PHASELESS SUPER-RESOLUTION USING MASKS

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ABSTRACT

Phaseless super-resolution is the problem of reconstructing a signal from its *low-frequency* Fourier *magnitude* measurements. It is the combination of two classic signal processing problems: phase retrieval and super-resolution. Due to the absence of phase and high-frequency measurements, additional information is required in order to be able to uniquely reconstruct the signal of interest.

In this work, we use masks to introduce redundancy in the phaseless measurements. We develop an analysis framework for this setup, and use it to show that *any* super-resolution algorithm can be seamlessly extended to solve phaseless super-resolution (up to a global phase), when measurements are obtained using a certain set of masks. In particular, we focus our attention on a robust semidefinite relaxation-based algorithm, and provide reconstruction guarantees. Numerical simulations complement our theoretical analysis.

Index Terms— Phaseless super-resolution, masks, minimum separation, semidefinite relaxation.

1. INTRODUCTION

In many measurement systems, the magnitude-square of the Fourier transform is the measurable quantity. For example, in optics, detection devices like CCD cameras and photosensitive films measure the photon flux, which is proportional to the magnitude-square of the Fourier transform of the underlying signal. The problem of reconstructing a signal from its phaseless Fourier measurements is known as *phase retrieval*. This reconstruction problem occurs in many areas, such as optics [1], X-ray crystallography [2] and astronomical imaging [3]. We refer the readers to [4, 5] for a survey of classic approaches. Recent surveys can be found in [6, 7].

In the aforementioned applications, it is often very difficult to obtain high-frequency measurements due to physical limitations. For example, in optics, there is a fundamental resolution limit due to diffraction. We therefore consider the problem of *phaseless super-resolution*, which is the problem of reconstructing a signal from its low-frequency Fourier magnitude measurements.

In order to solve phaseless super-resolution, it is necessary to overcome the well-known uniqueness and algorithmic issues of phase retrieval [12]. In this regard, popular approaches include sparsity prior [13–17] and additional magnitude-only measurements (e.g., masks [18–20], STFT [21, 22]). In this work, we use masks and obtain additional structured measurements.

Super-resolution, which is the problem of reconstructing a signal from its low-frequency Fourier measurements, has a rich history and a wide variety of techniques have been proposed [8–11]. In this work, we show that any super-resolution algorithm can be extended to solve phaseless super-resolution (up to a global phase), when measurements are obtained using three particular masks. Consequently, k-sparse signals can be reconstructed from $6k + 3$ low-frequency phaseless measurements in the noiseless setting. We also develop a robust semidefinite relaxation-based algorithm, and show that signals which satisfy a minimum separation condition can be provably reconstructed.

Existing work: In [23], a combinatorial phaseless super-resolution algorithm was proposed to reconstruct sparse signals. The algorithm has two limitations: (i) In the noiseless setting, $\mathcal{O}(k^2)$ measurements are required to reconstruct a k-sparse signal. (ii) In the noisy setting, the algorithm is unstable due to error propagation.

The rest of the paper is organized as follows. In Section 2, we mathematically set up the problem of phaseless super-resolution using masks. The analysis framework, along with the semidefinite relaxation-based algorithm, is described in Section 3. Numerical simulations are presented in Section 4. Section 5 concludes the paper.

Acknowledgement: This work is inspired by ideas in [24].

2. PROBLEM SETUP

Let $\mathbf{x} = (x[0], x[1], \ldots, x[N-1])^T$ be a complex-valued signal of length N and sparsity k. For $0 \leq r \leq R-1$, let $\mathbf{D}_r$ be an $N \times N$ diagonal matrix, corresponding to the rth mask, with diagonal entries $(d_r[0], d_r[1], \ldots, d_r[N-1])$. Let $\mathbf{Z}$ denote the $K \times R$ magnitude-square measurements, such that the rth column of $\mathbf{Z}$ corresponds to the magnitude-square of the first K terms of the N point DFT of the masked signal $\mathbf{D}_r\mathbf{x}$. We consider the following reconstruction problem:

$$\begin{aligned} &\text{find} && \mathbf{x} \\ &\text{subject to} && Z[m,r] = \left|\langle \mathbf{f}_m, \mathbf{D}_r\mathbf{x}\rangle\right|^2 \\ &\quad\text{for} && 0 \le m \le K-1 \quad \text{and} \quad 0 \le r \le R-1, \end{aligned} \tag{1}$$

where $\mathbf{f}_m$ is the conjugate of the mth column of the N point DFT matrix and $\langle .,. \rangle$ is the standard inner product operator.

Let $\mathbf{F}$ denote the N point DFT matrix and $\mathbf{F}_K$ be the $K \times N$ submatrix of $\mathbf{F}$, constructed by considering the first K rows. Further, let $\mathbf{x}_0$ be the underlying signal and $\mathbf{X}_0 = \mathbf{x}_0\mathbf{x}_0^\star$.

3. METHODOLOGY

3.1. Semidefinite relaxation-based reconstruction

Note that (1) is a quadratic-constrained problem. A technique, popularly known as *lifting*, has enjoyed success in solving several quadratic-constrained problems [25, 26]. The steps to formulate such problems as a semidefinite program (SDP) are as follows: (i) Embed the problem in a higher dimensional space using the transformation $\mathbf{X} = \mathbf{x}\mathbf{x}^\star$, a process which converts the problem of recovering a signal with quadratic constraints into a problem of recovering a rank-one matrix with affine constraints. (ii) Relax the rank-one constraint to obtain a convex program.

The matrix we are interested in recovering is both sparse and low-rank. The most natural objective function to recover such a matrix is a linear combination of ℓ_1 norm and nuclear norm (same as *trace* norm for positive semidefinite matrices). Hence, we consider the following convex program:

$$\begin{aligned} &\text{minimize} && \|\mathbf{X}\|_1 + \lambda\, \text{trace}(\mathbf{X}) \\ &\text{subject to} && Z[m,r] = \text{trace}(\, \mathbf{D}_r^\star \mathbf{f}_m \mathbf{f}_m^\star \mathbf{D}_r \mathbf{X}\,) \\ &\quad\text{for} && 0 \le m \le K-1 \quad \text{and} \quad 0 \le r \le R-1, \\ &&& \mathbf{X} \succcurlyeq 0, \end{aligned} \tag{2}$$

for some regularizer λ.

3.2. Analysis Framework

Based on intuitions from compressed sensing, for $\mathbf{X}_0 = \mathbf{x}_0\mathbf{x}_0^\star$ to be the unique optimizer of (2), one would expect roughly $\mathcal{O}(k)$ generic measurements to be sufficient. However, in [27,28], it is shown that at least $\mathcal{O}(\min(k^2, N))$ generic measurements are necessary. For a discussion on this square-root bottleneck, we refer the readers to [29].

In order to overcome this bottleneck, inspired by ideas in [24], we first focus our attention on the constraint set of (2). For each r, let $(s_r[0], s_r[1], \ldots, s_r[N-1])^T$ denote the N point DFT of the vector $(d_r[0], d_r[1], \ldots, d_r[N-1])^T$ and for each l, let $\text{Diag}(\mathbf{f}_l)$ be an $N \times N$ diagonal matrix with diagonal entries $\mathbf{f}_l$. We have the following relationship:

$$\mathbf{f}_m^\star \mathbf{D}_r = \left(\sum_{l=0}^{N-1} s_r[l]\; \mathbf{f}_m^\star \text{Diag}(\mathbf{f}_l) \right).$$

Since $\mathbf{f}_m^\star \text{Diag}(\mathbf{f}_l) = \mathbf{f}_{m-l}^\star$, for every m and r, the aforementioned expression can be rewritten as

$$\mathbf{f}_m^\star \mathbf{D}_r = \sum_{l=0}^{N-1} s_r[l] \mathbf{f}_{m-l}^\star = \sum_{l=0}^{N-1} s_r[l] \mathbf{e}_{m-l}^\star \mathbf{F},$$

where $\mathbf{e}_m$ is the mth column of the identity matrix.

Suppose the masks are chosen such that $s_r[l] = 0$ for every $l \ge L$, for some parameter L. In the $L-1 \le m \le K-1$ regime, we can rewrite $\text{trace}(\mathbf{D}_r^\star \mathbf{f}_m \mathbf{f}_m^\star \mathbf{D}_r \mathbf{X})$ as:

$$\text{trace}\left(\left(\sum_{l=0}^{L-1} s_r^\star[l] \mathbf{e}_{m-l} \right) \left(\sum_{l=0}^{L-1} s_r[l] \mathbf{e}_{m-l}^\star \right) \mathbf{F}_K \mathbf{X} \mathbf{F}_K^\star \right). \tag{3}$$

By considering such measurements from a sufficient number of carefully chosen masks, if we ensure that (2) has only one positive semidefinite feasible matrix (which will be rank-one as $\mathbf{F}_K\mathbf{X}_0\mathbf{F}_K^\star$ is feasible), then the phaseless super-resolution problem reduces to the problem of reconstructing $\mathbf{X}_0$ from $\mathbf{Y} = \mathbf{F}_K\mathbf{X}_0\mathbf{F}_K^\star$. This is equivalent to the problem of reconstructing $\mathbf{x}_0$ from $e^{i\phi}\mathbf{F}_K\mathbf{x}_0$. Hence, by using such masks, any super-resolution algorithm can be extended to solve phaseless super-resolution up to a phase.

In particular, the convex program (2) becomes equivalent to the ℓ_1 minimization-based convex program proposed in [10, 11].

3.3. A Specific Example

Let $V = \{v_0, v_1, \ldots, v_{k-1}\}$ denote the support of $\mathbf{x}_0$, i.e., the set of indices where $\mathbf{x}_0$ has a non-zero value. The minimum-spacing, denoted by $\Delta(V)$, is defined as the closest distance between any two elements in V, i.e,

$$\Delta(V) = \min_{0 \le i,j \le k-1 : i \ne j} \left| v_i - v_j \right|. \tag{4}$$

Here, the distance is defined in a cyclic manner. For example, if $N = 100$, then the distance between $v_i = 90$ and $v_j = 10$ is 20.

In [20], measurements using three masks, defined by the following diagonal matrices:

$$\mathbf{D}_0 = \mathbf{I}, \quad \mathbf{D}_1 = \mathbf{I} + \mathbf{D}, \quad \mathbf{D}_2 = \mathbf{I} - i\mathbf{D}, \tag{5}$$

where $\mathbf{D}$ is a diagonal matrix with diagonal entries given by

$$d[n] = e^{i2\pi \frac{n}{N}} \quad \text{for} \quad 0 \le n \le N-1,$$

are considered. Using the analysis framework developed in the previous subsection, we provide the following reconstruction guarantee for the convex program (2):

Theorem 3.1. *The matrix $\mathbf{X}_0 = \mathbf{x}_0\mathbf{x}_0^\star$ is the unique optimizer of (2), and therefore $\mathbf{x}_0$ can be uniquely reconstructed up to a phase, if*

(i) $K \geq \frac{2N}{\Delta(V)}$

(ii) The first K values of the N point DFT of $\mathbf{x}_0$ are non-zero

(iii) The masks $\{\mathbf{D}_0, \mathbf{D}_1, \mathbf{D}_2\}$ defined in (5) are used.

Proof. For the masks defined in (5), the values of $s_r[l]$ are as follows:

$$s_0[l] = \begin{cases} 1 & \text{for} \quad l = 0 \\ 0 & \text{otherwise} \end{cases}$$

$$s_1[l] = \begin{cases} 1 & \text{for} \quad l = 0, 1 \\ 0 & \text{otherwise} \end{cases}$$

$$s_2[l] = \begin{cases} 1 & \text{for} \quad l = 0 \\ -i & \text{for} \quad l = 1 \\ 0 & \text{otherwise} \end{cases}$$

Substituting these values in (3), the measurement corresponding to $r = 0$ and any $0 \leq m \leq K-1$ fixes $Y[m,m]$. Similarly, the measurements corresponding to $r = 1, 2$ and any $1 \leq m \leq K-1$ fix the values of $Y[m-1, m-1] + Y[m-1, m] + Y[m, m-1] + Y[m,m]$ and $Y[m-1, m-1] + iY[m-1, m] - iY[m, m-1] + Y[m,m]$. These measurements, combined with the measurements corresponding to $r = 0$, fix $Y[m-1, m]$ and $Y[m, m-1]$.

Hence, the diagonal and the first off-diagonal entries of every feasible $\mathbf{Y}$ match the diagonal and the first off-diagonal entries of the matrix $(\mathbf{F}_K\mathbf{x}_0)(\mathbf{F}_K\mathbf{x}_0)^\star$. Since the entries are sampled from a rank one matrix with non-zero diagonal entries (the first K values of the N point DFT of $\mathbf{x}_0$ are non-zero), there is exactly one positive semidefinite completion, which is the rank one completion $(\mathbf{F}_K\mathbf{x}_0)(\mathbf{F}_K\mathbf{x}_0)^\star$.

In particular, due to the fact that $\mathbf{Y} \succcurlyeq 0$ and $\mathbf{F}_K\mathbf{x}_0$ is non-vanishing, $(\mathbf{F}_K\mathbf{x}_0)(\mathbf{F}_K\mathbf{x}_0)^\star$ is the only feasible $\mathbf{Y}$. As explained in the previous subsection, the reconstruction problem is reduced to the problem of reconstructing $\mathbf{x}_0$ from $e^{i\phi}\mathbf{F}_K\mathbf{x}_0$, for some ϕ.

Since the conditions of Corollary 1.4 in [10] are satisfied, $\mathbf{X}_0 = \mathbf{x}_0\mathbf{x}_0^\star$ is the unique optimizer of (2). □

In the noiseless setting, a k-sparse signal $\mathbf{x}_0$ can be reconstructed from $\mathbf{F}_K\mathbf{x}_0$ if $K \geq 2k+1$ (e.g., matrix pencil method [30]). Consequently, if measurements are obtained using masks defined in (5), then k-sparse signals can be reconstructed from $(2k+1) \times 3 = 6k+3$ low-frequency phaseless measurements.

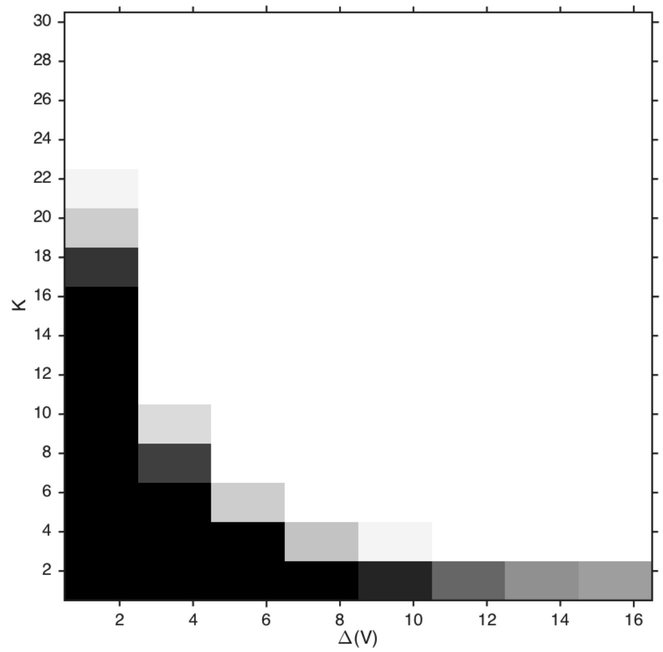

Fig. 1: Probability of successful reconstruction for $N = 32$ and various choices of K and $\Delta(V)$, using masks $\{\mathbf{D}_0, \mathbf{D}_1, \mathbf{D}_2\}$ defined in (5).

4. NUMERICAL SIMULATIONS

In this section, we demonstrate the performance of the proposed semidefinite relaxation-based algorithm (2) using numerical simulations.

We choose $N = 32$ and evaluate the performance for various choices of minimum-spacing $\Delta(V)$ and number of low-frequency measurements K. The masks $\{\mathbf{D}_0, \mathbf{D}_1, \mathbf{D}_2\}$ defined in (5) are used.

For each choice of $\Delta(V)$ and K, we perform 25 trials using the parser YALMIP and the solver SeDuMi. For each trial, a complex signal is randomly generated as follows: Starting from an empty V, 100 indices in the range 0 and $N-1$ are generated uniformly at random (with repetition) and sequentially added to V as long as the minimum-spacing criterion is not violated. The signal values in the support are drawn from a standard complex Gaussian distribution independently. The probability of successful reconstruction is plotted in Fig. 1. The black region corresponds to a success probability of 0 and the white region corresponds to a success probability of 1.

If $K \gtrsim \frac{N}{\Delta(V)}$, then the SDP-based algorithm reconstructs the signal with very high probability. Theorem 3.1 explains this behavior when $K \geq \frac{2N}{\Delta(V)}$, and hence is off from the simulations by a factor of ~ 2. This phenomenon was also observed in [10].

5. CONCLUSIONS AND FUTURE DIRECTIONS

We considered the problem of phaseless super-resolution using masks. We developed an analysis framework for this setup and used it to show that any super-resolution algorithm can be extended to solve phaseless super-resolution, when measurements are obtained using a certain set of masks. For a particular choice of three masks, we argued that order-wise optimal number of low-frequency phaseless measurements are sufficient in the noiseless setting. We also developed a robust semidefinite relaxation-based algorithm and provided reconstruction guarantees for signals which satisfy a minimum separation condition. Numerical simulations are in accordance with our theoretical predictions.

In this work, we did not consider the impact of measurement noise on signal reconstruction. A theoretical analysis of the estimation error in the noisy setting is one direction for future study. A characterization of the performance when measurements are obtained using various sets of masks is another interesting direction.

6. REFERENCES

[1] A. Walther, "The question of phase retrieval in optics," Journal of Modern Optics 10, no. 1 (1963): 41-49.

[2] R. P. Millane, "Phase retrieval in crystallography and optics," JOSA A 7, no. 3 (1990): 394-411.

[3] J. C. Dainty and J. R. Fienup, "Phase retrieval and image reconstruction for astronomy," Image Recovery: Theory and Application (1987): 231-275.

[4] R. W. Gerchberg and W. O. Saxton, "A practical algorithm for the determination of the phase from image and diffraction plane pictures," Optik 35 (1972): 237.

[5] J. R. Fienup, "Phase retrieval algorithms: A comparison," Applied Optics 21, no. 15 (1982): 2758-2769.

[6] Y. Shechtman, Y. C. Eldar, O. Cohen, H. N. Chapman, J. Miao and M. Segev, "Phase retrieval with application to optical imaging," IEEE Signal Processing Magazine 32, no. 3 (2015): 87-109.

[7] K. Jaganathan, Y. C. Eldar and B. Hassibi, "Phase retrieval: An overview of recent developments," arXiv:1510.07713 (2015).

[8] R. Roy and T. Kailath, "ESPRIT-estimation of signal parameters via rotational invariance techniques," Acoustics, Speech and Signal Processing, IEEE Transactions on 37.7 (1989): 984-995.

[9] R. O. Schmidt, "Multiple emitter location and signal parameter estimation," Antennas and Propagation, IEEE Transactions on 34.3 (1986): 276-280.

[10] E. J. Candes and C. Fernandez-Granda, "Towards a mathematical theory of super-resolution," Communications on Pure and Applied Mathematics 67.6 (2014): 906-956.

[11] G. Tang, B. N. Bhaskar, P. Shah and B. Recht, "Compressed sensing off the grid," Information Theory, IEEE Transactions on 59, no. 11 (2013): 7465-7490.

[12] E. M. Hofstetter, "Construction of time-limited functions with specified autocorrelation functions," Information Theory, IEEE Transactions on 10.2 (1964): 119-126.

[13] Y. M. Lu and M. Vetterli, "Sparse spectral factorization: Unicity and reconstruction algorithms," IEEE International Conference on Acoustics, Speech and Signal Processing (2011): 5976-5979.

[14] Y. Shechtman, Y. C. Eldar, A. Szameit and M. Segev, "Sparsity based sub-wavelength imaging with partially incoherent light via quadratic compressed sensing," Optics Express 19 (2011): 14807-14822.

[15] K. Jaganathan, S. Oymak and B. Hassibi, "Recovery of sparse 1-D signals from the magnitudes of their Fourier transform," IEEE International Symposium on Information Theory Proceedings (2012): 1473-1477.

[16] Y. Shechtman, A. Beck and Y. C. Eldar, "GESPAR: Efficient Phase Retrieval of Sparse Signals," Signal Processing, IEEE Transactions on 62, no. 4 (2014): 928-938.

[17] K. Jaganathan, S. Oymak and B. Hassibi, "Sparse phase retrieval: Uniqueness guarantees and recovery algorithms," arXiv:1311.2745 (2015).

[18] E. J. Candes, X. Li, and M. Soltanolkotabi, "Phase retrieval from coded diffraction patterns," Applied and Computational Harmonic Analysis (2014).

[19] K. Jaganathan, Y. C. Eldar and B. Hassibi, "Phase retrieval with masks using convex optimization," IEEE International Symposium on Information Theory Proceedings (2015): 1655-1659 .

[20] E. J. Candes, Y. C. Eldar, T. Strohmer and V. Voroninski, "Phase retrieval via matrix completion," SIAM Journal on Imaging Sciences 6.1 (2013): 199-225.

[21] K. Jaganathan, Y. C. Eldar and B. Hassibi, "STFT Phase Retrieval: Uniqueness Guarantees and Recovery Algorithms," http://arxiv.org/abs/1508.02820 (2015).

[22] Y. C. Eldar, P. Sidorenko, D. G. Mixon, S. Barel and O. Cohen, "Sparse phase retrieval from short-time Fourier measurements," IEEE Signal Processing Letters 22, no. 5 (2015): 638-642.

[23] Y. Chen, Y. C. Eldar and A. J. Goldsmith, "An algorithm for exact super-resolution and phase retrieval," IEEE International Conference on Acoustics, Speech and Signal Processing (2014): 754-758.

[24] S. Bahmani and J. Romberg, "Efficient compressive phase retrieval with constrained sensing vectors," arXiv:1507.08254 (2015).

[25] M. X. Goemans and D. P. Williamson, “Improved approximation algorithms for maximum cut and satisfiability problems using semidefinite programming,” Journal of the ACM (JACM), 42(6), 1115-1145 (1995).

[26] I. Waldspurger, A. d’Aspremont and S. Mallat, “Phase recovery, maxcut and complex semidefinite programming,” Mathematical Programming 149.1-2 (2015): 47-81.

[27] X. Li and V. Voroninski, “Sparse signal recovery from quadratic measurements via convex programming,” SIAM Journal on Mathematical Analysis 45, no. 5 (2013): 3019-3033.

[28] S. Oymak, A. Jalali, M. Fazel, Y. C. Eldar and B. Hassibi, “Simultaneously structured models with application to sparse and low-rank matrices,” arXiv:1212.3753 (2012).

[29] K. Jaganathan, S. Oymak and B. Hassibi, “Sparse phase retrieval: Convex algorithms and limitations,” Information Theory Proceedings (ISIT), IEEE International Symposium on (2013): 1022-1026.

[30] Y. Hua and T. K. Sarkar, “Matrix pencil method for estimating parameters of exponentially damped/undamped sinusoids in noise.” Acoustics, Speech and Signal Processing, IEEE Transactions on 38, no. 5 (1990): 814-824.