A SWISS ARMY KNIFE FOR FINITE RATE OF INNOVATION SAMPLING THEORY

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ABSTRACT

Finite–Rate–of–Innovation (FRI) sampling theory prescribes a procedure for exact recovery of Dirac impulses from linear measurements in the form of orthogonal projections of streams of Dirac impulses onto the subspace of Fourier–bandlimited functions. This enables recovery of a continuous time sparse signals at sub-Nyquist rates. In many cases, the transform domain of interest may be more general than the Fourier domain. Recent work has extended FRI sampling theory to the spherical Fourier Transform, fractional Fourier Transform and the Laplace Transform. In this paper, we develop a broad FRI framework applicable to a general class of transformations that includes Fourier, Laplace, Fresnel, fractional Fourier, Bargmann and Gauss–Weierstrass transforms, among others. For this purpose, we consider the Special Affine Fourier Transform (SAFT) which parametrically generalizes a number of well known unitary transforms linked with signal processing and optics. We first derive a version of Shannon's sampling theory based on the convolution structure tailored for the SAFT domain. Having identified the subspace of SAFT–bandlimited functions, we apply FRI sampling theory to the SAFT and study recovery of sparse signals, thus providing a unified view of FRI sampling theory for a large class of disparately studied operations.

Index Terms— Finite–rate–of–innovation (FRI), fractional Fourier domain, Shannon, sampling and special affine Fourier Transform.

Table 1: SAFT, Unitary Transformations and Operations

SAFT Parameters ($\mathbf{\Lambda}_{\mathsf{S}}$)	Corresponding Transform
$\left[\begin{array}{cc\|c} a & b & 0 \\ c & d & 0 \end{array}\right] = \mathbf{\Lambda}_{\mathsf{L}}$	**Linear Canonical Transform (LCT)**
$\left[\begin{array}{cc\|c} \cos\theta & \sin\theta & p \\ -\sin\theta & \cos\theta & q \end{array}\right] = \mathbf{\Lambda}_{\theta}^{\mathsf{O}}$	**Offset Fractional Fourier Transform**
$\left[\begin{array}{cc\|c} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \end{array}\right] = \mathbf{\Lambda}_{\theta}$	**Fractional Fourier Transform (FrFT)**
$\left[\begin{array}{cc\|c} 0 & 1 & 0 \\ -1 & 0 & 0 \end{array}\right] = \mathbf{\Lambda}_{\mathsf{FT}}$	**FourierTransform (FT)**
$\left[\begin{array}{cc\|c} 0 & 1 & p \\ -1 & 0 & q \end{array}\right] = \mathbf{\Lambda}_{\mathsf{FT}}^{\mathsf{O}}$	**Offset Fourier Transform**
$\left[\begin{array}{cc\|c} 0 & \jmath & 0 \\ \jmath & 0 & 0 \end{array}\right] = \mathbf{\Lambda}_{\mathsf{LT}}$	**Laplace Transform (LT)**
$\left[\begin{array}{cc\|c} \jmath\cos\theta & \jmath\sin\theta & 0 \\ \jmath\sin\theta & -\jmath\cos\theta & 0 \end{array}\right]$	**Fractional Laplace Transform**
$\left[\begin{array}{cc\|c} 1 & b & 0 \\ 0 & 1 & 0 \end{array}\right]$	**Fresnel Transform**
$\left[\begin{array}{cc\|c} 1 & \jmath b & 0 \\ \jmath & 1 & 0 \end{array}\right]$	**Bilateral Laplace Transform**
$\left[\begin{array}{cc\|c} 1 & -\jmath b & 0 \\ 0 & 1 & 0 \end{array}\right], b \geq 0$	**Gauss–Weierstrass Transform**
$\frac{1}{\sqrt{2}}\left[\begin{array}{cc\|c} 0 & e^{-\jmath\pi/2} & 0 \\ -e^{-\jmath\pi/2} & 1 & 0 \end{array}\right]$	**Bargmann Transform**
SAFT Parameters ($\mathbf{\Lambda}_{\mathsf{S}}$)	Corresponding Signal Operation
$\left[\begin{array}{cc\|c} 1/\alpha & 0 & 0 \\ 0 & \alpha & 0 \end{array}\right] = \mathbf{\Lambda}_{\alpha}$	**Time Scaling**
$\left[\begin{array}{cc\|c} 1 & 0 & \tau \\ 0 & 1 & 0 \end{array}\right] = \mathbf{\Lambda}_{\tau}$	**Time Shift**
$\left[\begin{array}{cc\|c} 1 & 0 & 0 \\ 0 & 1 & \xi \end{array}\right] = \mathbf{\Lambda}_{\xi}$	**Frequency Shift/Modulation**
SAFT Parameters ($\mathbf{\Lambda}_{\mathsf{S}}$)	Corresponding Optical Operation
$\left[\begin{array}{cc\|c} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \end{array}\right] = \mathbf{\Lambda}_{\theta}$	**Rotation**
$\left[\begin{array}{cc\|c} 1 & 0 & 0 \\ \tau & 1 & 0 \end{array}\right] = \mathbf{\Lambda}_{\tau}$	**Lens Transformation**
$\left[\begin{array}{cc\|c} 1 & \eta & 0 \\ 0 & 1 & 0 \end{array}\right] = \mathbf{\Lambda}_{\eta}$	**Free Space Propagation**
$\left[\begin{array}{cc\|c} e^{\beta} & 0 & 0 \\ 0 & e^{-\beta} & 0 \end{array}\right] = \mathbf{\Lambda}_{\beta}$	**Magnification**
$\left[\begin{array}{cc\|c} \cosh\alpha & \sinh\alpha & 0 \\ \sinh\alpha & \cosh\alpha & 0 \end{array}\right] = \mathbf{\Lambda}_{\eta}$	**Hyperbolic Transformation**

1. INTRODUCTION

The year 2016 marks the Claude Shannon centenary. One of his many elegant results is linked with the topic of *Sampling Theory*. This result is at the heart of analog-to-digital conversion and states that a function bandlimited in the Fourier domain is completely characterized by its discrete measurements obtained at uniform time instants that are at least separated by an interval inversely proportional to twice the maximum frequency. This result has been extended far and wide. Two recurrent themes ask the following questions:

Q. 1 Can we recover non–bandlimited signals?

Q. 2 What if the domain of investigation is something other than the Fourier Transform?

Over the years both of these questions, as well as their combination, have received a lot of interest. The advent of wavelet transforms revolutionized the way we think about sampling theory [1, 2]. The introduction of shift–invariant subspaces lead to an approximation theoretic formalization of sampling theory which relaxed the Fourier–bandlimitedness constraint to a more general requirement of square–integrability of functions [3]. With respect to sampling of non-bandlimited signals, Vetterli/Blu [4, 5] and co–workers introduced the concept of *finite-rate-of-innovation* (FRI). The key idea is to consider signals that are characterized by their degrees of freedom. For example, a Dirac distribution is characterized by two degrees of freedom—its location and its weight. Similarly, piecewise polynomials are defined by the location and height of their discontinuities. Hence, recovery of non–bandlimited signals in the context of FRI modeling amounts to estimation of parameters linked with the degrees of freedom. A striking feature of the FRI conceptualization is that a signal with finite degrees of freedom can be recovered by sampling at sub-Nyquist rates [6]. Prior to the emergence of the FRI model, the work of Li and Speed [7] discussed the recovery of Dirac impulses given its low–pass/bandlimited samples. Their work relies on the observation that estimating Dirac impulses is the Fourier dual of the spectral estimation problem [8].

The work of Y. Eldar was funded by the European Union's Horizon 2020 research and innovation program under grant agreement ERC-BNYQ, by the Israel Science Foundation under Grant no. 335/14, and by ICore: the Israeli Excellence Center 'Circle of Light?'.

$$\kappa_{\mathbf{\Lambda}_{\mathsf{S}}}(t,\omega) = K_b^* \exp\left(-\frac{\jmath}{2b}\left(at^2 + d\omega^2 + 2t(p-\omega) - 2\omega(dp-bq)\right)\right), \qquad K_b = \frac{1}{\sqrt{\jmath 2\pi b}} \exp\left(\jmath \frac{dp^2}{2b}\right) \tag{1}$$

After the introduction of FRI in [4], an extension to exponential spline families was proposed by Dragotti, Blu and Vetterli in [9]. This paved the way for annihilation in time domain (as opposed to Fourier domain [4]) via computation of moments. Shukla and Dragotti worked on FRI principles linked with the *Radon Transform* in [10]. Bhandari and Marziliano [11] proposed a generalization of FRI to the *fractional Fourier Transform* (FrFT) domain. This result considered [4] as a special case. A further generalization in *phase space* is due to Bhandari, Eldar and Raskar [12] which deals with FRI in the context of super–resolution in which the annihilation algorithm is substituted by a convex program. An extension to *spherical harmonic basis functions* was studied by Deslauriers and Marziliano in [13] and later, by Dokmanić and Lu in [14].

While complex exponentials—the Fourier basis functions—are efficient for modeling periodic phenomenon and are the eigen functions of linear systems, in many areas of science and engineering, the constituent ingredients of the signal of interest are more general and assume the form of polynomial phase signals. Such signals can model non–stationary/time-varying phenomenon and may be thought of as generalized complex exponentials, $e^{\jmath\varphi(t)}$. For example, radar and sonar [18] often transmit chirps for which case, $\varphi(t) \propto \alpha t^2 + \beta t + \gamma$. Non–destructive testing [19] uses chirps in the form of FrFT basis functions. Similarly, holography benefits from the Fresnel Transform [20]. Other examples include quantum optics and wave–physics. Consequently, a number of models have been proposed in the literature that involve polynomial phase based basis functions [15, 21].

The introduction of the fractional Fourier Transform (FrFT) to the signal processing community by Almeida [15] led to several extensions of Shannon's sampling theory. For example, [16] and references therein discuss a number of results involved with sampling theory of signals bandlimited in the FrFT domain. In [17], Bhandari and Zayed provide the first characterization of shift–invariant subspaces associated with the FrFT domain.

In this paper, we extend the FRI sampling theory to a wide variety of unitary transformations that frequently occur in signal processing and optics and are capable of modeling sinusoidal as well as polynomial phase signals. We do so by repurposing the FRI concept for the Special Affine Fourier Transform (or the SAFT) which parametrically generalizes all the unitary transformations and operations listed in Table. 1. We show that a continuous time sparse signal of the form

$$s(t) = \sum\nolimits_{k=0}^{K-1} \mu_k \delta(t - t_k) \tag{2}$$

can be recovered from its orthogonal projection onto the subspace of SAFT–bandlimited functions. More precisely, we show that we can recover $s(t)$ from $2K+1$ equidistant samples of its low–pass version in the sense of the SAFT domain. As in the case of the Fourier domain, the orthogonal projection step amounts to low–pass filtering based on a convolution structure devised for the SAFT domain. All our results are backward compatible with previously known results linked with the Fourier Transform as well as the fractional Fourier Transform.

2. THE SPECIAL AFFINE FOURIER TRANSFORM (SAFT)

Abe and Sheridan first introduced the SAFT in the context of mathematical physics and harmonic analysis [22, 23]. Mathematically, the forward SAFT operation, that is, the mapping $\mathscr{T}_{\mathsf{SAFT}} : f \to \widehat{f}_{\mathbf{\Lambda}_{\mathsf{S}}}$ is defined by

Special Affine Fourier Transform (SAFT)

$$\widehat{f}_{\mathbf{\Lambda}_{\mathsf{S}}}(\omega) = \begin{cases} \langle f, \kappa_{\mathbf{\Lambda}_{\mathsf{S}}}(\cdot,\omega)\rangle & b \neq 0 \\ \sqrt{d} e^{\jmath \frac{cd}{2}(\omega-p)^2 + \jmath\omega q} x(d(\omega-p)) & b = 0 \end{cases} \tag{3}$$

where:

- ▷ $\mathbf{\Lambda}_{\mathsf{S}}^{(2\times 3)}$ is the SAFT parameter matrix $\mathbf{\Lambda}_{\mathsf{S}} = [\ \mathbf{\Lambda} \ | \ \underline{\boldsymbol{\lambda}}\ ]$ formed by augmenting the Linear Canonical Transform (LCT) matrix $\mathbf{\Lambda}_{\mathsf{L}}$ (see Table 1, cf. [24]) and an offset vector $\underline{\boldsymbol{\lambda}}$ such that,

$$\mathbf{\Lambda} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \text{ with } ad - bc = 1 \quad \text{and} \quad \underline{\boldsymbol{\lambda}} = \begin{bmatrix} p \\ q \end{bmatrix}.$$

- ▷ $\kappa_{\mathbf{\Lambda}_{\mathsf{S}}}(t,\omega)$ in (1) is the SAFT kernel parameterized by $\mathbf{\Lambda}_{\mathsf{S}}$.

Much in the same way as the FT, the Dirac distribution is non–bandlimited in the SAFT domain:

$$\widehat{\delta}_{\mathbf{\Lambda}_{\mathsf{S}}}(\omega) \stackrel{(3)}{=} K_b \exp\left(\frac{\jmath}{2b}\left(d\omega^2 - 2\omega(dp - bq)\right)\right). \tag{4}$$

Due to the additive property of the SAFT [25], the inverse–SAFT is an SAFT evaluated using matrix $\mathbf{\Lambda}_{\mathsf{S}}^{\mathrm{inv}}$ with parameters,

$$\mathbf{\Lambda}_{\mathsf{S}}^{\mathrm{inv}} \stackrel{\mathrm{def}}{=} \left[\begin{array}{cc|c} +d & -b & bq - dp \\ -c & +a & cp - aq \end{array}\right] = \left[\begin{array}{cc|c} +d & -b & p_0 \\ -c & +a & q_0 \end{array}\right]. \tag{5}$$

We define the inverse transform/iSAFT more compactly using,

$$f(t) = c_{\mathbf{\Lambda}_{\mathsf{S}}^{\mathrm{inv}}} \left\langle \widehat{f}_{\mathbf{\Lambda}_{\mathsf{S}}}, \kappa_{\mathbf{\Lambda}_{\mathsf{S}}^{\mathrm{inv}}}(\cdot, t)\right\rangle \tag{6}$$

where $c_{\mathbf{\Lambda}_{\mathsf{S}}^{\mathrm{inv}}} = \exp\left(\frac{\jmath}{2}\left(cdp^2 + abq^2 - 2adpq\right)\right)$.

2.1. SAFT Convolution Theorem

As is well known, convolution of two functions amounts to a multiplication of their respective spectrums in Fourier domain. This is known as the *convolution theorem*. This is not the case for the SAFT domain [26, 27]. We resolve this problem by adopting a version of the convolution operator for the SAFT domain, denoted by "$*_{\mathbf{\Lambda}_{\mathsf{S}}}$", such that, $\mathscr{T}_{\mathsf{SAFT}}[f *_{\mathbf{\Lambda}_{\mathsf{S}}} g] \propto \mathscr{T}_{\mathsf{SAFT}}[f]\, \mathscr{T}_{\mathsf{SAFT}}[g]$.

Definition 1 (SAFT Convolution/Filtering [27]) *Let f and g be two given functions and $*$ denote the usual convolution operator. The SAFT convolution is defined by*

$$(f *_{\mathbf{\Lambda}_{\mathsf{S}}} g)(t) = K_b e^{-\jmath \frac{at^2}{2b}} \left(f(t) e^{\jmath \frac{at^2}{2b}} * g(t) e^{\jmath \frac{at^2}{2b}}\right). \tag{7}$$

The convolution–product theorem for the SAFT [27] proves that

$$\underbrace{h(t) \stackrel{(7)}{=} (f *_{\mathbf{\Lambda}_{\mathsf{S}}} g)(t)}_{\text{SAFT Convolution}} \xrightarrow{\text{SAFT}} \underbrace{\widehat{h}_{\mathbf{\Lambda}_{\mathsf{S}}}(\omega) = \Phi_{\mathbf{\Lambda}_{\mathsf{S}}}(\omega)\, \widehat{f}_{\mathbf{\Lambda}_{\mathsf{S}}}(\omega)\, \widehat{g}_{\mathbf{\Lambda}_{\mathsf{S}}}(\omega)}_{\text{Product of SAFT Spectrums}},$$

where $\Phi_{\mathbf{\Lambda}_{\mathsf{S}}}(\omega) = K_b^{-1} \widehat{\delta}^*_{\mathbf{\Lambda}_{\mathsf{S}}}(\omega)$ and $\widehat{\delta}_{\mathbf{\Lambda}_{\mathsf{S}}}(\omega)$ is given by (4).

2.2. Subspace of SAFT–bandlimted Functions

We now show that the orthogonal projection onto the subspace of SAFT–bandlimited functions amounts to low–pass filtering using the SAFT convolution in (7) followed by sampling. By SAFT–bandlimited function, we refer to a signal which is bandlimited in the SAFT domain.

Let $\Delta = b/\Omega$ and let $\mathrm{sinc}(t) = \frac{\sin(\pi t)}{\pi t}$. The family of functions,

Orthonormal Subspace of SAFT–bandlimted Functions

$$\varphi_n(t) = \frac{1}{\Delta} e^{-\jmath \frac{a\left(t^2-(n\Delta)^2\right)}{2b}} e^{-\jmath p \frac{t-n\Delta}{b}} \mathrm{sinc}\left(\frac{t}{\Delta} - n\right) \quad (8)$$

forms an orthonormal subspace of SAFT–bandlimted functions [28] with maximum admissible frequency $\omega_{max} = \pi\Omega = b\pi/\Delta$. To verify orthonormality, assume that $\Delta = 1$, for simplicity. Then,

$$\langle \varphi_n, \varphi_k \rangle = e^{\jmath \frac{n^2-k^2}{2b}} e^{\jmath p \frac{n-k}{b}} \delta_{n-k} = \delta_{n-k}.$$

To show bandlimitedness, we define the low–pass kernel $\varphi_{\mathsf{LP}} = \varphi_0$. Then, we have the bandlimitedness property,

$$\widehat{\varphi}_{\mathsf{LP}}(\omega) = K_b e^{\jmath \frac{d\omega^2}{2b}} e^{-\frac{\omega}{b}(dp-bq)} \underbrace{\Pi\left(\frac{\omega}{2\pi\Omega}\right)}_{\text{Bandlimited}} \quad (9)$$

where $\Pi(\omega) = 1, |\omega| \leqslant 1/2$ and zero elsewhere.

Consequently, any function that belongs to the subspace of SAFT–bandlimited functions may be exactly recovered via its orthogonal projection onto this subspace, that is, $f = \mathscr{P}_\varphi f$. This results in an extension of Shannon's Sampling Theorem for the SAFT–domain [28]:

Extension of Shannon's Sampling Theorem to SAFT Domain

$$\begin{aligned}\mathscr{P}_\varphi f &= \sum_{n\in\mathbb{Z}} \langle f, \varphi_n \rangle \varphi_n(t) \\ &= \frac{e^{-\jmath \frac{at^2}{2b}}}{\Delta} \sum_{n\in\mathbb{Z}} f(n\Delta) e^{\jmath \frac{a(n\Delta)^2}{2b}} e^{-\jmath p \frac{t-n\Delta}{b}} \mathrm{sinc}\left(\frac{t}{\Delta} - n\right).\end{aligned}$$

In fact, the inner–product $\langle f, \varphi_n \rangle$ is linked with the SAFT convolution operation. Let $\psi(t) = (\Delta K_b)^{-1} e^{-\jmath \frac{at^2}{2b}} e^{-\jmath p \frac{t}{b}} \mathrm{sinc}\left(-\frac{t}{\Delta}\right)$ be an SAFT–bandlimited function. It is easy to verify that,

$$\langle f, \varphi_n \rangle = \left(f *_{\mathbf{\Lambda}_\mathsf{S}} \psi_n\right)\big|_{t=n\Delta, n\in\mathbb{Z}}. \quad (10)$$

Hence $\langle f, \varphi_n \rangle$ amounts to low–pass filtering followed by sampling.

3. SAMPLING FRI SIGNALS IN THE SAFT–DOMAIN

Let $\varphi_{\mathsf{LP}} = \varphi_0 = \Delta^{-1} e^{-\jmath \mathcal{Q}(t)} \mathrm{sinc}(t/\Delta)$ (cf. (8)) be a $\pi\Omega$–bandlimited sampling kernel with

$$\mathcal{Q}(t) = \left(at^2 + 2pt\right)/2b, \quad (11)$$

and $s(t)$ be the sparse/FRI signal defined in (2). Also, let

$$y(t) = \left(s *_{\mathbf{\Lambda}_\mathsf{S}} \varphi_{\mathsf{LP}}\right)(t) \quad (12)$$

be low–pass filtered measurements of (2) where $\Delta = b/\Omega$ is the sampling rate and $*_{\mathbf{\Lambda}_\mathsf{S}}$ is the SAFT convolution operation (cf. (7)). Suppose that we observe N discrete measurements of the form,

$$y(n\Delta), \quad n = 0, \ldots, N-1. \qquad \text{(Low–Pass Samples)} \quad (13)$$

We then show that we can recover $s(t)$ from $\{y(n\Delta)\}_{n=0}^{N-1}$.

To show this, we first develop a series based representation of the sparse signal in (2) which turns out to be a parametric representation. Then, based on the series representation, we show that recovery of a sparse signal from its low-pass measurements amounts to the well known frequency estimation problem in the Fourier domain.

3.1. Special Affine Fourier Series (SAFS)

In the same way that $\left\{e^{\jmath k\omega_0 t}\right\}_{k\in\mathbb{Z}}$ are the basis functions for Fourier Series of a $T = 2\pi/\omega_0$–periodic function, we are interested in developing a series representation for the SAFT domain.

We start with the identification of time domain basis functions associated with the SAFT series. In order to define the basis functions, we compute the time-domain response of a harmonic component $\delta(\omega - n\omega_0)$. This is given by,

$$\Phi_{\mathbf{\Lambda}_\mathsf{S}}(t) = \left\langle \delta(\omega - n\omega_0), \kappa_{\mathbf{\Lambda}_\mathsf{S}^{\mathrm{inv}}}(\omega, t) \right\rangle \equiv \kappa^*_{\mathbf{\Lambda}_\mathsf{S}^{\mathrm{inv}}}(n\omega_0, t). \quad (14)$$

For a time–limited signal $s(t), t \in [0, T)$, our goal is to obtain a series expansion of the form,

$$s(t) = \sum\nolimits_n \widehat{s}_{\mathbf{\Lambda}_\mathsf{S}}[n] \, \kappa^*_{\mathbf{\Lambda}_\mathsf{S}^{\mathrm{inv}}}(n\omega_0, t), \quad (15)$$

where the SAFS coefficients are defined by

$$\widehat{s}_{\mathbf{\Lambda}_\mathsf{S}}[n] = \langle s, \kappa_{\mathbf{\Lambda}_\mathsf{S}}(\cdot, n\omega_0) \rangle_{[0,T]}. \quad (16)$$

For this to be possible, we must enforce the orthonormality condition on the basis functions involved, that is,

$$\left\langle \kappa_{\mathbf{\Lambda}_\mathsf{S}}(t, n\omega_0), \kappa_{\mathbf{\Lambda}_\mathsf{S}^{\mathrm{inv}}}(k\omega_0, t) \right\rangle_{[0,T]} = \delta_{n-k}.$$

Developing this equation and constraining the orthonormality property, leads to the requirement $\omega_0 = 2\pi b/T$. Furthermore, we compute the scaling constant that enforces orthonormality condition. Let,

$$m_{n,k} = \left\langle \kappa_{\mathbf{\Lambda}_\mathsf{S}}(t, n\omega_0), \kappa_{\mathbf{\Lambda}_\mathsf{S}^{\mathrm{inv}}}(k\omega_0, t) \right\rangle.$$

With $n = k$, $m_{0,0} = T/2\pi|b|$. Hence, we scale the basis functions by $1/\sqrt{m_{0,0}}$ to obtain an orthonormal basis. For simplicity, we will assume that the constant $K_b/\sqrt{m_{0,0}}$ has been absorbed into $\widehat{s}_{\mathbf{\Lambda}_\mathsf{S}}[n]$.

3.2. SAFT Series of Sparse Signals

We now use the SAFT series to obtain a series expansion of the of the sparse signal $s(t)$ specified in (2). Let $T = |\max t_k - \min t_k|$ since $s(t)$ is time–limited. As a result, we compute the coefficients,

$$\begin{aligned}\widehat{s}_{\mathbf{\Lambda}_\mathsf{S}}[n] &\overset{(15)}{=} \langle s, \kappa_{\mathbf{\Lambda}_\mathsf{S}}(t, n\omega_0) \rangle_{[0,T]} = \int_0^T s(t) \kappa^*_{\mathbf{\Lambda}_\mathsf{S}}(t, n\omega_0)\, dt \\ &= \sum\nolimits_{k=0}^{K-1} \mu_k \kappa^*_{\mathbf{\Lambda}_\mathsf{S}}(t_k, n\omega_0), \quad \omega_0 = 2\pi b/T. \end{aligned} \quad (17)$$

Back substituting this result into (15), we obtain the SAFT series,

$$s(t) \overset{(16)}{=} \sum_{n\in\mathbb{Z}} \sum_{k=0}^{K-1} \mu_k \kappa^*_{\mathbf{\Lambda}_\mathsf{S}}(t_k, n\omega_0) \kappa^*_{\mathbf{\Lambda}_\mathsf{S}^{\mathrm{inv}}}(n\omega_0, t). \quad (18)$$

Using the definition of the SAFT kernel in (1), we simplify the product $\kappa^*_{\mathbf{\Lambda}_\mathsf{S}}(t_k, n\omega_0)\, \kappa^*_{\mathbf{\Lambda}_\mathsf{S}^{\mathrm{inv}}}(n\omega_0, t)$ to obtain,

$$\kappa^*_{\mathbf{\Lambda}_\mathsf{S}}(t_k, n\omega_0)\, \kappa^*_{\mathbf{\Lambda}_\mathsf{S}^{\mathrm{inv}}}(n\omega_0, t) \overset{(1)}{=} e^{-\jmath \mathcal{Q}(t) - \mathcal{Q}(t_k)} e^{\jmath \frac{\omega_0 n}{b}(t-t_k)} \quad (19)$$

where $\mathcal{Q}(t)$ is defined in (11). Plugging into (18) results in,

$$s(t) = e^{-\jmath\mathcal{Q}(t)} \sum\nolimits_{n\in\mathbb{Z}} \sum\nolimits_{k=0}^{K-1} \underbrace{\mu_k e^{\jmath\mathcal{Q}(t_k)}}_{\rho_k} \underbrace{e^{-\jmath\frac{\omega_0 n t_k}{b}}}_{u_k^n} e^{\jmath\frac{\omega_0 n t}{b}}$$
$$= e^{-\jmath\mathcal{Q}(t)} \sum\nolimits_{n\in\mathbb{Z}} \widehat{h}[n] e^{\jmath\frac{\omega_0 n t}{b}}, \quad (20)$$

where $\widehat{h}[n]$ is a sum of complex exponentials:

$$\widehat{h}[n] = \sum\nolimits_{k=0}^{K-1} \mu_k e^{\jmath\mathcal{Q}(t_k)} e^{\jmath\frac{\omega_0 n}{b}t_k} = \sum\nolimits_{k=0}^{K-1} \rho_k u_k^n. \quad (21)$$

A re-arrangement of the terms above shows an underlying Fourier Series that is linked with the sparse signal,

$$\underbrace{s(t) e^{\jmath\mathcal{Q}(t)}}_{h(t)} = \underbrace{\sum\nolimits_{n\in\mathbb{Z}} \widehat{h}[n] e^{\jmath\frac{2\pi}{T}nt}}_{\text{Fourier Series}} \equiv h(t). \quad (22)$$

Hence, even when dealing with something very general such as the SAFT, the sparse signal is characterized by an underlying Fourier Series whose coefficients are linked to a sum of complex exponentials.

3.3. Sampling and Reconstruction of FRI Signals in SAFT–domain

Our main result takes form of the theorem below. We describe a constructive procedure which leads to a recovery method for the sparse signal $s(t)$. This is based on the fact that it is possible to estimate $\widehat{h}[n]$ in (21) from low–pass samples in (13). Having estimated $\widehat{h}[n]$, we can use the annihilating filter [4] or any other FRI technique to estimate the FRI parameters $\{\mu_k, t_k\}_{k=0}^{K-1}$.

Theorem 1 (FRI Sampling in SAFT Domain) *Let $s(t)$ be a continuous–time, FRI/sparse signal (2) and let $\varphi_{\mathsf{LP}}(t) = \Delta^{-1} e^{-\jmath\mathcal{Q}(t)} \operatorname{sinc}(t/\Delta)$ be the low–pass filter associated with the SAFT domain with $\mathcal{Q}(t) = (at^2 + 2pt)/2b$ and $\Delta = b/\Omega$. Suppose that we observe low–pass filtered samples $y(n\Delta) = (s *_{\mathbf{\Lambda}_{\mathsf{S}}} \varphi_{\mathsf{LP}})(n\Delta), n = 0, \ldots, N-1$. Provided that $\mathbf{\Lambda}_{\mathsf{S}}$ is known and $N \geq T/\Delta + 1$, the samples $y(n\Delta)$ are a sufficient characterization of the FRI signal $s(t)$ in (2).*

First, by using the definition of $s(t)$ in (20) and following (12),

$$y(t) = \frac{e^{-\jmath\frac{at^2}{2b}}}{\Delta} \left(e^{-\jmath\frac{pt}{b}} h(t) * e^{-\jmath\frac{pt}{b}} \operatorname{sinc}(\Delta^{-1}t)\right)$$
$$= \frac{e^{-\jmath\mathcal{Q}(t)}}{\Delta} \sum_{m\in\mathbb{Z}} \widehat{h}[m] e^{\jmath\frac{\omega_0 m t}{b}} \int \operatorname{sinc}(\Delta^{-1}t) e^{-\jmath\frac{\omega_0 m t}{b}} dt$$
$$= e^{-\jmath\mathcal{Q}(t)} \sum_{|m|\leqslant f_c} \widehat{h}[m] e^{\jmath\frac{\omega_0 m t}{b}}, \quad f_c = \lfloor \Omega T/2b \rfloor.$$

Next, we modulate the samples by $e^{\jmath\mathcal{Q}(t)}$ to obtain a Fourier Series:

Low–Pass Samples of FRI Signal

$$\boxed{g_n = \underbrace{y(n\Delta) e^{\jmath\mathcal{Q}(n\Delta)}}_{\text{Modulated Low–Pass Samples}} = \sum_{|m|\leqslant f_c} \widehat{h}[m] e^{\jmath\frac{\omega_0 m}{b}n\Delta}.} \quad (23)$$

From (21), the coefficients $\widehat{h}[m], |m| \leqslant f_c = \lfloor \Omega T/2b \rfloor$ are a linear combination of complex exponentials linked with FRI parameters.

We can now treat our recovery problem in two steps. First, given $\{g_n\}_{n=0}^{N-1}$, we recover $\widehat{h}[m]$ from (23). With $\widehat{h}[m]$ known, we estimate the FRI parameters $\{\mu_k, t_k\}_{k=0}^{K-1}$ using the annihilation principle [4,8].

Let $\mathbf{g} = \begin{bmatrix} g_0 & \cdots & g_{N-1} \end{bmatrix}^\top$ be the vector of N samples. Then we can write $\mathbf{g} \overset{(23)}{=} \mathbf{V}\mathbf{h}$, where $\mathbf{V}$ is a $N \times 2f_c+1$ Vandermonde matrix with element $[\mathbf{V}]_{n,m} = e^{\jmath\frac{2\pi\Delta}{T}mn}$ and $\mathbf{h} = \begin{bmatrix} \widehat{h}_{-f_c} & \cdots & \widehat{h}_{+f_c} \end{bmatrix}^\top$ is the vector of $2f_c+1$ coefficients we seek. With $f_c = \lfloor \Omega T/2b \rfloor$ we see that $T/\Delta = 2f_c$. Provided that $N \geq 2f_c + 1 \equiv N \geqslant T/\Delta + 1$, we can compute $\mathbf{h} = \mathbf{V}^+\mathbf{g}$ where $(\cdot)^+$ denotes the pseudo–inverse. With $\mathbf{h}$ known, we can estimate $\{\mu_k, t_k\}_{k=0}^{K-1}$ associated with the sparse signal in (20) using any of the spectral estimation methods [8].

Central to the theme of estimation of innovation parameters $\{\mu_k, t_k\}_{k=0}^{K-1}$ is the observation that the sequence $\mathbf{h}$ admits an auto–regressive solution,

$$\widehat{h}[m] + \sum\nolimits_{k=1}^{K} r[k] \widehat{h}[m-k] = 0, \quad (24)$$

where the filter r is associated the *annihilating polynomial* [4, 5],

$$\mathcal{R}(z) = \prod\nolimits_{k=0}^{K-1} (1 - u_k/z) \equiv \sum\nolimits_{k=0}^{K} r[k] z^{-k}.$$

The annihilation equation (24) can be solved provided that $\widehat{h}[m], m \in [-K, K]$ is known, implying that $f_c = \lfloor \Omega T/2b \rfloor \geqslant K$ or,

$$N \geqslant T/\Delta + 1. \quad (25)$$

Whenever this sampling condition is true, we can estimate the $K+1$ coefficient FIR filter r in (24) with which we construct $\mathcal{R}(z)$. The roots of this polynomial are nothing but $\{\widetilde{u}_k\}_{k=0}^{K-1}$ [4]. We then estimate innovation parameter $\widetilde{t}_k = (b/\omega_0 m) \angle\widetilde{u}_k$. With $\widetilde{t}_k$ known, we construct the quadratic polynomial $\mathcal{Q}(t_k) = (at_k^2 + 2pt_k)/2b$. Now it remains to estimate weights μ_k which are simply the solution to the following linear least–squares problem:

$$\{\widetilde{\mu}_k\}_{k=0}^{K-1} = \min_{\mu_k} \sum_m \left|\widehat{h}[m] - \sum\nolimits_{k=0}^{K-1} \mu_k e^{\jmath\mathcal{Q}(\widetilde{t}_k)} \widetilde{u}_k^m\right|^2.$$

Remark (Generalization and Backward Compatibility) Even though our result is quite general, the sampling condition (25) is the same for the Fourier Domain. This is because dealing with SAFT domain still allows us to use operations based on Fourier Series (20). Furthermore, by appropriately parameterizing $\mathbf{\Lambda}_{\mathsf{S}}$, one may now use the FRI result for any of the operations described in Table 1. For example, with $\mathbf{\Lambda}_{\mathsf{S}} = \mathbf{\Lambda}_{\mathsf{FT}}$ and $\mathbf{\Lambda}_{\mathsf{S}} = \mathbf{\Lambda}_{\mathsf{FrFT}}$, we obtain the results of [4] and [11], respectively.

4. CONCLUSION

We have described a very general recipe for extension of FRI sampling theory to a wide class of mathematical operations listed in Table 1. This was possible by developing a sampling theory related result for the Special Affine Fourier Transforms (SAFT). More precisely, we showed that the orthogonal projection of a signal onto the subspace of SAFT–bandlimited functions is equivalent to low–pass filtering followed by sampling. Based on this equivalence, we studied the representation of FRI signals in the SAFT domain. Such signals assume a parametric representation in the SAFT domain and the parameters may be estimated using annihilating filters. Our work finds many interesting extensions including the study of Strang–Fix kernels [9, 29] as well as analysis of polynomial phase systems [18, 19] via FRI principles.

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