DEALING WITH UNCERTAIN MODELS IN WIRELESS COMMUNICATIONS

Ezio Biglieri

Departament TIC, Universitat Pompeu Fabra, Barcelona, Spain — e.biglieri@ieee.org

ABSTRACT

We advocate the use of interval-type bounds on cumulative distribution functions, called probability boxes, to handle several problems arising when a wireless communication system performance must be assessed in the presence of model uncertainties, poorly known or unknown dependencies of random variables, or imprecisely specified probability distributions. The amount of uncertainty is evaluated quantitatively by using sharp bounds on performance parameters.

***Index Terms*—** Wireless channel models; Probability boxes; Uncertainty propagation; Dependence bounds

1. INTRODUCTION AND MOTIVATION OF THE WORK

A good deal of research activity in wireless communications is devoted in finding accurate statistical models of the channel. A major problem here is caused by the fact that no single model (e.g., Rayleigh or Rice fading distributions) is accurate enough for a wide variety of physical channels. Considerable effort has been spent in the search for general classes of probability density functions (pdfs) that are physically justified and flexible enough to fit a large mass of experimental results. Nonetheless, wireless system analysis and design should in some way account for the uncertainty intrinsic in the use of an inaccurate channel model, which adds to that caused by the randomness in system behavior but differs from it in a substantial way. In [6], it is suggested that we should treat in a different way the uncertainties due to randomness and those due to ignorance. This distinction has generated several techniques to deal with the latter type of uncertainty: among them, we recall random-set theory [3, 8], fuzzy-set theory [20], Dempster–Shafer theory [14], and probability boxes (relations among these techniques, and some equivalence results, are discussed in [8, 12, 18]). Probability boxes use standard probability theory, and for this reason we choose these among other techniques. The present paper shows the application of this theory to wireless communications, an area which was not considered before.

Probability boxes (or, for short, p-boxes) are interval-type bounds on cumulative distribution functions (cdfs) that can handle several problems, like model uncertainties, poorly known or unknown dependencies of random variables (RVs), or imprecisely specified distributions [6, p. 12–14]. The key point here is that the use of p-boxes in lieu of individual cdfs results into performance analyses not relying on unwarranted distribution assumptions. With this approach, instead of selecting a single distribution from a class of distributions matching a set of constraints, the whole class of these distributions is used. In addition, instead of deriving upper and lower bounds on performance parameters only at the end of calculations (see, for example, [5]), all relevant calculations keep track of the uncertainties implicit in the entities used (this approach is similar in spirit to interval analysis [9]).

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This paper is organized as follows: after defining probability boxes, Section 2 shows some examples of actual constructions and combination of p-boxes, while Section 3 examines the important case when two random variables (RVs) with unknown dependence are combined by a binary operation. Conclusions are drawn in Section 4.

2. PROBABILITY BOXES

Probability boxes are defined in terms of upper and lower probabilities $\overline{P}$ and $\underline{P}$, respectively [12]. Once upper and lower probabilities have been assigned to every event in the probability space, upper and lower cumulative distribution functions (cdf) of the RV X are defined as [12, p. 24]

$$\overline{F}_X(x) \triangleq \overline{P}(X \leqslant x), \qquad \underline{F}_X(x) \triangleq \underline{P}(X \leqslant x) \tag{1}$$

respectively. (Here, unless otherwise specified, we restrict our attention to RVs defined on the positive real axis $\mathbb{R}^+$.)

When the pair $[\overline{F}_X, \underline{F}_X]$ circumscribes an imprecisely known cdf $F_X(x)$, it is called a p-box for $F_X(x)$. As a special case, an interval $[a, b]$ can be identified by a p-box with $\overline{F}(x) = u(x - a)$ and $\underline{F}(x) = u(x - b)$, where $u(\cdot)$ denotes the unit-step function. A precise cdf has $\underline{F} = \overline{F} = F$. Fig. 1 below shows an example of a p-box, to be illustrated later.

As a possible use of a p-box, one may choose a representative cdf within it, and use it to determine the relevant performance metric. A common approach consists of choosing the *worst* cdf within the p-box, which is often unduly pessimistic, and may detract from robustness [17]. Another approach consists of choosing a small number of cdfs within the box, and determining a separate performance metric for each one. This approach does not allow an overall synthesis of the modeling effort.

2.1. Generating probability boxes

Probability boxes can be generated whenever sharp upper and lower bounds on the cdf of a RV can be obtained on the basis of the knowledge available about the RV itself. The width of a p-box yields a quantitative indication of the effect of the model uncertainty on the distribution of the RV and the performance parameters derived from it. Hence, a wide p-box may not be caused by a weakness of the theory, but rather reflects the amount of model uncertainty. P-boxes can also show the robustness of a performance parameter against the choice of a model.

2.1.1. Using moment bounds

Given a RV X whose $N + 1$ moments $\mu_k \triangleq \mathbb{E}[X^k]$, $k = 0, \dots, N$, are known, it is possible to evaluate upper and lower bounds to the cdf of X. The classical solution to this problem, obtained through *moment bound theory* [7], yields bounds that are *sharp*, i.e., such that there exists a pair of RVs that have moments $\mu_k, k = 0, \dots, N$, and whose cdfs coincide with these upper and lower bounds. For example [4], assume that the mean μ_1 and the variance $\sigma_X^2 \triangleq \mathbb{E}[X - \mu_1]^2$ are known for a RV X taking values in the finite interval $[a, b]$. Then we have

$$
\begin{aligned}
&0 \leqslant F_X(x) \leqslant \frac{\sigma_X^2}{(\mu_1 - x)^2 + \sigma_X^2}, \quad a \leqslant x \leqslant \overline{x}_1 \\
&\frac{(x-\mu_1)(b-\mu_1) + \sigma_X^2}{(x-a)(b-a)} \leqslant F_X(x) \leqslant \\
&\quad \leqslant 1 - \frac{(\mu_1 - x)(\mu_1 - a) + \sigma_X^2}{(b-x)(x-a)}, \quad \overline{x}_1 \leqslant x \leqslant \underline{x}_2 \\
&\frac{(x-\mu_1)^2}{(x-\mu_1)^2 + \sigma_X^2} \leqslant F_X(x) \leqslant 1, \quad \underline{x}_2 \leqslant x \leqslant b
\end{aligned}
\tag{2}
$$

where

$$
\underline{x}_1 = a, \quad \overline{x}_1 = \mu_1 - \frac{\sigma_X^2}{b - \mu_1}, \quad \underline{x}_2 = \mu_1 + \frac{\sigma_X^2}{\mu_1 - a}, \quad \overline{x}_2 = b \tag{3}
$$

It is also possible to solve the same problem when the moments of X take values that are only known within an interval. [1]

2.1.2. Using parameter intervals

Sometimes a model cdf can be assessed with reasonable accuracy, except for its parameters that are known only within an interval. In this case a p-box can be generated as the envelope of all cdfs whose parameters lie in that interval. As an example, Fig. 1 shows the p-box generated by Nakagami cdfs [16, p. 24] whose parameters are $m \in (0.5, 5.), \Omega \in (1., 2.)$.

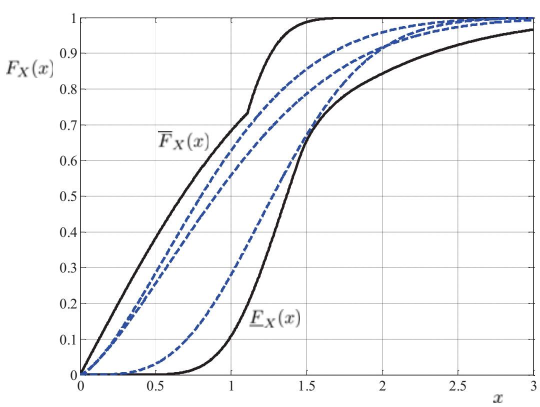

Fig. 1. *Probability box $[\overline{F}_X, \underline{F}_X]$ generated by Nakagami-m cdfs with parameters $m \in (0.5, 5.), \Omega \in (1., 2.)$. The dashed curves correspond to three Nakagami-m cdfs with randomly selected values of m and Ω within their intervals.*

2.1.3. Combining probability boxes

Given a set of p-boxes referring to an individual modeling problem, they can be aggregated in several ways: [6, p. 67 ff.].

1. **Intersection.** If it is assumed that all p-boxes contain the correct model cdf, then one may generate their intersection: Given the n p-boxes $[\overline{F}_i(x), \underline{F}_i(x)]$, $1 \leqslant i \leqslant n$, the intersection p-box is $[\overline{F}^*(x), \underline{F}^*(x)]$, where

$$
\begin{aligned}
\overline{F}^*(x) &= \min\big(\overline{F}_1(x), \dots, \overline{F}_n(x)\big) \\
\underline{F}^*(x) &= \max\big(\underline{F}_1(x), \dots, \underline{F}_n(x)\big)
\end{aligned}
\tag{4}
$$

2. **Envelope.** If *at least one* of the p-boxes contains the correct model cdf, then the p-box $[\overline{F}^*(x), \underline{F}^*(x)]$ may be used, where

$$
\begin{aligned}
\overline{F}^*(x) &= \max\big(\overline{F}_1(x), \dots, \overline{F}_n(x)\big) \\
\underline{F}^*(x) &= \min\big(\underline{F}_1(x), \dots, \underline{F}_n(x)\big)
\end{aligned}
\tag{5}
$$

3. UNCERTAINTY PROPAGATION: P-BOXES OF COMBINATIONS OF RANDOM VARIABLES

An important application of p-boxes arises when a RV is obtained as the result of combining other RVs described through their p-boxes. In several instances, the resulting p-boxes can be computed by simple application of the definitions. As an example, consider the p-boxes of the maximum and minimum of n independent RVs X_i, $i = 1, \dots, n$, each with cdf $F_i(x)$ included in the p-box $[\overline{F}_i(x), \underline{F}_i(x)]$. The cdf of the maximum is, with obvious notations, $F_{\max}(x) = \prod_{i=1}^n F_i(x)$, and the cdf of the minimum is $F_{\min}(x) = 1 - \prod_{i=1}^n (1 - F_i(x))$. Simple interval arithmetic [9] yields the p-boxes

$$
\begin{aligned}
F_{\max}(x) &\in \left[\prod_{i=1}^n \overline{F}_i(x), \prod_{i=1}^n \underline{F}_i(x)\right] \\
F_{\min}(x) &\in \left[1 - \prod_{i=1}^n (1 - \overline{F}_i(x)), 1 - \prod_{i=1}^n (1 - \underline{F}_i(x))\right]
\end{aligned}
\tag{6}
$$

General results can be obtained by considering binary operations that are monotonic in both arguments, and in particular the four operations, denoted $\circ$, of the set $\{+, -, \times, \div\}$. Consider first the case of two independent RVs whose p-boxes of cdfs are given. We denote by $\oplus$, $\ominus$, $\otimes$, and $\oslash$, respectively, the "generalized convolutions" that combine F_X and F_Y to generate $F_{X \circ Y}$, so that if for example $Z = X \times Y$ we have $F_{X \times Y}(z) = (F_X \otimes F_Y)(z) = \int_0^z F_X(z/t)\, dF_Y(t)$. In these conditions it can be easily proved [18, p. 100] that the p-boxes for F_Z are given by

$$
\begin{aligned}
\underline{F}_{X+Y} &= \underline{F}_X \oplus \underline{F}_Y & \overline{F}_{X+Y} &= \overline{F}_X \oplus \overline{F}_Y \\
\underline{F}_{X-Y} &= \underline{F}_X \ominus \overline{F}_Y & \overline{F}_{X-Y} &= \overline{F}_X \ominus \underline{F}_Y \\
\underline{F}_{X\times Y} &= \underline{F}_X \otimes \underline{F}_Y & \overline{F}_{X\times Y} &= \overline{F}_X \otimes \overline{F}_Y \\
\underline{F}_{X\div Y} &= \underline{F}_X \oslash \overline{F}_Y & \overline{F}_{X\div Y} &= \overline{F}_X \oslash \underline{F}_Y
\end{aligned}
\tag{7}
$$

3.1. Dependence bounds

A more interesting case arises when the RVs that are combined are not independent, and their dependence is unknown or only partially known. In this case, p-boxes can be obtained using upper and lower bounds to cdfs obtained by bounding the copulas connecting the two marginal pdfs into their joint pdf. This approach allows one to determine the width of the performance range caused by possibly unwarranted independence assumptions.

A *copula* is a function $\mathsf{C}(x, y)$ that links the marginal cdfs of two random variables X and Y to their joint cdf. If F_{XY} denotes a two-dimensional cdf with marginals F_X, F_Y, then a copula C exists such that

$$
F_{XY}(x, y) = \mathsf{C}\big(F_X(x), F_Y(y)\big) \tag{8}
$$

Hence, the copula C contains all the information about the dependence of X and Y. For example, $\mathsf{C}(x, y) = xy$ indicates that X and Y are independent RVs [10, p. 25]. All copulas satisfy the inequalities

$$\mathsf{W}(x, y) \leqslant \mathsf{C}(x, y) \leqslant \mathsf{M}(x, y), \qquad \forall (x, y) \in [0, 1] \times [0, 1] \quad (9)$$

where both W and M are copulas, defined as [10, p. 14]:

$$\mathsf{W}(x, y) \triangleq \max(x + y - 1, 0) \text{ and } \mathsf{M}(x, y) \triangleq \min(x, y) \quad (10)$$

Combining (8)–(10) one can obtain the *Fréchet–Hoeffding bounds* on a joint cdf in terms of its marginals [10, 18, 19]:

$$\max [F_X(x)+F_Y(y)-1, 0] \leqslant F_{XY}(x, y) \leqslant \min [F_X(x), F_Y(y)] \quad (11)$$

3.1.1. Using copula bounds to generate dependence p-boxes

Consider a RV Z obtained as a composition of X and Y, We have the following key result [19, Theorem 1]: Let X, Y denote two RVs defined on the extended real line $\mathbb{R}^* \triangleq \mathbb{R} \cup \{-\infty, \infty\}$, and $\mathcal{L}$ the set of binary operations mapping $\mathbb{R}^* \times \mathbb{R}^*$ to $\mathbb{R}^*$ which are nondecreasing in each place and continuous except possibly at $(0, \infty)$ and $(\infty, 0)$. If $Z \triangleq X \circ Y$, where $\circ \in \mathcal{L}$ and $\underline{\mathsf{C}}_{XY}$ is any lower bound on copula C_{XY}, then two functions $\mathsf{ldb}_{\underline{\mathsf{C}}_{XY}}$ (the "lower dependence bound") and $\mathsf{udb}_{\underline{\mathsf{C}}_{XY}}$ (the "upper dependence bound") exist such that, $\forall z \in \mathbb{R}^*$,

$$\mathsf{ldb}_{\underline{\mathsf{C}}_{XY}}(F_X, F_Y, \circ)(z) \leqslant F_Z(z) \leqslant \mathsf{udb}_{\underline{\mathsf{C}}_{XY}}(F_X, F_Y, \circ)(z) \quad (12)$$

where [18, p. 76]

$$\mathsf{ldb}_{\underline{\mathsf{C}}_{XY}}(F_X, F_Y, \circ)(z) \triangleq \sup_{x \circ y = z} \underline{\mathsf{C}}_{XY}(F_X(x), F_Y(y)) \quad (13)$$

$$\mathsf{udb}_{\underline{\mathsf{C}}_{XY}}(F_X, F_Y, \circ)(z) \triangleq \inf_{x \circ y = z} \underline{\mathsf{C}}^{\partial}_{XY}(F_X(x), F_Y(y)) \quad (14)$$

where the superscript ∂ indicates *dual* copula [10].

Here, $\underline{\mathsf{C}}_{XY}$ summarizes what we know about the dependence of X and Y. If no information is available, since generally $\underline{\mathsf{C}}_{XY} \geqslant \mathsf{W}$, one may use the loosest bound obtained by choosing $\underline{\mathsf{C}}_{XY} = \mathsf{W}$, which yields the following expressions for ldb and udb:

$$\mathsf{ldb}_{\mathsf{W}} = \sup_{x \circ y = z} \mathsf{W}\big(F_X(x), F_Y(y)\big) \quad (15)$$

$$\mathsf{udb}_{\mathsf{W}} = \inf_{x \circ y = z} \mathsf{W}^{\partial}\big(F_X(x), F_Y(y)\big) \quad (16)$$

where $\mathsf{W}^{\partial} = \min(x + y, 1)$ is the dual of W. In this case, instead of $\mathsf{ldb}_{\mathsf{W}}$ and $\mathsf{udb}_{\mathsf{W}}$ we may write ldb and udb, respectively.

The special case $\circ \in \{+, -, \times, \div\}$ yields the following results, valid on $\mathbb{R}^+$ [18, p. 77-78]:

$$\begin{aligned}
\underline{F}_{X+Y}(z) &= \sup_{x+y=z} \max[F_X(x) + F_Y(y) - 1, 0] \\
\overline{F}_{X+Y}(z) &= \inf_{x+y=z} \min[F_X(x) + F_Y(y), 1] \\
\underline{F}_{X-Y}(z) &= \sup_{x+y=z} \max[F_X(x) - F_Y(-y), 0] \\
\overline{F}_{X-Y}(z) &= 1 + \inf_{x+y=z} \min[F_X(x) - F_Y(-y), 0] \\
\underline{F}_{X\times Y}(z) &= \sup_{xy=z} \max[F_X(x) + F_Y(y) - 1, 0] \\
\overline{F}_{X\times Y}(z) &= \inf_{xy=z} \min[F_X(x) + F_Y(y), 1] \\
\underline{F}_{X\div Y}(z) &= \sup_{xy=z} \max[F_X(x) - F_Y(1/y) - 1, 0] \\
\overline{F}_{X\div Y}(z) &= 1 + \inf_{xy=z} \min[F_X(x) - F_Y(1/y), 0]
\end{aligned} \quad (17)$$

These bounds are sharp, i.e., cannot be further improved [18, p. 78 ff.], [12, p.6]. Notice also that the results in (17) and referring to $\circ = +$ and $\circ = -$ hold generally in $\mathbb{R}$ and not only in $\mathbb{R}^+$ [18, p. 78], [12, p.6].

3.1.2. Using p-boxes of F_X and F_Y

Finally, if F_X, F_Y are cdfs in p-boxes $[\overline{F}_X, \underline{F}_X]$ and $[\overline{F}_Y, \underline{F}_Y]$, then $F_{X\circ Y}$ is contained in p-box $[\overline{F}_{X\circ Y}, \underline{F}_{X\circ Y}]$, where [19, p. 111]

$$\begin{aligned}
\underline{F}_{X+Y}(z) &= \mathsf{ldb}(\underline{F}_X, \underline{F}_Y, +) & \overline{F}_{X+Y}(z) &= \mathsf{udb}(\overline{F}_X, \overline{F}_Y, +) \\
\underline{F}_{X-Y}(z) &= \mathsf{ldb}(\underline{F}_X, \overline{F}_Y, -) & \overline{F}_{X-Y}(z) &= \mathsf{udb}(\overline{F}_X, \underline{F}_Y, -) \\
\underline{F}_{X\times Y}(z) &= \mathsf{ldb}(\underline{F}_X, \underline{F}_Y, \times) & \overline{F}_{X\times Y}(z) &= \mathsf{udb}(\overline{F}_X, \overline{F}_Y, \times) \\
\underline{F}_{X\div Y}(z) &= \mathsf{ldb}(\underline{F}_X, \overline{F}_Y, \div) & \overline{F}_{X\div Y}(z) &= \mathsf{udb}(\overline{F}_X, \underline{F}_Y, \div)
\end{aligned}$$

3.2. Example 1

Our first example derives the p-box of the cdf of $Z \triangleq X_1^2 + X_2^2$, where $X_1, X_2 \sim \mathcal{N}(0, 1)$ are two RVs with correlation coefficient ρ. Using a standard procedure, we rewrite Z in the form $Z = \lambda_1 U_1^2 + \lambda_2 U_2^2$, where U_1, U_2 are independent $\sim \mathcal{N}(0, 1)$, and λ_1, λ_2 are the eigenvalues of the covariance matrix of X_1, X_2, i.e., $\lambda_1 = 1 - \rho$ and $\lambda_2 = 1 + \rho$. Thus, Z is the sum of two χ^2 RVs with one degree of freedom. Using Eq. (5.8) of [15], $F_Z(z)$ can be determined. Fig. 2 shows $F_Z(z)$ for correlation values $\rho = 0$, $\rho = .5$, and $\rho = 1$ (the latter value corresponds to the case $X_1 = X_2$). If the joint distribution of X_1, X_2 is unknown, the p-box of F_Z is obtained from the first two equations of (17). The marginals here are, for $i = 1, 2$: $F_{X_i^2}(x_i) = \operatorname{erf}\left(\sqrt{x_i/2}\right)$, $x_i \geqslant 0$. Simple calculations lead to the analytic expressions, valid for $z \geqslant 0$:

$$\underline{F}(z) = \max\left\{2 \operatorname{erf}\left(\sqrt{z/4}\right) - 1, 0\right\} \text{ and } \overline{F}(z) = \operatorname{erf}\left(\sqrt{z/2}\right) \quad (18)$$

It is noticed that the p-box is not the envelope of the cdfs corresponding to the whole set of values of ρ, because normal marginals do not imply a normal joint cdf (see, e.g., [11, p. 128]). Thus, the gap enclosing the three innermost curves of Fig. 2 reflects the uncertainty in the knowledge of the value of ρ for jointly normal X_1, X_2, while the wider gap reflects the uncertainty about their joint distribution.

To evaluate the impact on performance of the uncertainty on the joint distribution, we examine (see Fig. 3) the error probability of binary antipodal transmission over a wireless channel with fading envelope $R = \sqrt{Z}$. The lower curve is obtained under the assumption of independent X_1, X_2, which yields a Rayleigh-distributed envelope, and hence error probability [2, p. 89] $P(e) = (1 - \sqrt{\eta/(1 + \eta)})/2$, where η is the signal-to-noise ratio. The upper curve is obtained by computer simulation, using $\overline{F}_Z$ as the cumulative distribution function of Z.

3.3. Example 2

In this example, we examine the potentially strong impact on the performance of diversity combining caused by different joint distributions having the same marginals. We assume two-branch diversity with Rayleigh fading gains R_1 and R_2 on both branches, and maximal-ratio combining. The equivalent fading $\check{R}$ generated by the combination is given by the square root of $R_1^2 + R_2^2$ [2, p. 113]. With R_1, R_2 independent, the cdf of $\check{R}^2$ is $F_{\check{R}^2}(z) = 1 - e^{-z}(1+z)$, $z \geqslant 0$. With unknown dependence, one can use the bounds of the first

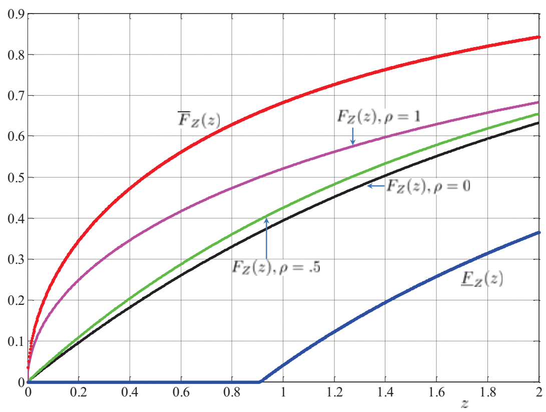

Fig. 2. *Probability box* $[\overline{F}_Z, \underline{F}_Z]$ *generated by* $Z \triangleq X_1^2 + X_2^2$, *with* $X_1, X_2 \sim \mathcal{N}(0,1)$. *The inner cdfs are exact, and refer to* X_1, X_2 *jointly normal with correlation coefficient* ρ.

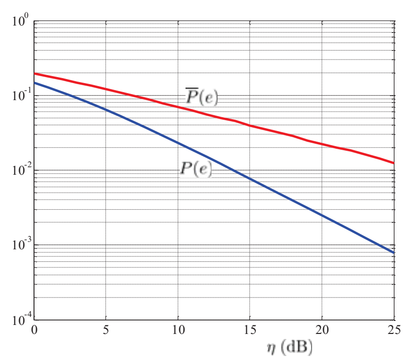

Fig. 3. *Error probability of binary antipodal transmission over a channel with fading envelope* $\sqrt{X_1^2 + X_2^2}$, *where* $Z \triangleq X_1^2 + X_2^2$, *with* $X_1, X_2 \sim \mathcal{N}(0,1)$. *Lower curve:* X_1 *and* X_2 *are independent. Upper curve: Obtained from the upper dependence bound distribution* $\overline{F}_Z$.

two equations of (17), with marginals $F_{R_i^2}(x_i) = 1 - e^{-x_i}$, $x_i \geqslant 0$, $i = 1, 2$, which yields, for $z \geqslant 0$,

$$\underline{F}(z) = \max\left\{1 - 2e^{-z/2}, 0\right\} \quad \text{and} \quad \overline{F}(z) = 1 - e^{-z} \tag{19}$$

Fig. 4 shows the p-box of $R_1^2 + R_2^2$, while Fig. 5 compares the error probabilities of binary antipodal transmission with independent diversity [2, p. 113] and worst-case dependent branches.

Related calculations were performed in [13], based on a parametric family of copulas (the *Clayton copula*) rather than dependence bounds. As observed in [19, p. 134], the physical implications of the choice of a copula family may not be immediately apparent.

4. CONCLUSIONS

We have examined how the use of p-boxes as interval-type bounds on cumulative distribution functions is able to handle several problems

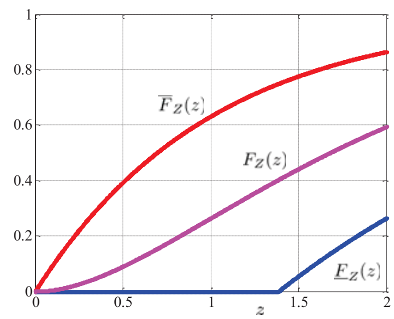

Fig. 4. *Probability box* $[\overline{F}_Z, \underline{F}_Z]$ *generated by* $Z \triangleq R_1^2 + R_2^2$, *with Rayleigh-distributed* R_1, R_2. *The inner cdf is exact, and refers to independent* R_1, R_2.

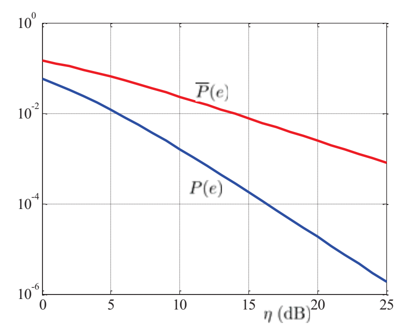

Fig. 5. *Error probability of binary antipodal transmission over a channel with diversity* 2, *Rayleigh fading and maximal-ratio combining. Lower curve: Independent fading on the two branches. Upper curve: Obtained from the upper dependence bound distribution.*

arising when a wireless communication system performance must be assessed in the presence of model uncertainties. In particular, it is shown how lower and upper bounds on required performance parameters can be derived even if their distribution cannot be calculated, as it is only known that it belongs to a certain class. Dependence bounds are especially informative, as a wide p-box may indicate the need for additional modeling effort intended to gather information about the joint cdf of the random variables affecting system performance.

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