularly shifts each vector of an eigen-family to the left by k times. Due to our ordering convention on the eigenvectors in (7), we have the following

$$\boldsymbol{\Omega}^k\boldsymbol{V} = \Big[\boldsymbol{v}_{1,1+k} \cdots \boldsymbol{v}_{1,M+k} \;\cdots\; \boldsymbol{v}_{N/M,1+k} \cdots \boldsymbol{v}_{N/M,M+k} \Big]. \quad (14)$$

Notice that this permutation of the columns of $\boldsymbol{V}$ can also be written with a column permutation matrix. Therefore we have

$$\boldsymbol{\Omega}^k \, \boldsymbol{V} = \boldsymbol{V} \, \boldsymbol{\Pi}_k, \quad (15)$$

where

$$\boldsymbol{\Pi}_k = \boldsymbol{I}_{N/M} \otimes \boldsymbol{C}_M^k = \left[\begin{array}{ccc} \boldsymbol{C}_M^k & \cdots & \boldsymbol{0} \\ \boldsymbol{0} & \ddots & \boldsymbol{0} \\ \boldsymbol{0} & \cdots & \boldsymbol{C}_M^k \end{array} \right], \quad (16)$$

where $\boldsymbol{C}_M$ is the size M cyclic permutation matrix as in Eq. (12) of [13]. Using (15), frequency domain operation, $\boldsymbol{G}$, can be written as:

$$\boldsymbol{G} = \boldsymbol{I}_{N/M} \otimes \frac{1}{M} \sum_{k=0}^{M\text{-}1} \boldsymbol{C}_M^k. \quad (17)$$

Since the consecutive M powers of cyclic permutation matrix of size M add up to matrix with all 1 entries, $\boldsymbol{G}$ further simplifies to

$$\boldsymbol{G} = \boldsymbol{I}_{N/M} \otimes \frac{1}{M} \, \mathbb{1}_M \mathbb{1}_M^T, \qquad (18)$$

where $\mathbb{1}_M$ denotes the column vector of size M with all 1 entries.

To be consistent with double indexing of the eigenvectors, we will stick to that scheme for the frequency components of a graph signal. That is to say,

$$\widehat{\boldsymbol{x}} = [\widehat{x}_{1,1} \cdots \widehat{x}_{1,M} \; \cdots \; \widehat{x}_{N/M,1} \cdots \widehat{x}_{N/M,M}]^T. \qquad (19)$$

Due to (18), we have the following relation between the graph Fourier transform of the original signal and the graph Fourier transform of the downsampled-then-upsampled signal

$$\widehat{y}_{i,1} = \widehat{y}_{i,2} = \cdots = \widehat{y}_{i,M} = \frac{1}{M} \sum_{j=1}^{M} \widehat{x}_{i,j}, \qquad (20)$$

for all $1 \leqslant i \leqslant N/M$. We state this result in the following theorem.

Theorem 1 (Spectrum folding in graph signals). *Let $\boldsymbol{A}$ be the adjacency matrix of a graph. Assume that $\boldsymbol{A}$ diagonalizable and has the eigenvector structure in (9) as indexed in (7) with arbitrary eigenvalues. Let $\boldsymbol{x}$ be a signal on the graph and $\boldsymbol{y} = \boldsymbol{D}^T \boldsymbol{D} \boldsymbol{x}$ where $\boldsymbol{D}$ is as in (3). Then, the graph Fourier transforms of $\boldsymbol{x}$ and $\boldsymbol{y}$ are related as:*

$$\widehat{\boldsymbol{y}} = \frac{1}{M} \left(\boldsymbol{I}_{N/M} \otimes \mathbb{1}_M \mathbb{1}_M^T \right) \widehat{\boldsymbol{x}}. \qquad (21)$$

Thus the DU operation results in the phenomenon described by (20) in the frequency domain. This is similar to *aliasing* or *spectral folding* because multiple frequency components of the input overlap into the same frequency component of the output. This is similar to the effect of decimation in classical signal processing [25]. From the folded spectrum (20) we cannot in general recover the original signal, which is consistent with the fact that decimation is in general a information-lossy operation. It should be remembered however that the expression (20) has been derived only for graphs $\boldsymbol{A}$ for which the eigenvectors have the restricted structure (9).

A similar result for 2-fold decimation is derived in [8] where the analysis is built upon the graph Laplacian. Our result is based on the adjacency matrix of the graph and is applicable to *any* M.

4. GRAPH FILTER BANKS ON GRAPHS WITH THE EIGENVECTOR STRUCTURE

In this section we consider filter banks on graphs and study a class of filter banks which are analogous to ideal brickwall filters (bandlimited filter banks in the classical case) and show that perfect reconstruction can be achieved under some constraints on the eigenstructure of the graph $\boldsymbol{A}$. Such constructions on graphs are insightful. More general systems will be considered elsewhere.

Fig. 2 shows an M-channel maximally decimated graph filter bank where the analysis filters $H_k(\boldsymbol{A})$ and the synthesis filters $F_k(\boldsymbol{A})$ are polynomials in $\boldsymbol{A}$, and $\boldsymbol{D}$ is as in (3). If we allow non-polynomial filters then the problem of constructing PR filter banks becomes rather trivial. It can be shown that polynomial filter banks can be designed to have much smaller complexity that unrestricted graph filter banks, whenever the graph $\boldsymbol{A}$ has simple entries. These details are beyond the scope of this paper and will be elaborated elsewhere.
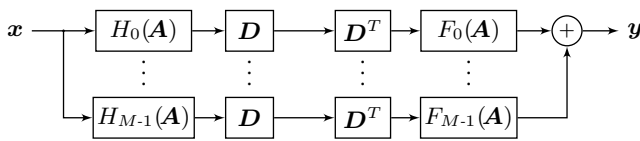

Fig. 2. An M-channel maximally decimated filter bank on a graph $\boldsymbol{A}$. Overall response of the filter bank is denoted as $T(\boldsymbol{A})$.

In the following, we will assume that eigenvectors of the adjacency matrix satisfy (9) and let eigenvalues be arbitrary but distinct.

In Section 5, we will show how this constraint on eigenvectors can be removed by generalizing the definition of the decimator $\boldsymbol{D}$.

Remember from (20) that DU operation, $\boldsymbol{D}^T \boldsymbol{D}$, results in aliasing for an arbitrary graph signal. Nonetheless, we can still recover the input signal from the output of $\boldsymbol{D}^T \boldsymbol{D}$ if the input signal has zeros in its graph Fourier transform. To discuss this further, we define *band-limited* signals on graphs as follows:

Definition 2 (Band-limited graph signals). *Let $\boldsymbol{A}$ be the adjacency matrix of a graph with the following eigenvalue decomposition $\boldsymbol{A} = \boldsymbol{V} \boldsymbol{\Lambda} \boldsymbol{V}^{-1}$, where columns of $\boldsymbol{V}$ satisfy (9) with the indexing scheme in (7) and eigenvalues are arbitrary. A signal $\boldsymbol{x}$ on the graph $\boldsymbol{A}$ said to be k^{th}-band-limited when its graph Fourier transform, $\widehat{\boldsymbol{x}} = \boldsymbol{V}^{-1} \boldsymbol{x}$, has zeros in the following way:*

$$\widehat{x}_{i,j} = 0, \qquad 1 \leqslant j \leqslant M, \quad j \neq k, \quad 1 \leqslant i \leqslant N/M. \qquad (22)$$

where we used the double indexing scheme similar to (6) and (7) to denote the graph Fourier coefficients $\widehat{x}_{i,j}$. Thus, only the N/M quantities $\{\widehat{x}_{1,k}, \; \widehat{x}_{2,k}, \; \cdots, \; \widehat{x}_{N/M,k}\}$ can be nonzero.

In the literature, there are different notions and definitions for band-limited graph signals [10–12, 16, 17]. Under an appropriate re-indexing of the eigenvalues (and the eigenvectors), our notion of kth band-limited signal is similar to the one in [16] with bandwidth N/M. This is consistent with our purpose of constructing M-channel graph filter banks.

When a graph signal is k^{th}-band-limited according to Definition 2, due to (20), the output spectrum of DU operation becomes as follows.

$$\widehat{y}_{i,1} = \widehat{y}_{i,2} = \cdots = \widehat{y}_{i,M} = \frac{1}{M} \widehat{x}_{i,k}. \qquad (23)$$

In this case we can recover the input signal from the output of the DU operator since only one of the aliasing frequency components is non-zero. For this purpose, let $\boldsymbol{F}$ be a linear filter with Fourier domain operation $\boldsymbol{V}^{-1} \boldsymbol{F} \boldsymbol{V} = \boldsymbol{I}_{N/M} \otimes M \, \boldsymbol{e}_k \boldsymbol{e}_k^T$, where $\boldsymbol{e}_k$ is the kth element of the standard basis for $\mathcal{C}^M$. Then, consider the following system,

$$\boldsymbol{z} = \boldsymbol{F} \boldsymbol{D}^T \boldsymbol{D} \, \boldsymbol{x}. \qquad (24)$$

Due to (18) and the construction of $\boldsymbol{F}$ above, in the graph Fourier domain (24) translates to the following,

$$\begin{aligned} \widehat{\boldsymbol{z}} &= \left(\boldsymbol{I}_{N/M} \otimes M \, \boldsymbol{e}_k \boldsymbol{e}_k^T \right) \left(\boldsymbol{I}_{N/M} \otimes \frac{1}{M} \mathbb{1} \mathbb{1}^T \right) \widehat{\boldsymbol{x}} \\ &= \left(\boldsymbol{I}_{N/M} \otimes \boldsymbol{e}_k \mathbb{1}^T \right) \widehat{\boldsymbol{x}}. \end{aligned} \qquad (25)$$

Since $\widehat{\boldsymbol{x}}$ is assumed to be k^{th}-band-limited according to Definition 2, we get $\widehat{\boldsymbol{z}} = \widehat{\boldsymbol{x}}$. That is, we can reconstruct the original signal from its decimated version using the linear reconstruction filter $\boldsymbol{F}$. When the graph is assumed to have distinct eigenvalues, $\boldsymbol{F}$ can be realized as a polynomial filter [5,22]. We have therefore proved the following theorem.

Theorem 2 (Polynomial interpolation filters). *Let $\boldsymbol{A}$ be the adjacency matrix of a graph with the following eigenvalue decomposition $\boldsymbol{A} = \boldsymbol{V} \boldsymbol{\Lambda} \boldsymbol{V}^{-1}$, where columns of $\boldsymbol{V}$ satisfy (9) with the indexing scheme in (7) and eigenvalues are arbitrary but distinct. Let $\boldsymbol{x}$ be a kth-band-limited signal on the graph. Then, there exists a polynomial interpolation filter $F(\boldsymbol{A})$ that recovers $\boldsymbol{x}$ from $\boldsymbol{D} \boldsymbol{x}$.*

The recovery of missing samples in graph signals is also discussed in [16, 29, 30].

In the following, we will discuss how we can write an arbitrary full-band signal as a sum of kth-band-limited signals. This will lead to the construction of a maximally decimated PR filter bank on the graph. For this purpose, consider the following identity,

$$\boldsymbol{I}_N = \sum_{k=1}^{M} \boldsymbol{I}_{N/M} \otimes \boldsymbol{e}_k \boldsymbol{e}_k^T. \qquad (26)$$

Then we can write $\widehat{\boldsymbol{x}} = \sum_{k=1}^{M} \widehat{\boldsymbol{x}}_k$, where

$$\widehat{\boldsymbol{x}}_k = \left(\boldsymbol{I}_{N/M} \otimes \boldsymbol{e}_k \boldsymbol{e}_k^T\right)\widehat{\boldsymbol{x}}. \tag{27}$$

With the construction in (27), $\widehat{\boldsymbol{x}}_k$ is a kth-band-limited signal. Notice that (27) is a frequency domain relation. In the graph signal domain, assume a linear filter $\boldsymbol{H}_{k-1}$ with Fourier domain operation

$$\widehat{\boldsymbol{H}}_{k\text{-}1} = \boldsymbol{V}^{-1}\,\boldsymbol{H}_{k\text{-}1}\,\boldsymbol{V} = \boldsymbol{I}_{N/M} \otimes \boldsymbol{e}_k \boldsymbol{e}_k^T. \tag{28}$$

Then, we have $\boldsymbol{x} = \sum_{k=1}^{M} \boldsymbol{x}_k$, where $\boldsymbol{x}_k = \boldsymbol{H}_{k\text{-}1}\,\boldsymbol{x}$. Since $\boldsymbol{x}_k$ is a kth-band-limited graph signal, we can reconstruct it from its decimated version using the interpolation filter discussed in Theorem 2. For this purpose, let $\boldsymbol{F}_{k-1}$ be a linear filter with Fourier domain operation $\boldsymbol{V}^{-1}\boldsymbol{F}_{k\text{-}1}\boldsymbol{V} = \boldsymbol{I}_{N/M} \otimes M\,\boldsymbol{e}_k \boldsymbol{e}_k^T$. We will then have:

$$\boldsymbol{x}_k = \boldsymbol{F}_{k\text{-}1}\,\boldsymbol{D}^T \boldsymbol{D}\,\boldsymbol{x}_k = \boldsymbol{F}_{k\text{-}1}\,\boldsymbol{D}^T \boldsymbol{D}\,\boldsymbol{H}_{k\text{-}1}\,\boldsymbol{x}. \tag{29}$$

So we have proved the following theorem.

Theorem 3 (Ideal PR filter banks). *Let $\boldsymbol{A}$ be the adjacency matrix of a graph with the following eigenvalue decomposition $\boldsymbol{A} = \boldsymbol{V}\boldsymbol{\Lambda}\boldsymbol{V}^{-1}$, where columns of $\boldsymbol{V}$ satisfy (9) with the indexing scheme in (7) and eigenvalues are arbitrary but distinct. Now consider the maximally decimated filter bank of Fig. 2 with polynomial analysis and synthesis filters as follows:*

$$H_{k\text{-}1}(\boldsymbol{A}) = \boldsymbol{V}\left(\boldsymbol{I}_{N/M} \otimes \boldsymbol{e}_k \boldsymbol{e}_k^T\right)\boldsymbol{V}^{-1}, \quad F_{k\text{-}1}(\boldsymbol{A}) = M\,H_{k\text{-}1}(\boldsymbol{A}) \tag{30}$$

for $1 \leqslant k \leqslant M$. This is a perfect reconstruction system, that is, $T(\boldsymbol{A})\,\boldsymbol{x} = \boldsymbol{x}$ for all graph signals $\boldsymbol{x}$.

Notice that Fourier domain operation of each filter, (30), has N/M nonzero values and N-(N/M) zero values. These are similar to "ideal" band-limited filters (with bandwidth $2\pi/M$) in classical theory. In classical filter bank theory it is well known that an M channel maximally decimated filter bank has perfect reconstruction if the filters are ideal "brickwall" filters chosen as

$$H_k(e^{j\omega}) = \begin{cases} 1, & 2\pi k/M \leqslant \omega \leqslant 2\pi(k{+}1)/M, \\ 0, & \text{otherwise,} \end{cases} \tag{31}$$

and $F_k(e^{j\omega}) = M\,H_k(e^{j\omega})$. The result of Theorem 3 for graph filter banks is analogous to that classical result. Notice that Theorems 2 and 3 hold only for graphs with certain assumptions on $\boldsymbol{A}$ as stated above. Notice also that in classical theory, ideal filters have infinite duration impulse responses, whereas the graph filters are polynomials in $\boldsymbol{A}$ with at most N taps [5, 22].

5. GENERALIZED DECIMATOR AND EXPANDER

The results in Theorems 1-3 are restricted to graphs with the eigenvector structure (9). However this restriction can be removed completely under the mild assumption that $\boldsymbol{A}$ be diagonalizable. The basic idea is to work with a *similarity-transformed* graph matrix $\widetilde{\boldsymbol{A}} = \boldsymbol{Q}\boldsymbol{A}\boldsymbol{Q}^{-1}$. Here we choose $\boldsymbol{Q} = \boldsymbol{E}\boldsymbol{V}^{-1}$ where $\boldsymbol{V}$ is as in (2) and $\boldsymbol{E}$ is an invertible matrix that has the property (9) on its columns.

With this selection of $\boldsymbol{Q}$, $\widetilde{\boldsymbol{A}} = \boldsymbol{E}\boldsymbol{\Lambda}\boldsymbol{E}^{-1}$ has the property (9) on its eigenvectors, hence Theorems 1-3 are applicable to $\widetilde{\boldsymbol{A}}$. Given the original graph signal $\boldsymbol{x}$ we will then define a modified signal $\widetilde{\boldsymbol{x}} = \boldsymbol{Q}\,\boldsymbol{x}$ and use the modified filter bank $\{H_k(\widetilde{\boldsymbol{A}}), F_k(\widetilde{\boldsymbol{A}})\}$ with canonical decimator (3) to process it. Then the filter bank output $\widetilde{\boldsymbol{y}}$ is transformed back to $\boldsymbol{y} = \boldsymbol{Q}^{-1}\widetilde{\boldsymbol{y}}$ (see Fig. 3(a)).

Alternatively, we can integrate $\boldsymbol{Q}$ into the decimator and $\boldsymbol{Q}^{-1}$ into the expander and use the same FB on $\boldsymbol{A}$. The resulting system is schematically described in Fig. 3(b) where the *generalized decimator*, $\widetilde{\boldsymbol{D}}$, and the *generalized expander*, $\widetilde{\boldsymbol{U}}$, are defined as follows:

(a)

(b)
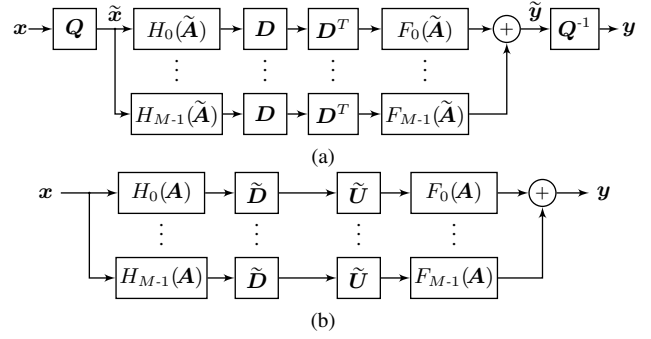
Fig. 3. (a) A FB on the similarity-transformed adjacency matrix $\widetilde{\boldsymbol{A}}$ sandwiched between $\boldsymbol{Q}$ and $\boldsymbol{Q}^{-1}$. (b) The FB with generalized decimator and expander on the original adjacency matrix $\boldsymbol{A}$. $\widetilde{\boldsymbol{D}}$ and $\widetilde{\boldsymbol{U}}$ are as in (32).

$$\widetilde{\boldsymbol{D}} = \boldsymbol{D}\boldsymbol{Q} = \boldsymbol{D}\boldsymbol{E}\boldsymbol{V}^{-1}, \quad \widetilde{\boldsymbol{U}} = \boldsymbol{Q}^{-1}\boldsymbol{D}^T = \boldsymbol{V}\boldsymbol{E}^{-1}\boldsymbol{D}^T. \tag{32}$$

It is important to note that the FB in Fig. 3(b) does not require any structure on the eigenvectors of the given $\boldsymbol{A}$, yet it still provides PR since the filters in Fig. 3(a) are designed to result in a PR system. Equivalence of the systems in Fig. 3(a) and Fig. 3(b) follows from the following theorem, and the polynomial nature of filters.

Theorem 4 (Similarity-transform invariance of polynomial filters). *Polynomial graph filters are invariant to invertible similarity transforms. That is, the following is true for an arbitrary invertible matrix $\boldsymbol{Q}$ for all polynomial filters.*

$$H(\boldsymbol{A}) = \boldsymbol{Q}^{-1}\,H(\widetilde{\boldsymbol{A}})\,\boldsymbol{Q}, \quad \text{where} \quad \widetilde{\boldsymbol{A}} = \boldsymbol{Q}\boldsymbol{A}\,\boldsymbol{Q}^{-1}. \tag{33}$$

Notice that Theorem 4 generalizes the permutation invariance property of polynomial graph filters in [5] to invertible transforms.

6. CONCLUDING REMARKS AND FUTURE WORK

In this paper we considered the concept of aliasing in graph signals and constructed M-channel maximally decimated ideal filter banks. In this regard we started with the canonical definition of the decimator and identified the corresponding expander. We then introduced M-block cyclic graphs. The unique eigenstructure of such graphs was also shown, and the concept of spectrum folding in such graphs was thereby established. We then considered M-channel maximally decimated filter banks on graphs. We found that perfect reconstruction (PR) can be achieved with polynomial filters for graphs which satisfy certain eigenstructure conditions. It was later shown that the necessary eigenvector structure can be relaxed when we use generalized decimator and expander.

Although it is theoretically possible to construct ideal PR filter banks for graph signals, they may not be practical to implement for the following reasons. Firstly, when the graph is large, filters become lengthy, hence computationally expensive, and numerical stability comes as an issue. Secondly, even a slight change in the eigenvalues will result in a drastic deviation in the filter coefficients. As a result any mismatch in the eigenvalues of the adjacency matrix will violate the PR property. It is of future interest to construct more practical PR systems with smaller polynomial orders for the filters in order to address above mentioned problems.

Results of this work open up some questions: How does this filter bank perform for practical graph signals, and how do they compare with alternative ways of processing these graph signals? Many such practical questions remain to be addressed. We plan to explore these aspects in future studies.

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