SEQUENCE DESIGN TO MINIMIZE THE PEAK SIDELOBE LEVEL

Junxiao Song, Prabhu Babu, and Daniel P. Palomar

Department of Electronic and Computer Engineering, Hong Kong University of Science and Technology, Hong Kong

ABSTRACT

Sequences with low aperiodic autocorrelation sidelobes are well-known to have extensive applications in active sensing and communication systems. In this paper, we consider the sequence design problem of minimizing the ℓ_p-norm of the autocorrelation sidelobes, which can then be used to minimize the peak sidelobe level (PSL) criterion. An algorithm based on the general majorization-minimization method is developed to tackle the problem. The proposed algorithm can be implemented by means of fast Fourier transform (FFT) operations and thus is computationally efficient in practice. Numerical experiments show that the proposed algorithm can produce very long sequences with impulse-like autocorrelation and with much smaller PSL compared with some well-known analytical sequences.

***Index Terms*—** Autocorrelation, majorization-minimization, peak sidelobe level, unimodular sequences.

1. INTRODUCTION

Sequences with good autocorrelation properties lie at the heart of many active sensing and communication systems. Important applications include synchronization of digital communication systems (e.g., GPS receivers or CDMA cellular systems), pilot sequences for channel estimation, coded sonar and radar systems, and even cryptography for secure systems [1–5]. In practice, due to the limitations of sequence generation hardware components (such as the maximum signal amplitude clip of analog-to-digital converters and power amplifiers), unimodular sequences are of special interest because of their maximum energy efficiency [5].

Let $\{x_n\}_{n=1}^N$ denote a complex unimodular sequence of length N, then the aperiodic autocorrelations of $\{x_n\}_{n=1}^N$ are defined as

$$r_k = \sum_{n=1}^{N-k} x_n^* x_{n+k} = r_{-k}^*,\ k = 0, \ldots, N-1. \tag{1}$$

The problem of sequence design for good autocorrelation properties usually arises when small autocorrelation sidelobes (i.e., $k \neq 0$) are required. To measure the goodness of the autocorrelation property of a sequence, a commonly used metric is the peak sidelobe level (PSL)

$$\text{PSL} = \max\{|r_k|\}_{k=1}^{N-1}. \tag{2}$$

Owing to the practical importance of sequences with low autocorrelation sidelobes, a lot of effort has been devoted to identifying such sequences. Binary Barker sequences, with their peak sidelobe level (PSL) no greater than 1, are perhaps the most well-known such sequences [6]. However, it is generally accepted that they do not exist for lengths greater than 13. In 1965, Golomb and Scholtz [7] started to investigate more general sequences called generalized Barker sequences, which obey the same PSL maximum, but may have complex (polyphase) elements. Since then, a lot of work has been done to extend the list of polyphase Barker sequences [8–10], and the longest one ever found is of length 77. It is still unknown whether there exist longer polyphase Barker sequences. Apart from searching for longer polyphase Barker sequences, some families of polyphase sequences with good autocorrelation properties that can be constructed in closed-form have also been proposed in the literature, such as the Frank sequences [11], the Chu sequences [12], and the Golomb sequences [13].

In this paper, we develop an efficient optimization algorithm to minimize the ℓ_p-norm of the autocorrelation sidelobes, which can be used to design sequences with low PSL metric. The proposed algorithm is derived based on the general majorization-minimization (MM) method and can be implemented by means of FFT operations. Numerical experiments show that the proposed algorithm can generate long sequences with very low PSL.

Notation: $\text{Re}(\cdot)$ and $\text{Im}(\cdot)$ denote the real and imaginary part, respectively. $\arg(\cdot)$ denotes the phase of a complex number. The superscripts $(\cdot)^T$, $(\cdot)^*$ and $(\cdot)^H$ denote transpose, complex conjugate, and conjugate transpose, respectively. $\circ$ denotes the Hadamard product. x_i denotes the i-th element of vector $\mathbf{x}$. $\mathbf{X}_{i:j,k:l}$ denotes the submatrix of $\mathbf{X}$ from $X_{i,k}$ to $X_{j,l}$. $\text{Tr}(\cdot)$ denotes the trace of a matrix. $\text{vec}(\mathbf{X})$ is a column vector consisting of all the columns of $\mathbf{X}$ stacked. $\mathbf{I}_n$ denotes an $n \times n$ identity matrix.

2. PROBLEM FORMULATION

Let $\{x_n\}_{n=1}^N$ denote the complex unimodular sequence to be designed and $\{r_k\}_{k=1}^{N-1}$ be the aperiodic autocorrelations of $\{x_n\}_{n=1}^N$ as defined in (1). In this paper, we consider the general ℓ_p-norm metric of the autocorrelation sidelobes defined as

$$\left(\sum_{k=1}^{N-1} |r_k|^p\right)^{1/p} \tag{3}$$

with $2 \le p < \infty$. The motivation is that by choosing different p values, we may get different metrics of particular interest. Especially, by choosing $p \to +\infty$, the ℓ_p-norm metric tends to the ℓ_∞-norm of the autocorrelation sidelobes, which is known as the PSL. Then the problem of interest is the following ℓ_p-norm ($2 \le p < \infty$) metric minimization problem:

$$\begin{array}{ll} \underset{x_n}{\mathsf{minimize}} & \left(\sum_{k=1}^{N-1} |r_k|^p\right)^{1/p} \\ \mathsf{subject\ to} & |x_n| = 1,\ n = 1, \ldots, N, \end{array} \tag{4}$$

which is equivalent to

$$\begin{array}{ll} \underset{x_n}{\mathsf{minimize}} & \sum_{k=1}^{N-1} |r_k|^p \\ \mathsf{subject\ to} & |x_n| = 1,\ n = 1, \ldots, N. \end{array} \tag{5}$$

If we choose $p = 2$, problem (5) reduces to the integrated sidelobe level (ISL) minimization problem considered in [14, 15].

This work was supported by the Hong Kong RGC 16206315 research grant.

3. PSL MINIMIZATION VIA MAJORIZATION-MINIMIZATION

The majorization-minimization (MM) method is an iterative approach to solve optimization problems that are too difficult to solve directly. Interested readers may refer to [16] and references therein for more details.

Let $f(\mathbf{x})$ denote the objective function of the problem (5) and $\mathcal{X} \in \mathbf{C}^N$ be the constraint set. Instead of minimizing $f(\mathbf{x})$ directly, the MM approach optimizes a sequence of approximate objective functions that majorize $f(\mathbf{x})$. Formally, a function $u(\mathbf{x}, \mathbf{x}^{(l)})$ is said to majorize the function $f(\mathbf{x})$ at the point $\mathbf{x}^{(l)}$ over $\mathcal{X}$ provided

$$u(\mathbf{x}, \mathbf{x}^{(l)}) \geq f(\mathbf{x}), \quad \forall \mathbf{x} \in \mathcal{X}, \tag{6}$$

$$u(\mathbf{x}^{(l)}, \mathbf{x}^{(l)}) = f(\mathbf{x}^{(l)}). \tag{7}$$

In other words, the majorization function $u(\mathbf{x}, \mathbf{x}^{(l)})$ is an upper bound of $f(\mathbf{x})$ over $\mathcal{X}$ and coincides with $f(\mathbf{x})$ at $\mathbf{x}^{(l)}$.

To tackle the problem (5) via majorization-minimization, we need to construct a majorization function of the objective and the idea is to first majorize each $|r_k|^p, k = 1, \ldots, N-1$ by a quadratic function of $|r_k|$ locally based on the following lemma [17].

Lemma 1. *Let $f(x) = x^p$ with $p \geq 2$ and $x \in [0, t]$. Then for any given $x_0 \in [0, t)$, $f(x)$ is majorized at x_0 over the interval $[0, t]$ by the following quadratic function:*

$$ax^2 + (px_0^{p-1} - 2ax_0)x + ax_0^2 - (p-1)x_0^p, \tag{8}$$

where

$$a = \frac{t^p - x_0^p - px_0^{p-1}(t - x_0)}{(t - x_0)^2}. \tag{9}$$

Given $\left|r_k^{(l)}\right|$ at iteration l, according to Lemma 1, we know that $|r_k|^p$ $(p \geq 2)$ is majorized at $\left|r_k^{(l)}\right|$ over $[0, t]$ by

$$a_k |r_k|^2 + b_k |r_k| + a_k \left|r_k^{(l)}\right|^2 - (p-1)\left|r_k^{(l)}\right|^p, \tag{10}$$

where

$$a_k = \frac{t^p - \left|r_k^{(l)}\right|^p - p\left|r_k^{(l)}\right|^{p-1}(t - \left|r_k^{(l)}\right|)}{(t - \left|r_k^{(l)}\right|)^2}, \tag{11}$$

$$b_k = p\left|r_k^{(l)}\right|^{p-1} - 2a_k\left|r_k^{(l)}\right|. \tag{12}$$

Since the objective decreases at every iteration in the MM framework, it is sufficient to majorize $|r_k|^p$ over the set on which the objective is smaller than the current objective, i.e., $\sum_{k=1}^{N-1} |r_k|^p \leq \sum_{k=1}^{N-1} \left|r_k^{(l)}\right|^p$, which implies $|r_k| \leq \left(\sum_{k=1}^{N-1} \left|r_k^{(l)}\right|^p\right)^{\frac{1}{p}}$. Hence we can choose $t = \left(\sum_{k=1}^{N-1} \left|r_k^{(l)}\right|^p\right)^{\frac{1}{p}}$ in (11). Then the majorized problem of (5) is given by (ignoring the constant terms)

$$\begin{array}{ll} \underset{x_n}{\text{minimize}} & \sum_{k=1}^{N-1} \left(a_k |r_k|^2 + b_k |r_k|\right) \\ \text{subject to} & |x_n| = 1, \, n = 1, \ldots, N. \end{array} \tag{13}$$

Let us define $\mathbf{U}_k, \; k = 1-N, \ldots, N-1$ to be $N \times N$ Toeplitz matrices with the kth diagonal elements being 1 and 0 elsewhere, i.e.,

$$[\mathbf{U}_k]_{i,j} = \begin{cases} 1 & \text{if } j - i = k \\ 0 & \text{if } j - i \neq k, \end{cases} \quad i, j = 1, \ldots, N. \tag{14}$$

Noticing that

$$r_k = \mathbf{x}^H \mathbf{U}_k \mathbf{x} = \text{Tr}(\mathbf{U}_k \mathbf{x}\mathbf{x}^H), \; k = 1-N, \ldots, N-1, \tag{15}$$

we can rewrite the objective of problem (13) as follows:

$$\begin{aligned} &\sum_{k=1}^{N-1} \left(a_k \left|\text{Tr}(\mathbf{U}_k \mathbf{x}\mathbf{x}^H)\right|^2 + b_k |r_k|\right) \\ =&\frac{1}{2} \sum_{k=1-N}^{N-1} \left(a_k \left|\text{vec}(\mathbf{x}\mathbf{x}^H)^H \text{vec}(\mathbf{U}_k)\right|^2 + b_k |r_k|\right) \\ =&\frac{1}{2} \text{vec}(\mathbf{x}\mathbf{x}^H)^H \mathbf{L} \text{vec}(\mathbf{x}\mathbf{x}^H) + \frac{1}{2} \sum_{k=1-N}^{N-1} b_k |r_k|, \end{aligned} \tag{16}$$

where $a_{-k} = a_k$, $b_{-k} = b_k$, $a_0 = b_0 = 0$ and

$$\mathbf{L} = \sum_{k=1-N}^{N-1} a_k \text{vec}(\mathbf{U}_k) \text{vec}(\mathbf{U}_k)^H. \tag{17}$$

It has been shown that the maximum eigenvalue of $\mathbf{L}$ can be computed in closed form as follows [17]:

$$\lambda_{\max}(\mathbf{L}) = \max_k \{a_k(N-k) | k = 1, \ldots, N-1\}. \tag{18}$$

Then we can further majorize the first term of the objective based on the following simple result [15].

Lemma 2. *Let $\mathbf{L}$ be an $n \times n$ Hermitian matrix and $\mathbf{M}$ be another $n \times n$ Hermitian matrix such that $\mathbf{M} - \mathbf{L} \succeq \mathbf{0}$. Then for any point $\mathbf{x}_0 \in \mathbf{C}^n$, the quadratic function $\mathbf{x}^H \mathbf{L} \mathbf{x}$ is majorized by $\mathbf{x}^H \mathbf{M} \mathbf{x} + 2\text{Re}\left(\mathbf{x}^H (\mathbf{L} - \mathbf{M}) \mathbf{x}_0\right) + \mathbf{x}_0^H (\mathbf{M} - \mathbf{L}) \mathbf{x}_0$ at $\mathbf{x}_0$.*

By choosing $\mathbf{M} = \lambda_{\max}(\mathbf{L})\mathbf{I}$ in Lemma 2, we know that the first term in (16) is majorized by the following function at $\mathbf{x}^{(l)}$:

$$\begin{aligned} &u_1(\mathbf{x}, \mathbf{x}^{(l)}) \\ =& \frac{1}{2} \lambda_{\max}(\mathbf{L}) \text{vec}(\mathbf{x}\mathbf{x}^H)^H \text{vec}(\mathbf{x}\mathbf{x}^H) \\ &+ \text{Re}\big(\text{vec}(\mathbf{x}\mathbf{x}^H)^H (\mathbf{L} - \lambda_{\max}(\mathbf{L})\mathbf{I}) \text{vec}(\mathbf{x}^{(l)}\mathbf{x}^{(l)H})\big) \\ &+ \frac{1}{2} \text{vec}(\mathbf{x}^{(l)}\mathbf{x}^{(l)H})^H (\lambda_{\max}(\mathbf{L})\mathbf{I} - \mathbf{L}) \text{vec}(\mathbf{x}^{(l)}\mathbf{x}^{(l)H}), \end{aligned} \tag{19}$$

which can be simplified as

$$\mathbf{x}^H \left(\mathbf{R} - \lambda_{\max}(\mathbf{L}) \mathbf{x}^{(l)} (\mathbf{x}^{(l)})^H\right) \mathbf{x} + \text{const}, \tag{20}$$

where $\mathbf{R} = \sum_{k=1-N}^{N-1} a_k r_{-k}^{(l)} \mathbf{U}_k$ and const is some constant that does not depend on $\mathbf{x}$ (here we use the fact that $|x_n| = 1, \; n = 1, \ldots, N$).

For the second term, since it can be shown that $b_k \leq 0$, we have

$$\frac{1}{2} \sum_{k=1-N}^{N-1} b_k |r_k| \leq \frac{1}{2} \sum_{k=1-N}^{N-1} b_k \text{Re} \left\{ r_k^* \frac{r_k^{(l)}}{\left|r_k^{(l)}\right|} \right\} \tag{21}$$

$$= \frac{1}{2} \sum_{k=1-N}^{N-1} b_k \text{Re} \left\{ \mathbf{x}^H \mathbf{U}_{-k} \mathbf{x} \frac{r_k^{(l)}}{\left|r_k^{(l)}\right|} \right\} \tag{22}$$

$$= \frac{1}{2} \mathbf{x}^H \left(\sum_{k=1-N}^{N-1} b_k \frac{r_k^{(l)}}{\left|r_k^{(l)}\right|} \mathbf{U}_{-k} \right) \mathbf{x}. \tag{23}$$

By adding the two majorization functions, i.e., (20) and (23), and defining

$$w_{-k} = w_k = a_k + \frac{b_k}{2\left|r_k^{(l)}\right|} = \frac{p}{2}\left|r_k^{(l)}\right|^{p-2}, k = 1, \dots, N-1, \tag{24}$$

we have the majorized problem of (13) given by

$$\begin{array}{ll} \underset{x_n}{\text{minimize}} & \mathbf{x}^H\left(\tilde{\mathbf{R}} - \lambda_{\max}(\mathbf{L})\mathbf{x}^{(l)}(\mathbf{x}^{(l)})^H\right)\mathbf{x} \\ \text{subject to} & |x_n| = 1,\, n = 1, \dots, N, \end{array} \tag{25}$$

where $\tilde{\mathbf{R}} = \sum_{k=1-N}^{N-1} w_k r_{-k}^{(l)} \mathbf{U}_k$ is a Hermitian Toeplitz matrix.

It is clear that the objective function in (25) is quadratic in $\mathbf{x}$, but problem (25) is still hard to solve directly. So we propose to majorize the objective function of problem (25) based on Lemma 2 again to further simplify the problem. This time we need the following result regarding the bound of the extreme eigenvalue of a Hermitian Toeplitz matrix [18].

Lemma 3. *Let $\mathbf{T}$ be an $N \times N$ Hermitian Toeplitz matrix defined by $\{t_k\}_{k=0}^{N-1}$ as follows*

$$\mathbf{T} = \begin{bmatrix} t_0 & t_1^* & \dots & t_{N-1}^* \\ t_1 & t_0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & t_1^* \\ t_{N-1} & \dots & t_1 & t_0 \end{bmatrix}$$

and $\mathbf{F}$ be a $2N \times 2N$ FFT matrix with $F_{m,n} = e^{-j\frac{2mn\pi}{2N}}, 0 \le m, n < 2N$. Let $\mathbf{c} = [t_0, t_1, \cdots, t_{N-1}, 0, t_{N-1}^*, \cdots, t_1^*]^T$ and $\boldsymbol{\mu} = \mathbf{F}\mathbf{c}$ be the discrete Fourier transform of $\mathbf{c}$. Then

$$\lambda_{\max}(\mathbf{T}) \le \frac{1}{2}\left(\max_{1\le i\le N} \mu_{2i} + \max_{1\le i\le N} \mu_{2i-1}\right), \tag{26}$$

Since the matrix $\tilde{\mathbf{R}}$ is Hermitian Toeplitz, according to Lemma 3, we know that

$$\lambda_{\max}\left(\tilde{\mathbf{R}}\right) \le \frac{1}{2}\left(\max_{1\le i\le N} \mu_{2i} + \max_{1\le i\le N} \mu_{2i-1}\right), \tag{27}$$

where $\boldsymbol{\mu} = \mathbf{F}\mathbf{c}$ and

$$\mathbf{c} = [0, w_1 r_1^{(l)}, \dots, w_{N-1} r_{N-1}^{(l)}, 0, w_{N-1} r_{1-N}^{(l)}, \dots, w_1 r_{-1}^{(l)}]^T. \tag{28}$$

Let us denote the right hand side of (27) by λ_u, it is easy to see that

$$\lambda_u \ge \lambda_{\max}\left(\tilde{\mathbf{R}}\right) \ge \lambda_{\max}\left(\tilde{\mathbf{R}} - \lambda_{\max}(\mathbf{L})\mathbf{x}^{(l)}(\mathbf{x}^{(l)})^H\right). \tag{29}$$

Thus, we may choose $\mathbf{M} = \lambda_u \mathbf{I}$ in Lemma 2 and perform one more majorization step, and finally get the majorized problem

$$\begin{array}{ll} \underset{\mathbf{x}}{\text{minimize}} & -\text{Re}(\mathbf{y}^H\mathbf{x}) \\ \text{subject to} & |x_n| = 1,\, n = 1, \dots, N, \end{array} \tag{30}$$

where

$$\mathbf{y} = (\lambda_{\max}(\mathbf{L})N + \lambda_u)\,\mathbf{x}^{(l)} - \tilde{\mathbf{R}}\mathbf{x}^{(l)}. \tag{31}$$

It is easy to see that problem (30) has a closed form solution, which is given by

$$x_n = e^{j\arg(y_n)},\, n = 1, \dots, N. \tag{32}$$

Now we can summarize the overall algorithm and it is given in Algorithm 1. Note that, to avoid numerical issue, we have used the normalized a_k and w_k (i.e., divided by t^p) in Algorithm 1, which is equivalent to divide the objective in (13) by t^p during the derivation. Also note that since $\tilde{\mathbf{R}}$ is Hermitian Toeplitz, the matrix vector multiplication $\tilde{\mathbf{R}}\mathbf{x}^{(l)}$ has been implemented in terms of FFT operations [17].

Algorithm 1 Monotonic minimizer for the ℓ_p-metric of autocorrelation sidelobes ($p \ge 2$).

Require: sequence length N, parameter $p \ge 2$

Set $l = 0$, initialize $\mathbf{x}^{(0)}$.

repeat

- $\mathbf{f} = \mathbf{F}[\mathbf{x}^{(l)T}, \mathbf{0}_{1\times N}]^T$
- $\mathbf{r} = \frac{1}{2N}\mathbf{F}^H|\mathbf{f}|^2$
- $t = \|\mathbf{r}_{2:N}\|_p$
- $a_k = \frac{1+(p-1)\left(\frac{|r_{k+1}|}{t}\right)^p - p\left(\frac{|r_{k+1}|}{t}\right)^{p-1}}{(t-|r_{k+1}|)^2}, k = 1, \dots, N-1$
- $w_k = \frac{p}{2t^2}\left(\frac{|r_{k+1}|}{t}\right)^{p-2}, k = 1, \dots, N-1$
- $\lambda_L = \max_k\{a_k(N-k) | k = 1, \dots, N-1\}$
- $\mathbf{c} = \mathbf{r} \circ [0, w_1, \dots, w_{N-1}, 0, w_{N-1}, \dots, w_1]^T$
- $\boldsymbol{\mu} = \mathbf{F}\mathbf{c}$
- $\lambda_u = \frac{1}{2}\left(\max_{1\le i\le N}\mu_{2i} + \max_{1\le i\le N}\mu_{2i-1}\right)$
- $\mathbf{y} = \mathbf{x}^{(l)} - \frac{\mathbf{F}^H_{:,1:N}(\boldsymbol{\mu}\circ\mathbf{f})}{2N(\lambda_L N + \lambda_u)}$
- $x_n^{(l+1)} = e^{j\arg(y_n)}, n = 1, \dots, N$
- $l \leftarrow l + 1$

until convergence

4. NUMERICAL EXPERIMENTS

In this section, we present numerical results on applying the proposed algorithm to design unimodular sequences with low PSL. The acceleration scheme described in [17] (Algorithm 3) was applied in our implementation of the algorithm. All experiments were performed on a PC with a 3.20 GHz i5-3470 CPU and 8 GB RAM.

To apply the algorithm, we need to choose the parameter p. To examine the effect of the parameter p, we first apply the algorithm (denoted as MM-PSL) with four different p values, i.e., $p = 10, 100, 1000$ and 10000, to design a sequence of length $N = 400$. Frank sequences [11] are used to initialize the algorithm, which are known to be sequences with good autocorrelation. More specifically, Frank sequences are defined for lengths that are perfect squares and the Frank sequence of length $N = M^2$ is given by

$$x_{nM+k+1} = e^{j2\pi nk/M}, n, k = 0, 1, \dots, M-1. \tag{33}$$

For all p values, we stop the algorithm after 5×10^4 iterations and the evolution curves of the PSL are shown in Fig. (1). From the figure, we can see that smaller p values lead to faster convergence. However, if p is too small, it may not decrease the PSL at a later stage, as we can see that $p = 100$ finally gives smaller PSL compared with $p = 10$. It may be explained by the fact that ℓ_p-norm with larger p values approximates the ℓ_∞-norm better. So in practice, gradually increasing the p value is probably a better approach.

In the second experiment, we consider both an increasing scheme of p (MM-PSL-adaptive) and the fixed p scheme with $p = 100$. For the increasing scheme, we apply the MM-PSL algorithm with increasing p values $2, 2^2, \dots, 2^{13}$. For each p value, the stopping criterion was chosen to be $\left|\text{obj}(\mathbf{x}^{(l+1)}) - \text{obj}(\mathbf{x}^{(l)})\right| / \text{obj}(\mathbf{x}^{(l)}) \le$

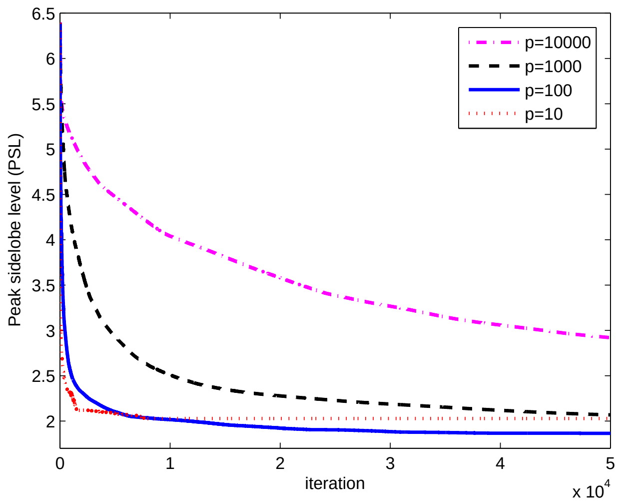

Fig. 1. The evolution curves of the peak sidelobe level (PSL).

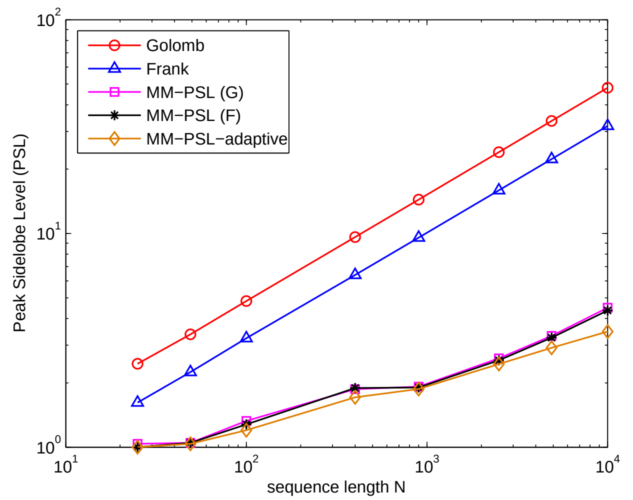

Fig. 2. Peak sidelobe level (PSL) versus sequence length. MM-PSL(G) and MM-PSL(F) denote the MM-PSL algorithm initialized by Golomb and Frank sequences, respectively.

$10^{-5}/p$, with $\text{obj}(\mathbf{x})$ being the objective in (4), and the maximum allowed number of iterations was set to be 5×10^3. For $p = 2$, the algorithm is initialized by the Frank sequence and for larger p values, it is initialized by the solution obtained at the previous p. For the fixed p scheme, the stopping criterion was chosen to be $\left|\text{obj}(\mathbf{x}^{(l+1)}) - \text{obj}(\mathbf{x}^{(l)})\right| / \text{obj}(\mathbf{x}^{(l)}) \leq 10^{-10}$, and the maximum allowed number of iterations was 2×10^5. In this case, in addition to the Frank sequence, the Golomb sequence [13] was also used as the initial sequence, which is also known for its good autocorrelation properties. In contrast to Frank sequences, Golomb sequences are defined for any positive integer and a Golomb sequence $\{x_n\}_{n=1}^N$ of length N is given by

$$x_n = e^{j\pi(n-1)n/N}, n = 1, \ldots, N. \tag{34}$$

The two schemes are applied to design sequences of the following lengths: $N = 5^2, 7^2, 10^2, 20^2, 30^2, 50^2, 70^2, 100^2$, and the PSL's of the resulting sequences are shown in Fig. 2. From the figure, we can see that for all lengths, the MM-PSL(G) and MM-PSL(F) sequences give nearly the same PSL; both are much smaller than the PSL of Golomb and Frank sequences, while a bit larger than the PSL of MM-PSL-adaptive sequences. For example, when $N = 10^4$, the PSL values of the MM-PSL(F) and MM-PSL-adaptive sequences are 4.36 and 3.48, while the PSL values of Golomb and Frank sequences are 48.03 and 31.84, respectively. The correlation level of the Golomb, Frank and the MM-PSL-adaptive sequences are shown in Fig. 3. We can notice that the autocorrelation sidelobes of the Golomb and Frank sequences are relatively large for k close to 0 and $N-1$, while the MM-PSL-adaptive sequence has much more uniform autocorrelation sidelobes across all lags.

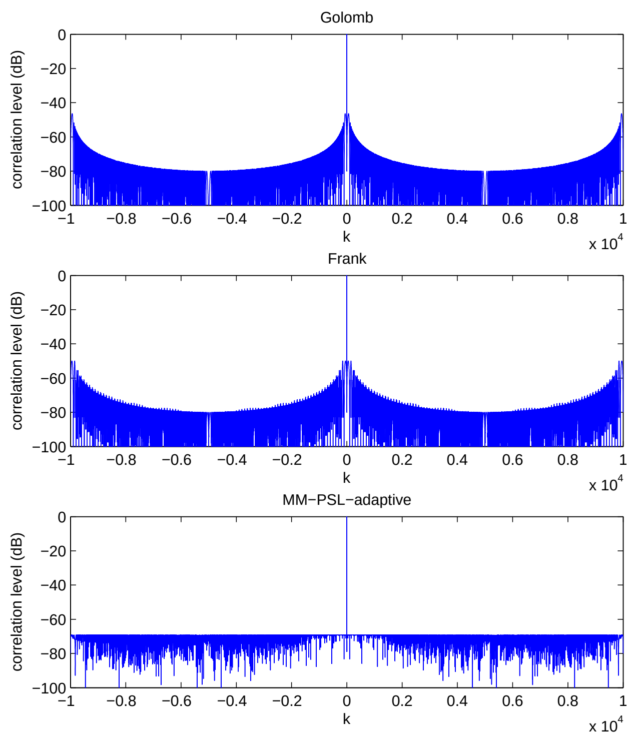

Fig. 3. Correlation level of the Golomb, Frank and MM-PSL-adaptive sequences of length $N = 10^4$.

5. CONCLUSION

We have developed an efficient algorithm for the design of unimodular sequences with low PSL metric based on the general MM method. The proposed algorithm is derived via applying three successive majorization steps. It can be implemented by means of FFT operations and thus is computationally very efficient in practice. It has been shown by numerical examples that the proposed algorithm can produce long sequences with much more uniform autocorrelation sidelobes and much smaller PSL compared with Frank and Golomb sequences, which are known for their good autocorrelation properties.

6. REFERENCES

[1] R. Turyn, “Sequences with small correlation,” *Error correcting codes*, pp. 195–228, 1968.

[2] P. Spasojevic and C. Georghiades, “Complementary sequences for ISI channel estimation,” *IEEE Transactions on Information Theory*, vol. 47, no. 3, pp. 1145–1152, Mar. 2001.

[3] N. Levanon and E. Mozeson, *Radar Signals*. John Wiley & Sons, 2004.

[4] S. W. Golomb and G. Gong, *Signal Design for Good Correlation: For Wireless Communication, Cryptography, and Radar*. Cambridge University Press, 2005.

[5] H. He, J. Li, and P. Stoica, *Waveform Design for Active Sensing Systems: A Computational Approach*. Cambridge University Press, 2012.

[6] R. Barker, "Group synchronizing of binary digital systems," *Communication theory*, pp. 273–287, 1953.

[7] S. W. Golomb and R. A. Scholtz, "Generalized Barker sequences," *IEEE Transactions on Information Theory*, vol. 11, no. 4, pp. 533–537, 1965.

[8] N. Zhang and S. W. Golomb, "Sixty-phase generalized Barker sequences," *IEEE Transactions on Information Theory*, vol. 35, no. 4, pp. 911–912, 1989.

[9] P. Borwein and R. Ferguson, "Polyphase sequences with low autocorrelation," *IEEE Transactions on Information Theory*, vol. 51, no. 4, pp. 1564–1567, 2005.

[10] C. Nunn and G. Coxson, "Polyphase pulse compression codes with optimal peak and integrated sidelobes," *IEEE Transactions on Aerospace and Electronic Systems*, vol. 45, no. 2, pp. 775–781, Apr. 2009.

[11] R. L. Frank, "Polyphase codes with good nonperiodic correlation properties," *IEEE Transactions on Information Theory*, vol. 9, no. 1, pp. 43–45, Jan. 1963.

[12] D. Chu, "Polyphase codes with good periodic correlation properties (corresp.)," *IEEE Transactions on Information Theory*, vol. 18, no. 4, pp. 531–532, 1972.

[13] N. Zhang and S. W. Golomb, "Polyphase sequence with low autocorrelations," *IEEE Transactions on Information Theory*, vol. 39, no. 3, pp. 1085–1089, 1993.

[14] P. Stoica, H. He, and J. Li, "New algorithms for designing unimodular sequences with good correlation properties," *IEEE Transactions on Signal Processing*, vol. 57, no. 4, pp. 1415–1425, 2009.

[15] J. Song, P. Babu, and D. P. Palomar, "Optimization methods for designing sequences with low autocorrelation sidelobes," *IEEE Transactions on Signal Processing*, vol. 63, no. 15, pp. 3998–4009, Aug. 2015.

[16] D. R. Hunter and K. Lange, "A tutorial on MM algorithms," *The American Statistician*, vol. 58, no. 1, pp. 30–37, 2004.

[17] J. Song, P. Babu, and D. P. Palomar, "Sequence design to minimize the weighted integrated and peak sidelobe levels," *accepted in IEEE Transactions on Signal Processing*, 2015.

[18] P. J. S. G. Ferreira, "Localization of the eigenvalues of Toeplitz matrices using additive decomposition, embedding in circulants, and the Fourier transform," in *Proceedings of the 10th IFAC Symposium on System Identification*, Copenhagen, Denmark, Jul. 1994, pp. 271–276.