DEGRADEDNESS AND STOCHASTIC ORDERS OF FAST FADING GAUSSIAN BROADCAST CHANNELS WITH STATISTICAL CHANNEL STATE INFORMATION AT THE TRANSMITTER

Pin-Hsun Lin[†], Eduard A. Jorswieck[†], Rafael F. Schaefer^{*}, Carsten Janda[†], Martin Mittelbach[†]

[†]Technische Universität Dresden Communications Theory Group {Pin-Hsun.Lin, Eduard.Jorswieck}@tu-dresden.de {Carsten.Janda, Martin.Mittelbach}@tu-dresden.de *Technische Universität Berlin, Information Theory and Applications Group rafael.schaefer@tu-berlin.de

ABSTRACT

The capacity regions of Gaussian broadcast channels depends on the knowledge of channel state information (CSI). When there is only statistical CSI at the transmitter and full CSI at the receiver, the ergodic capacity region is unknown in general. In this paper we investigate the relation between the degradedness and stochastic orders among channels from the transmitter to different receivers. We derive criteria to identify the degradedness for single and multiple-antenna cases when the channels belong to the usual stochastic order or the increasing convex order. Examples illustrate the usage of the derived criteria. We also show a case in which the channel enhancement technique can be applied even when there is only statistical CSIT.

I. INTRODUCTION

When a 2-receiver broadcast channel (BC) is degraded, we know that the capacity region of it [1] is the union, over all V, X satisfying the Markov chain $V \rightarrow X \rightarrow Y_1 Y_2$ of rate pairs (R_1, R_2) such that

$$\begin{cases} R_1 & \leq I(X;Y_1|V) \\ R_2 & \leq I(V;Y_2), \end{cases}$$

for some p(u,x).

Note that for non-degraded BC, only the inner and outer bounds are known, e.g., Marton's inner bound [2] and Nair-El Gamal outer bound [3]. Therefore it is much easier to characterize the performance of broadcast channels if we can identify the degradedness of it.

The capacity region of an additive white Gaussian noise (AWGN) BC (GBC) is known for both fixed and fading channels when the channel state information is known at the transmitter (CSIT) as well as at the receivers (CSIR). For multiple antennas GBC with perfect CSIT and CSIR, [4] invented the *channel enhancement* technique to form the degradedness and it is proved that Gaussian input is optimal. Immense endeavors have be made to solve the input covariance matrix of the Gaussian input, e.g., [5] [6] [7]. However, the problem is open when there is no perfect CSIT in general any only limited cases are known [8]. The fading BC with only perfect CSIR but no perfect CSIT lacks the degraded structure in general for arbitrary fading distributions, which makes it a challenging problem. In particular, the order of channel realizations to different receivers in fast fading broadcast channels vary within a codeword length. Therefore, intuitively we are not able to compare

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the channels as in full CSIT cases to identify the degradedness. Note that, even for degraded BCs, for example fading Gaussian BCs with coherent fading and statistical CSIT, Gaussian input is not optimal [9].

Contrary to [9], in which the Hermite polynomial is used to give a condition on what kind of fading distributions and degradedness that non-Gaussian input must be used, in this paper, given only statistical CSIT we investigate the order of channels between the transmitter to different receivers by stochastic orders [10] in order to compare the channels stochastically and to identify the degradedness. We consider fast fading GBCs with only statistical CSIT. For each individual channel the fading process is identically and independently distributed (i.i.d.) and memoryless with arbitrary distributions. More specifically, we prove that the usual stochastic order is sufficient but not necessary to result in a degraded Gaussian broadcast channel. We also identify the relation between the increasing convex order and the degradedness for single antenna cases, which is a more general stochastic order than the usual stochastic order. We then extend the result partially to multipleantenna cases. The derived results can help to show the existence of degraded broadcast channels which can further simplify the characterization of the capacity performance when there is only statistical CSIT.

The rest of the paper is organized as follows. In Section II we introduce the preliminaries and the considered system model. In Section III we discuss the derived conditions to identify degraded Gaussian broadcast channels with single antenna. Cases with multiple-antenna are discussed in Section IV. Finally Section V concludes this paper.

Notation: upper case normal/bold letter denote random variables/either random vectors or matrices, which will be defined when they are first mentioned; lower case bold letters denote vectors. The mutual information between two random variables is denoted by I(.;.). $X \to Y \to Z$ means X, Y, Z form a Markov chain. The complementary cumulative density function (CCDF) is denoted by $\overline{F}_X(x) = 1 - F_X(x)$, where $F_X(x)$ is the CDF of X. And we denote the probability mass function (PMF) and probability density function (PDF) by P and f, respectively. $X \sim F$ denotes that the random variable X follows the distribution F. vec(\mathbf{X}) concatenates the column vectors of \mathbf{X} as a super vector. \mathbf{I}_n is the identity matrix with dimension n.

II. PRELIMINARIES AND SYSTEM MODEL

In this section we first introduce the considered system model and then the underlying background knowledge for this work. We consider the case in which each node is equipped with a single antenna.

We assume that there is full CSI at the receivers such that they can compensate the phase rotation of their own channels, respectively, without changing the capacity to form real channels. Therefore, the considered L-receiver fast fading Gaussian broadcast channel can be stated as

$$Y_k = \sqrt{H_k} X + Z_k, \, k = 1 \cdots L, \tag{1}$$

where H_k is a real-valued non-negative independent random variable denoting the square of receiver k's fading channel with complementary cumulative distribution functions (CCDF) \bar{F}_{H_k} . The channel input is denoted by X. We consider the channel input power constraint as $E[X^2] \leq P_T$. The noises $\{Z_k\}$ at the corresponding receivers are independent additive white Gaussian noises (AWGN) with zero mean and unit variance. We assume that the transmitter only knows the statistics but not the instantaneous realizations of $\{H_k\}$. In the following discussion of both single and multipleantenna cases, we will consider the two-receiver case first then extend to the *L*-receiver case.

The following definition are important to derive the main results in this work.

Definition 1. A two user broadcast channel is physically degraded if the transition distribution function satisfies $f_{Y_1Y_2|X_1}(\cdot,\cdot) = f_{Y_1|X}(\cdot) f_{Y_2|Y_1}(\cdot)$, i.e., X, Y_1 , and Y_2 form a Markov chain $X \to Y_1 \to Y_2$. A Gaussian broadcast channel is stochastically degraded if its conditional marginal distribution is the same as that of a physically degraded Gaussian broadcast channel, i.e., there exists a distribution $f_{Y_1|Y_2}(\cdot)$ such that $f_{Y_1|X(y_1|x)} = \sum_{y_2} f_{Y_1'|X}(y_2|x) f_{Y_1|Y_1'}(y_1|y_2)$ and $f_{Y_1'|X} = f_{Y_2|X}$.

Since the capacity regions of broadcast channels only depend on the marginal distributions, in the following discussions and derivations, the *degraded* means the stochastically degraded. The following are definitions of the considered stochastic orders.

Definition 2. [10, (1.A.3), (4.A.5)] For random variables X and Y, the usual stochastic order and increasing convex order are respectively defined as

$$\begin{aligned} X &\leq_{st} Y : F_X(x) \leq F_Y(x), \, \forall x, \\ X &\leq_{icx} Y : \int_t^\infty \bar{F}_X(x) dx \leq \int_t^\infty \bar{F}_Y(y) dy, \, \forall t \end{aligned}$$

Definition 3. Define the following sets $S_{st} = \{(H_1, H_2) : H_1 \ge_{st} H_2\}$ and $S_{icx} = \{(H_1, H_2) : H_1 \ge_{icx} H_2\}.$

III. FADING GAUSSIAN BROADCAST CHANNEL WITH SINGLE ANTENNA

In this section we introduce the relation between the existence of the equivalent degraded Gaussian broadcast channel and different stochastic orders between channels to different receivers. This relation is helpful to identify the existence of the ergodic capacity region.

Lemma 1. The condition $H_1 \ge_{st} H_2$ is sufficient to construct an stochastically equivalent degraded Gaussian broadcast channel.

Proof. From the same marginal property, we know that if there exist H'_1 and H'_2 such that $p_{Y'_1|X} = p_{Y_1|X}$ and $p_{Y'_2|X} = p_{Y_2|X}$, where $Y'_1 = H'_1X + Z_1$ and $Y'_2 = H'_2X + Z_2$, then these two broadcast channels are stochastically equivalent. Besides, from [11, Proposition 9.2.2]



Fig. 1. The considered Gaussian broadcast channel.

we know that if $H_1 \ge_{st} H_2$, then there exist random variables H'_1 and H'_2 having the same distributions as H_1 and H_2 , respectively, such that $P(H'_1 \ge H'_2) = 1$, which completes the proof.

The equivalent channels H'_1 and H'_2 can be explicitly constructed as follows, which is extended from [12, P.15].

Lemma 2. If $H_1 \geq_{st} H_2$, and if the inverse of CDFs $F_{H_1}^{-1}$ and $F_{H_2}^{-1}$ exist, then the equivalent channels H'_1 and H'_2 can be constructed by $H'_1 = F_{H_1}^{-1}(U)$ and $H'_2 = F_{H_2}^{-1}(U)$, respectively, such that $P(H'_1 \geq H'_2) = 1$, where $U \sim Unif(0, 1)$.

Remark 1. Originally, even though we know $H_1 \ge_{st} H_2$, the order of the channel realizations $H_1 = h_1$ and $H_2 = h_2$ may vary for each realization, which hinders the justification of degradedness. However, from Lemma 2 we know that we can virtually explicitly align all the channel realizations within a codeword length such that each channel gain realization of H_1 is no worse than that of H_2 , if $H_1 \ge_{st} H_2$. Then we can claim that there exists an equivalent degraded Gaussian broadcast channel.

Definition 4. Define the set of pairs of fast fading channels (H_1, H_2) as

 $S_{\mathcal{D}} = \{(H_1, H_2): \text{ the fast fading Gaussian broadcast channel is degraded.}\}.$

In the following, we classify the relation between $\mathcal{S}_{\mathcal{D}}$ and different stochastic orders.



Fig. 2. The relation between different stochastic orders and the set $S_{\mathcal{D}}$, which is encircled with shadowed area.

Lemma 3. The usual stochastic order $H_1 \ge_{st} H_2$ is not necessary to generate an equivalent degraded broadcast channel. The increasing convex order is not sufficient to guarantee $(H_1, H_2) \in S_{\mathcal{D}}$. Furthermore, $(H_1, H_2) \in S_{\mathcal{D}}$ does not necessarily imply $H_1 \ge_{icx} H_2$,

Sketch of proof: the detailed proof can be modified from [13]. To show that the usual stochastic order is not necessary, we can

construct an example in which H_1 has non-zero probability at zero magnitude while H_2 has no zero component. Let the support of the PDF of H_1 be $\in [h_0, h_1]$, where $h_0 > 0$ is the crossing point of \overline{F}_1 and \overline{F}_2 . By definition, this example does not satisfy the usual stochastic order. In addition, we can prove that this example satisfies $P(r \leq \tilde{H}_r \leq r + \varepsilon, e \leq \tilde{H}_e) = 0$, where $\varepsilon > 0$ is arbitrarily small and $e > r + \varepsilon$, by constructing an equivalent joint CCDF $\bar{F}_{\tilde{H}_r,\tilde{H}_e}(r,e) = \min\{\bar{F}_{\tilde{H}_r}(r),\bar{F}_{\tilde{H}_e}(e)\}$ [8]. Therefore, this example is degraded. To show the second statement of this lemma, we can exploit the above example with the additional condition $\int_0^{h_0} |\bar{F}_{H_1}(h) - \bar{F}_{H_2}(h)| dh \leq \int_{h_0}^{\infty} |\bar{F}_{H_1}(h) - \bar{F}_{H_2}(h)| dh$. Then we have an example which satisfies the increasing convex order. That means, there exists $(H_1, H_2) \in S_{icx}$ which is degraded. To show the third statement, we can show the existence of cases with orders more general than the increasing convex order which can result in degraded broadcast channel by constructing an example in which there are several discrete channel values with non-zero probabilities. By using the proof of usual stochastic order is not necessary for degradedness, we can complete the proof.

Combining the results in Lemmas 2 and 3, we obtain a Venn diagram as shown in Fig. 2 illustrating the relation between S_D , S_{st} , and S_{icx} .

Now we can further generalize Lemma 3 to the case in which fading channels are formed by clusters of scatterers. In particular, we consider the case in which we are only provided the statistics of each cluster but not the superimposed result as $\sqrt{H_k}$ in 1. Therefore, phases of channels of each cluster should be taken into account, i.e., we consider the *k*-th clusters of the first and second user as $\tilde{H}_{1k} = \tilde{H}_{1k,Re} + i \cdot \tilde{H}_{1k,Im} = \sqrt{H_{1k}}e^{-i\phi_{1k}}$ and $\tilde{H}_{2k} = \tilde{H}_{2k,Re} + i \cdot \tilde{H}_{2k,Im} = \sqrt{H_{2k}}e^{-i\phi_{2k}}$.

Lemma 4. Let M be the number of clusters of scatterers for both users 1 and 2, and denote the channels of users 1's and 2's kth clusters as \tilde{H}_{1k} and \tilde{H}_{2k} , respectively, where $k = 1, \dots, M$. The broadcast channel is degraded if user 1's scatterers are stronger than those of user 2's in the sense that $\tilde{H}_{\pi_{1k},Re}\tilde{H}_{\pi_{1j},Re} \ge_{st} \tilde{H}_{\pi_{2k},Re}\tilde{H}_{\pi_{2j},Re}$, and $\tilde{H}_{\pi_{1k},Im}\tilde{H}_{\pi_{1j},Im} \ge_{st} \tilde{H}_{\pi_{2k},Im}\tilde{H}_{\pi_{2j},Im}$, $\forall k, j \in \{1, \dots, M\}$, for some permutation π_1 and π_2 of users 1's and 2's clusters.

The main idea of the proof of this lemma is by the fact that *the usual stochastic order is closed under convolution* [10, Theorem 1.A.3], which claims that for two sets of independent random variables $\{A_k\}$ and $\{B_k\}$, if $A_k \geq_{st} B_k$, $\forall k \in \{1, \dots, M\}$ then for any increasing function $\phi : \mathbb{R}^M \to \mathbb{R}$, one has $\phi(A_1, \dots, A_M) \geq_{st} \phi(B_1, \dots, B_M)$. In particular, $\sum_{k=1}^M A_k \geq_{st} \sum_{k=1}^M B_k$. In our problem, from Lemma 1 we know that $|\sum_{k=1}^M \tilde{H}_{1k}|^2 \geq_{st} |\sum_{k=1}^M \tilde{H}_{2k}|^2$ is sufficient to attain a degraded GBC. After expanding the left and right hand sides in addition with [10, Theorem 1.A.3], we can get Lemma 4.

The condition of the number of clusters of channels 1 and 2 in Lemma 4 can be relaxed to two non-negative integer-valued random variables N and M, respectively, and the result is still valid, if $|\sum_{k=1}^{N} \tilde{H}_{1k}|^2 \ge_{st} |\sum_{k=1}^{M} \tilde{H}_{2k}|^2$ and if $N \ge_{st} M$, which can be proved with the aid of [10, Theorem 1.A.4].

In the following we can extend the above result to scenarios with more than two receivers.

Corollary 1. For an L-receiver fast fading Gaussian broadcast

Remark 2. Note that the above discussion can be easily extended to complex cases with the assumption of full CSI at the receivers, where the noises are circularly symmetric complex Gaussian. With the former assumption the channel phase can be absorbed into the noises, which does not change the distributions of noises by the second assumption. Then it is easy to see that the in-phase and quadrature channels form a pair of identical parallel real Gaussian broadcast channels as shown in (1).

III-A. Examples of channels within the set $S_{\mathcal{D}}$

In the following, several examples with practical fading channels based on Corollary 1 are provided to explain how to determine the existence of the capacity, given the distributions of the fading channels. Three receivers are considered.

 Assume the magnitudes of the three channels are independent Nakagami-*m* random variables with *shape parameters m*₁, *m*₂, and *m*₃, and *spread parameters w*₁, *w*₂, and *w*₃ [14], respectively. From Corollary 1 we know that the broadcast channel is degraded if

$$\frac{\gamma\left(m_1,\frac{m_1x}{w_1}\right)}{\Gamma(m_1)} \geq \frac{\gamma\left(m_2,\frac{m_2x}{w_2}\right)}{\Gamma(m_2)} \geq \frac{\gamma\left(m_3,\frac{m_3x}{w_3}\right)}{\Gamma(m_3)}, \forall x,$$

where $\gamma(s,x) = \int_0^x t^{s-1}e^{-t}dt$ is the incomplete gamma function and $\Gamma(s) = \int_0^\infty t^{s-1}e^{-t}dt$ is the ordinary gamma function. An example satisfying the above inequality is $(m_1, w_1) = (1,3), (m_2, w_2) = (1,2)$ and $(m_3, w_3) = (0.5, 1)$.

- 2) In Fig. 3 we show an example that not all channel pairs satisfy the usual stochastic order but still it results in a degraded broadcast channel. From the proof of Lemma 3 we know that if $\overline{F}_k(h) = c_k, h \in [0, h_k]$, where h_k is the crossing point of $\overline{F}_k(h)$ and $\overline{F}_3(h)$ and $\overline{F}_k(h) \geq \overline{F}_1(h), k = 1, 2$, then H_3 is degraded with respect to H_k . On the other hand, we can observe that $H_1 \geq_{st} H_2$ by definition. Therefore, H_2 is degraded with respect to H_1 , and this forms a degraded GBC. In this example, the non-zero probabilities of H_1 and H_2 at zero magnitude can be treated as the sensitivity of the analog front ends at the receivers.
- 3) For full CSIT cases under block fading, if the transmitter perfectly knows the channel realizations $H_1 = h_1$, $H_2 = h_2$, and $H_3 = h_3$, then the CCDFs are given by $\bar{F}_{H_k} = 1 u(h h_k)$, k = 1, 2, 3. From Lemma 3 we know that $h_1 \le h_2 \le h_3$ results in the degraded broadcast channel, which is consistent to the traditional way to check the degradedness when there is perfect CSIT.

IV. MULTIPLE ANTENNAS

Based on the observation from single antenna cases, we aim to extend the description of this relation to cases in which there are multiple antennas at all terminals. We assume all terminals are equipped with the same number of antennas n_T . The result can be extended to the case with different number of antennas at all terminals by the same skills used in [4]. The received signals at receiver 1 and receiver 2, respectively, can then be expressed as

$$Y_1 = H_1 X + Z_1$$
, and $Y_2 = H_2 X + Z_2$, (2)

where $\mathbf{Z}_1 \sim \mathsf{CN}(0, \mathbf{I}_{n_T})$ and $\mathbf{Z}_2 \sim \mathsf{CN}(0, \mathbf{I}_{n_T})$, $\mathbf{X} \in \mathbb{C}^{n_T}$, \mathbf{H}_1 and $\mathbf{H}_2 \in \mathbb{C}^{n_T \times n_T}$ with entries varying for each code symbol. For the



Fig. 3. An example that not all channel pairs satisfy the usual stochastic order but still results in a degraded broadcast channel.

MIMO case, we apply the multi-variate usual stochastic order to the eigenvalues of $\mathbf{H_1}^{-1}\mathbf{H_1}^{-H}$ and $\mathbf{H_2}^{-1}\mathbf{H_2}^{-H}$, which are real.

From [15] we know that the probability of a random matrix **H** with i.i.d entries following continuous distributions being low rank has measure zero, which implies that $\mathbf{H}^{-1}\mathbf{H}^{-H}$ having finite eigenvalues has measure one. Thus we may assume that the channel matrices are invertible. In addition with the assumption of full CSIR, we can normalize (2) equivalently as

$$Y_1' = X + Z_1'$$
, and $Y_2' = X + Z_2'$, (3)

where $\mathbf{Z_1}' \sim CN(0, \mathbf{A})$ and $\mathbf{Z_2}' \sim CN(0, \mathbf{B})$, $\mathbf{A} \triangleq \mathbf{H_1}^{-1}\mathbf{H_1}^{-H}$, $\mathbf{B} \triangleq \mathbf{H_2}^{-1}\mathbf{H_2}^{-H}$. For full CSIT and full CSIR cases, the constraint $\mathbf{B} - \mathbf{A} \succeq \mathbf{0}$ is sufficient¹ to make the Markov chain $\mathbf{X} \to \mathbf{Y_1}' \to \mathbf{Y_2}'$ valid. On the contrary, in the considered scenario we have full CSIR but only statistical CSIT. Therefore, we aim to construct equivalent channels \mathbf{H}'_1 and \mathbf{H}'_2 to show $P(\mathbf{B}' - \mathbf{A}' \succeq \mathbf{0}) = 1$ according to Lemma 2, where $\mathbf{A}' = (\mathbf{H}_1')^{-1}(\mathbf{H}_1')^{-H}$ and $\mathbf{B}' = (\mathbf{H}_2')^{-1}(\mathbf{H}_2')^{-H}$. Note that in [10, Theorem. 6.B.1] the usual stochastic order in vector (but not matrix) version is considered, where in vec(\mathbf{B}) \leq_{st} vec(\mathbf{A}), the inequality is element-wise, i.e., $b'_i \leq a'_i, \forall i$, for $P(\text{vec}(\mathbf{B}') \leq \text{vec}(\mathbf{A}')) = 1$. However, we can not directly apply the multivariate usual stochastic order to our scenario because it does not guarantee the positive definiteness of $\mathbf{B} - \mathbf{A}$. In the following we aim to find the relation of the degradedness and the stochastic order of the channels through the eigenvalues of \mathbf{A} and \mathbf{B} , which will be shown to be sufficient to identify the existence of \mathbf{A}' and \mathbf{B}' such that $P(\mathbf{B}' - \mathbf{A}' \succeq \mathbf{0}) = 1$.

In the following, we derive a result which allows us to stochastically compare the minimum and maximum eigenvalues of the covariance matrices of receiver 1's and receiver 2's channels.

Theorem 1. A sufficient condition to have a degraded MIMO Gaussian broadcast channel is

$$\lambda_{min}(\mathbf{H_1H_1}^H) \ge_{st} \lambda_{max}(\mathbf{H_2H_2}^H).$$
(4)

The proof is sketched as follows. From [16, Theorem 9H.1] we know that $\lambda_{max}(\mathbf{AB}^{-1}) \leq \lambda_{max}(\mathbf{A})\lambda_{max}(\mathbf{B}^{-1})$. If we enforce the upper bound of $\lambda_{max}(\mathbf{AB}^{-1})$ to be less than 1, then from

[17, 10.50(b)] we know that $\mathbf{B} - \mathbf{A} \succeq \mathbf{0}$ is valid. Thus we can get the sufficient condition of the degraded MIMO Gaussian broadcast channel by letting $\lambda_{max}(\mathbf{A})\lambda_{max}(\mathbf{B}^{-1}) \leq 1$. Since $\lambda_{max}(\mathbf{A})^{-1} = \lambda_{min}(\mathbf{A}^{-1}) > 0$, we have $\lambda_{max}(\mathbf{B}^{-1}) \leq \lambda_{min}(\mathbf{A}^{-1})$. After applying the property of multi-variate usual stochastic order we can get the result.

Theorem 1 can be generalized to N-user cases as follows.

Corollary 2. A sufficient condition for an L-receiver fast fading MIMO GBC to be a degraded one is that

$$\lambda_{\min}(\mathbf{H}_{\mathbf{k}}\mathbf{H}_{\mathbf{k}}^{H}) \geq_{st} \lambda_{\max}(\mathbf{H}_{\mathbf{k}+1}\mathbf{H}_{\mathbf{k}+1}^{H}), \forall k \in \{1 \cdots L-1\}.$$

In the following we provide another scenario resulting in the degradedness.

Theorem 2. Let $\mathbf{H}_1' = \Sigma_1^{1/2} \mathbf{H}_1$, $\mathbf{H}_2' = \Sigma_2^{1/2} \mathbf{H}_2$. If \mathbf{H}_1 and \mathbf{H}_2 are isotropically distributed and $\Sigma_1 \succeq \Sigma_2 \succ \mathbf{0}$, where Σ_1 and Σ_2 are the antenna correlation matrices at the receivers 1 and 2, respectively, then it is equivalent to a degraded Gaussian broadcast channel.

Sketch of proof: We can form an equivalent broadcast channel by absorbing $\Sigma_1^{1/2}$ and $\Sigma_2^{1/2}$ into noise covariance matrices. Since the vector of the ordered eigenvalues of the second equivalent noise covariance matrix majorizes that of the first one by the monotonicity theorem [18, Theorem 8.4.9], we can show that the new second received signal is stochastically degraded respect to the first one. With the property that both H_1 and H_2 are i.d., we can further prove that $f_{\tilde{Y}_2|X} = f_{Y_2|X}$. Thus we conclude that the original channel is equivalent to a degraded one.

Remark 3. The result in Theorem 2 can be easily extended to the case with L-receiver. The constraint $\Sigma_1 \succeq \Sigma_2$ in Theorem 2 can be relaxed to $\Sigma_1 \nleq \Sigma_2$ by deterministic channel enhancement [4]. The following is an example.

Example: For the Gaussian broadcast channel

$$\mathbf{Y}_1 = \boldsymbol{\Sigma}_1^{1/2} \mathbf{H} \mathbf{X} + \mathbf{Z}_1 \text{ and } \mathbf{Y}_2 = \boldsymbol{\Sigma}_2^{1/2} \mathbf{H} \mathbf{X} + \mathbf{Z}_2,$$
(5)

assume that the fading channel **H** has realizations $\{\mathbf{H}_0, \{\mathbf{H}_0\mathbf{B}\}: \mathbf{B} \in U(n_T^2)\}$, where U(n) is the unitary group with degree *n*. The distribution of **H** can be arbitrary. Then it is easy to see that we can apply the channel enhancement technique [4] to the pair of channel realizations $(\Sigma_1^{1/2}\mathbf{H}_0, \Sigma_2^{1/2}\mathbf{H}_0)$ to construct an equivalent degraded MIMO GBC. It is also to see that **B** can be absorbed into **X** when the optimal distribution of **X** is Gaussian [4], which is in fact the case when the channel enhancement is considered.

V. CONCLUSION

In this work we characterize the relation between the stochastic orders and the degradedness among different receivers of a Gaussian broadcast channel. In particular, we investigate the usual stochastic order and the increasing convex order. Both single and multiple antennas at all nodes are considered under fast fading with statistical CSIT. We derive criteria to check the degradedness of several commonly used Gaussian broadcast channels which is helpful to characterize the performance of the broadcast channel under statistical CSIT.

¹The reason that it is not necessary is, we may be able to use the channel enhancement scheme to obtain a degraded channel with $\mathbf{B} \not\succeq \mathbf{A}$.

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