COMPLEXITY REDUCTION OF SUMIS MIMO SOFT DETECTION BASED ON BOX OPTIMIZATION FOR LARGE SYSTEMS

M. Ángeles Simarro ⋆ *Victor M. García* † *F. J. Martínez-Zaldívar*⋆
Alberto González ⋆ Antonio Vidal †

iTEAM, Universitat Politècnica de València, Spain⋆
Dept. of Information Systems and Computing, Universitat Politècnica de València, Spain †

ABSTRACT

A new algorithm called SUMIS-BO is proposed for soft-output MIMO detection. This method is a meaningful improvement of the "*Subspace Marginalization with Interference Suppression*" (SUMIS) algorithm. It exhibits good performance with reduced complexity and has been evaluated and compared in terms of performance and efficiency with the SUMIS algorithm using different system parameters. Results show that the performance of the SUMIS-BO is similar to the SUMIS algorithm, however its efficiency is improved. The new algorithm is far more efficient than SUMIS, especially with large systems.

Index Terms— MIMO, SUMIS, Box optimization, large systems, complexity

1. INTRODUCTION

Multiple-Input Mutliple-Output (MIMO) communication systems have received considerable interest in recent years and have been adopted in many wireless communication standards, such as IEEE 802.11n/ac [1] and 3GPP Long Term Evolution Advanced [2]. It is known that MIMO systems can provide significant capacity improvement over other communication systems and their capacity increases with the minimum of the number of transmit and receive antennas [3]. At the receiver side of these systems, the use of a soft output detector concatenated with a soft input channel decoder provides significant increase of the reliability of wireless communications. A soft output detection algorithm provides reliability soft information expressed as log-likelihood ratios (LLRs), which is used by the channel decoder to carry out the final decision. In contrast to the many advantages of the MIMO systems, a strong limitation is the high computational cost of optimal detection to compute exactly the LLR values. This problem can be especially significant in large MIMO systems [4] and high order constellations. In these systems, a good balance between performance and complexity is critical.

In the above context, actual output detectors exhibit different tradeoffs between complexity and performance. The optimal detector, which computes the LLR exactly, holds prohibitively high computational complexity. For this reason several algorithms with more reduced complexity have been recently proposed. Max-log approximation detection is employed by the most common proposed detectors. Optimal max-log approximation is achieved by "*Single Tree Search*" (STS) [5] and "*Repeated Tree Search*" (RTS) [6] algorithms, both based on the "*Sphere Decoder*" (SD) method. The computational complexity of these algorithms varies depending on the channel and noise realizations and, even in the best cases, the RTS algorithm complexity is high. The STS algorithm exhibits lower the computational cost than the RTS algorithm, however, this algorithm is still computationally very expensive when the number of antennas or the constellation order is relatively high. Low complexity solutions such as: "*Soft Fixed Sphere Decoder*" (SFSD) [7], "*List Sphere Decoder*" (LSD) [8], and "*Soft Output k-Best*" [9] among others, have been proposed, however these suboptimal search methods give a certain performance loss.

An intermediate approach between the optimal detector and max-log approximation has been recently proposed based on "*Partial Marginalization*" (PM), reported originally in [10] and improved in [11], and the SUMIS [12] algorithm.

The SUMIS algorithm offers the best trade-off between exact and approximate computation of the LLR values. The work presented in [12] considers only the constellations where a "symbol" is equivalent to a "bit". The present paper extends the SUMIS algorithm to higher order constellations, furthermore, its main contribution is an improvement of this algorithm, resulting in a low performance degradation, nevertheless with less complexity than original SUMIS algorithm. The key idea is to incorporate the use of continuous constrained minimization techniques, also called box optimization (BO) [13] [14], combined with the "*Zero Forcing*" (ZF) algorithm to reduce the computational complexity of the original SUMIS. The employed BO algorithm is described in [13] and adapted in [15], obtaining an extremely tight bound on the solution and good results with reduced complexity. Through the rest of the paper, we will refer to this modification as SUMIS-BO detector.

The rest of this paper is organized as follows. In section 2, the MIMO system model is presented first (subsection 2.1) and then a brief review of previous detectors algorithms (subsection 2.2). The modifications proposed to the SUMIS algorithm and the simulation results are presented in section 3 and 4 respectively. Finally conclusions are given in Section 5.

2. BACKGROUND

2.1. System Model

Throughout this paper, we consider the real MIMO system model, using n_T transmit and n_R receive antennas with $n_T \leq n_R$. At the transmitter, the information bits are encoded, interleaved and then mapped to symbols. Each symbol s_j is taken independently from the M-ary constellation Ω. The symbol contains $k = \log_2(M)$ encoded and interleaved bits. The corresponding bits are denoted by $s_{j,b}$, where the indices refer to the bth bit associated with the jth

Thanks to the Spanish MINECO Grant RACHEL TEC2013-47141-C4-4-R and to the TEC2012-38142-C04-01 and the PROMETEO/2009/013 projects for funding.

symbol. At each signaling period, the relation between the transmitted symbol vector, $\mathbf{s} \in \mathbb{R}^{n_T}$, and the associated received vector, $\mathbf{y} \in \mathbb{R}^{n_R}$, can be expressed as

$$\mathbf{y} = \mathbf{H}\mathbf{s} + \mathbf{v}, \tag{1}$$

where $\mathbf{H} \in \mathbb{R}^{n_R \times n_T}$ denotes a fading channel matrix with independent elements $h_{j,i} \sim \mathcal{N}(0,1)$ and it is assumed to be perfectly known by the receiver. Vector $\mathbf{v} \sim \mathcal{N}(\mathbf{0}, \frac{N_o}{2}\mathbf{I})$ denotes an additive Gaussian noise (AWGN). Since separable complex value constellation can be considered (quadrature amplitude modulation (QAM)), we can easily rewrite the real system model (1) from the equivalent complex model.

At the receiver, the demodulator computes soft information in form of LLR values for each of the encoded and interleaved bit $s_{j,b}$ and is given by

$$L_{j,b} = \log \frac{P(s_{j,b} = 1|\mathbf{y})}{P(s_{j,b} = 0|\mathbf{y})}, \tag{2}$$

which expresses how likely is the hypothesis that the $s_{j,b}$ bit was equal to 1 or 0. Assuming equal a priori probabilities and using Bayes' theorem, Eq. (2) can be rewritten as

$$L_{j,b} = \log \frac{\sum_{\mathbf{s}\epsilon\chi^1_{j,b}} \exp(-\frac{1}{N_0}\|\mathbf{y} - \mathbf{H}\mathbf{s}\|^2)}{\sum_{\mathbf{s}\epsilon\chi^0_{j,b}} \exp(-\frac{1}{N_0}\|\mathbf{y} - \mathbf{H}\mathbf{s}\|^2)}, \tag{3}$$

where $\chi^u_{j,b}$ denotes the set of possible transmitted vectors for which $s_{j,b}$ bit is equal to u. The computational complexity of (3) grows exponentially with n_T and is polynomial with M. Thus, the exact MIMO detection scheme becomes prohibitive. The most common approach to cope with this limitation is the max-log approximation [8] where

$$L_{j,b} \approx \min_{\mathbf{s}\epsilon\chi^0_{j,b}} \frac{1}{N_o}\|\mathbf{y} - \mathbf{H}\mathbf{s}\|^2 - \min_{\mathbf{s}\epsilon\chi^1_{j,b}} \frac{1}{N_o}\|\mathbf{y} - \mathbf{H}\mathbf{s}\|^2. \tag{4}$$

However, the max-log approximation does not lead to a complexity reduction by itself. Eq (4) requires the computation of the same metrics than (3). Nonetheless, it can be exploited to design low-complexity algorithms [5]– [9]. In other works, some authors propose an alternative to max-log aproximation [10]– [12] and consider a new approach to the problem of computing (3). The basic idea is to define the following partitioning model, which is based in (1),

$$\mathbf{y} = \mathbf{H}\mathbf{s} + \mathbf{n} = [\overline{\mathbf{H}} \quad \widetilde{\mathbf{H}}] \quad [\overline{\mathbf{s}}^T \quad \widetilde{\mathbf{s}}^T]^T + \mathbf{v} = \overline{\mathbf{H}}\overline{\mathbf{s}} + \widetilde{\mathbf{H}}\widetilde{\mathbf{s}} + \mathbf{v} \tag{5}$$

where $\overline{\mathbf{H}} \in \mathbb{R}^{n_R \times n_s}$, $\widetilde{\mathbf{H}} \in \mathbb{R}^{n_R \times (n_T - n_s)}$, $\overline{\mathbf{s}} \in \Omega^{n_s}$ and $\widetilde{\mathbf{s}} \in \Omega^{n_T - n_s}$ for fixed $n_s \in 1, \cdots, n_T$.

The partitioned model carries intrinsically an optimal permutation of the columns of $\mathbf{H}$ that determines $\overline{\mathbf{H}}$ and $\widetilde{\mathbf{H}}$. It is important to note that this optimal permutation is difficult to find out and depends on the selected detection method.

2.2. SUMIS algorithm review

The SUMIS [12] algorithm employs the partitioning model (5), which is based on $\mathbf{H}^T\mathbf{H}$ as is explained in [12]. This algorithm is composed by two main stages. A first approximation to the LLR values is computed in Stage A and then, these approximate values are used to compute new refined LLRs values in Stage B. Here we give a brief review of SUMIS algorithm explained in [12]. It is important to note that in [12] referring to a symbol is equivalent to a bit, which is not the case here, since we extend the algorithm to higher-order constellations.

Stage A: The algorithm starts with the partitioned model (5) defining the new model as

$$\overline{\mathbf{y}} = \overline{\mathbf{H}}\overline{\mathbf{s}} + \mathbf{e} \tag{6}$$

where $\mathbf{e} = \widetilde{\mathbf{H}}\widetilde{\mathbf{s}} + \mathbf{v}$ is a Gaussian stochastic vector $\mathbf{e} \sim \mathcal{N}(\mathbf{0}, \mathbf{Q})$ with $\mathbf{Q} = \widetilde{\mathbf{H}}\widetilde{\mathbf{H}}^T + \frac{N_o}{2}\mathbf{I}$. Using the following operator $\|\mathbf{x}\|^2_{\mathbf{Q}} \triangleq \mathbf{x}^T\mathbf{Q}^{-1}\mathbf{x}$, we compute the approximate $\lambda_{j,b}$ LLR as

$$\lambda_{j,b} = \log \frac{\sum_{\overline{\mathbf{s}}\epsilon\chi^0_{j,b}} \exp(-\frac{1}{2}\|\mathbf{y} - \overline{\mathbf{H}}\overline{\mathbf{s}}\|^2_Q)}{\sum_{\overline{\mathbf{s}}\epsilon\chi^1_{j,b}} \exp(-\frac{1}{2}\|\mathbf{y} - \overline{\mathbf{H}}\overline{\mathbf{s}}\|^2_Q)} \tag{7}$$

Stage A is performed for all bits $b = 1, \cdots, k$ in all symbols $j = 1, \cdots, n_T$.

Stage B: In the second stage, the interfering vector $\widetilde{\mathbf{s}}$ is suppressed in (6) and then the LLR values are computed again over a purified model. In this context, the new model is given by

$$\mathbf{y}' \triangleq \mathbf{y} - \widetilde{\mathbf{H}}\mathbb{E}\{\widetilde{\mathbf{s}}|\mathbf{y}\} \approx \overline{\mathbf{H}}\overline{\mathbf{s}} + \mathbf{n}' \tag{8}$$

where $\mathbb{E}\{\widetilde{\mathbf{s}}|\mathbf{y}\}$ is the conditional expected value of vector $\widetilde{\mathbf{s}}$, and $n' \backsim \mathcal{N}(\mathbf{0}, \mathbf{Q}')$ with $\mathbf{Q}' \triangleq \widetilde{\mathbf{H}}\widetilde{\mathbf{\Upsilon}}\widetilde{\mathbf{H}}^T + \frac{N_o}{2}\mathbf{I}$. $\widetilde{\mathbf{\Upsilon}}$ is the conditional covariance matrix of $\widetilde{\mathbf{s}}$ and can be computed by

$$\widetilde{\mathbf{\Upsilon}} = \mathbb{E}\{\text{diag}(\widetilde{\mathbf{s}})^2|\mathbf{y}\} - \mathbb{E}\{\text{diag}(\widetilde{\mathbf{s}})|\mathbf{y}\}^2 \tag{9}$$

where diag($\triangle$) returns a diagonal matrix with the elements of $\triangle$ vector on its diagonal.
Hence, the refined LLR values can be computed as

$$L_{j,b} \approx \log \frac{\sum_{\overline{\mathbf{s}}\epsilon\chi^0_{j,b}} \exp(-\frac{1}{2}\|\mathbf{y}' - \overline{\mathbf{H}}\overline{\mathbf{s}}\|^2_{Q'})}{\sum_{\overline{\mathbf{s}}\epsilon\chi^1_{j,b}} \exp(-\frac{1}{2}\|\mathbf{y}' - \overline{\mathbf{H}}\overline{\mathbf{s}}\|^2_{Q'})}. \tag{10}$$

3. SUMIS-BO ALGORITHM

The SUMIS algorithm previously explained provides a clear trade-off between computational complexity and detection performance. In [12] the algorithm considers only the constellations where a "symbol" is equivalent to a "bit", however in the previous section we have given a brief summary of the algorithm extending the notation to constellations which a "symbol" is not equivalent to a "bit". For these higher constellations, the conditional expected value $\mathbb{E}\{s_j|\mathbf{y}\}$ of symbol s_j with $j = 1, \cdots, n_T$ should be computed as

$$\mathbb{E}\{s_j|\mathbf{y}\} \triangleq \sum_{\mathbf{s}\in\Omega} sP(s_j = s|\mathbf{y}) \approx \sum_{\mathbf{s}\in\Omega} sP(s_j = s|\overline{\mathbf{y}})|_{\overline{\mathbf{y}}=\mathbf{y}}$$
$$= \sum_{\mathbf{s}\in\Omega} s \prod_{b=1}^{k} \frac{1}{1 + e^{(-2s_{j,b}+1)\lambda_{j,b}}}. \tag{11}$$

The SUMIS algorithm computes the $\lambda_{j,b}$ in *Stage A* using (7), where the number of terms in the summation over $\overline{\mathbf{s}}$ is k^{n_s}. This implies that SUMIS algorithm has to compute the term $\exp(-\frac{1}{2}\|\mathbf{y} - \overline{\mathbf{H}}\overline{\mathbf{s}}\|^2_Q)$ a number of times $k^{n_s} \times n_T \times k$ to compute the total $\lambda_{j,b}$ values.

SUMIS-BO algorithm proposes to reduce the complexity of this part, computing approximate $\lambda_{j,b}$ values, not the exact ones. These

SUMIS\SUMIS-BO	$M = 4$	$M = 16$	$M = 64$
$n_T = 8$	128\32	2048\64	10368\96
$n_T = 16$	256\64	4096\128	20736\192
$n_T = 24$	384\96	6144\192	31104\288

Table 1: Number of terms $\frac{1}{2}(\|\mathbf{y} - \overline{\mathbf{H}}\bar{\mathbf{s}}\|_Q^2)$ that SUMIS\SUMIS-BO has to compute for $n_s = 3$.

values are not the final LLRs but are employed in an interference suppression mechanism in Stage B.

In the SUMIS-BO, we employed max-log approximation and a reduced complexity technique to compute the $\lambda_{j,b}$ values using (4). The algorithm finds a good approximation of $\min_{\bar{\mathbf{s}}\epsilon\chi^0_{j,b}} \|\mathbf{y} - \mathbf{H}\mathbf{s}\|^2$ and $\min_{\bar{\mathbf{s}}\epsilon\chi^1_{j,b}} \|\mathbf{y} - \mathbf{H}\mathbf{s}\|^2$, which we call as $\bar{\mathbf{s}}_0$ and $\bar{\mathbf{s}}_1$ respectively. Once $\bar{\mathbf{s}}_0$ and $\bar{\mathbf{s}}_1$ have been computed, the estimated λ_{jb} can be calculated by a slight modification of (4) given by

$$\lambda_{jb} = \frac{1}{2}(\|\mathbf{y} - \overline{\mathbf{H}}\bar{\mathbf{s}}_0\|_Q^2 - \|\mathbf{y} - \overline{\mathbf{H}}\bar{\mathbf{s}}_1\|_Q^2). \quad (12)$$

In this case to compute the $\lambda_{j,b}$ values we only need to compute two terms for the total $n_T \times k$ LLR values. In Table 1 we can see the number of terms that we have to compute for SUMIS and SUMIS-BO using different system parameters. Thus, it is clear that the number of operations to perform the computation of $\lambda_{j,b}$ values is drastically reduced.

Algorithm 1 *Stage A* SUMIS-BO pseudo-code

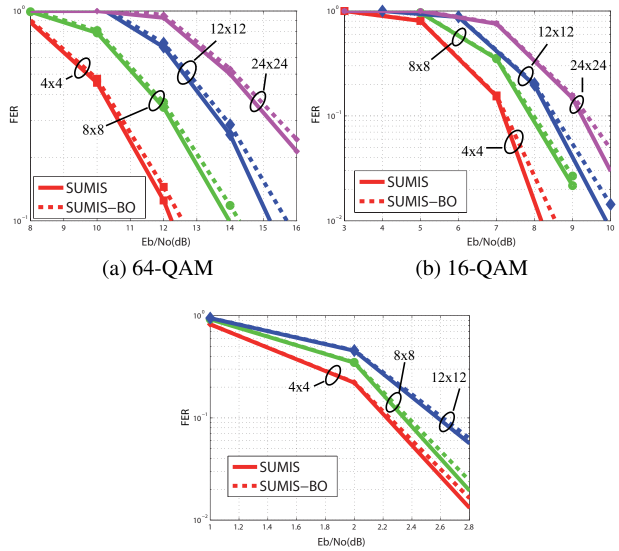
```
Input: H, y, k and n_s
for j = 1 to n_T do
    Decide a partitioning in (5).
    Calculate ŝ in (3).
    for b = 1 to k do
        Apply BO if (3) is outside the constellation.
        Calculate λ_{j,b} using (4).
    end for
    Calculate E{s_j|y} using (11).
end for
```

As we can see in the simulation result, the method employed in this calculation affects very slightly to the final result. The key of this technique to keep a good performance is the use of box optimization (BO) method. The complete explanation of the BO algorithm is complex and extensive and it can not be detailed here. Details of this algorithm and its good performance in MIMO detection systems can be found in [15].

Modified Stage A: In this work the main idea is to employ the BO method combined with the "*Zero Forcing*" (ZF) algorithm to compute $\bar{\mathbf{s}}_0$ and $\bar{\mathbf{s}}_1$ in the *Stage A*. Firstly the linear detector ZF is computed by

$$\hat{\bar{\mathbf{s}}} = \overline{\mathbf{H}}^{-1}\mathbf{y}. \quad (13)$$

The vector obtained after this process $\hat{\bar{\mathbf{s}}}$, known as ZF estimator, is a meaningful starting point for the BO algorithm. ZF estimate requires a matrix inversion which can be very complex for higher antenna dimension. However, the number of columns of $\overline{\mathbf{H}}$ is given by n_s, which it is relatively low. For this reason, we can use the QR decomposition of the real $\overline{\mathbf{H}}$ matrix and compute the ZF estimation of the equivalent problem. In this case, the matrix inverse in it will be of size $n_s \times n_s$, avoiding the problem for higher antenna dimension.



(a) 64-QAM (b) 16-QAM

(c) 4-QAM

Fig. 1: FER as function of E_b/N_0 with different system sizes and constellation orders.

It was shown that if any component is outside of the box that delimits the constellation, we can improve the accuracy of the ZF estimator by applying the BO method. Then, the new estimated vector is quantized to the nearest element of the constellation. On the other hand, in cases where all components of the ZF estimator are within the constellation, the BO method is not applied and the components of the ZF estimator are rounded to the nearest element of the constellation. It is important to note that the constellation used for the quantizing the ZF and BO estimation is restricted, using only the points of the constellations with the corresponding bit equal to one or zero, which depends on whether we calculate $\bar{\mathbf{s}}_1$ or $\bar{\mathbf{s}}_0$ respectively. Once $\bar{\mathbf{s}}_0$ and $\bar{\mathbf{s}}_1$ have been computed, the estimated λ_{jb} can be calculated using (12).

The *Stage A* of the SUMIS-BO algorithm is summarized in Algorithm 1 with generic pseudo-code. The *Stage B* remains equal than the SUMIS algorithm explained in [12].

4. SIMULATION AND ANALYSIS

We estimate the Bit Error Rate (BER) by means of Monte Carlo simulations varying the signal-to-noise ratio, defined as E_b/N_0. E_b is the transmitted energy per uncoded bit. A rate $1/2$ LDPC code of codeword size 1296 bits is also used. The LDPC encoding and decoding scheme comes from the IEEE 802.11n wireless LAN standard; and some software tools have been download from http://www.csl.cornell.edu/∽studer/software_ldpc.html. The selected decoding option is the sum-product algorithm. There is no iteration between the detector and the decoder, and the transmitted symbols are assumed to be uniformly distributed. We simulate 4×4, 8×8, 12×12 and 24×24 complex MIMO systems with M-QAM constellation, where M is $\{16, 64\}$, $\{4, 16, 64\}$, $\{4, 16, 64\}$ and $\{4, 16\}$ respectively. The number n_s over which the partitioning

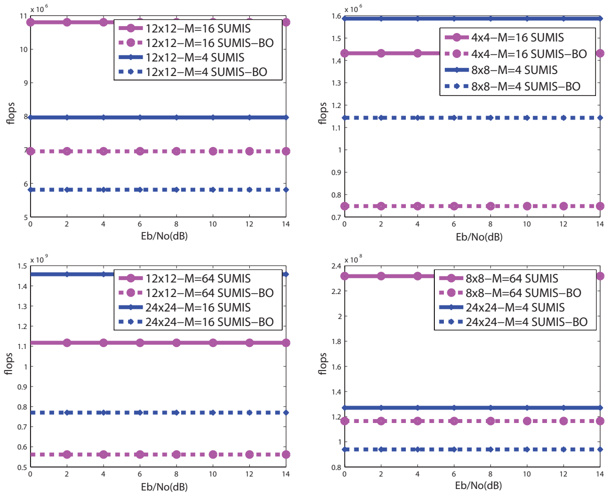


Fig. 2: Flops as function of E_b/N_0 with different system sizes and constellation orders.

$n_T \backslash M$	4	16	64
8	29.7%	47.7%	49.7%
16	27.9%	47.5%	49.7%
24	27%	47.5%	49.7%
48	26%	47.2%	47.2%

Table 2: Percentage of flops improvement of SUMIS-BO with respect to SUMIS.

model is done, is equal to 3 since in [12] this value offers a well defined performance-complexity tradeoff.

4.1. Simulation Setup

In order to evaluate our proposal, we have compared SUMIS-BO algorithm with the original SUMIS algorithm. Since the SUMIS method in [12] has been compared with others algorithms like PM [10], SFSD [7] or STS [5], we omit plotting its performance and complexity curves. The observation that SUMIS and SUMIS-BO in this work give better performance than the max-log based algorithms is relevant, because the most important competitors to SUMIS are based on the max-log approximation.

Fig. 1 shows a comparison between the algorithms SUMIS and SUMIS-BO with different number of antennas and constellation orders. The results in the figures clearly illustrate that for $4-$QAM the SUMIS-BO detector performs equals to SUMIS for all E_b/N_0 values and all cases. Note however, that for higher constellation orders the SUMIS-BO achieves slightly worse behavior regarding SUMIS algorithm. It is important to note that this performance loss is negligible and in contrast, the improvement in the computational cost of the algorithm is very high.

The computational cost is represented in Fig. 2 in terms of flops. For this purpose, the experiments were carried out varying the E_b/N_0 from 0 to 14, detecting 100 signal for each E_b/N_0 and the average number of flops were recorded. Table 2 shows that the SUMIS-BO algorithm reaches large advances over the SUMIS algorithm in terms of flops, especially for problems with higher constellations orders. Another important issue is the fixed complexity over E_b/N_0 that is exhibited.

5. CONCLUSION

We have proposed an improved version of the SUMIS MIMO detection method, SUMIS-BO. The proposed algorithm shows a slight performance loss in some cases, however SUMIS-BO reduces drastically the computational cost of the SUMIS for all the studied problem sizes and retains the fixed complexity of the SUMIS method. In conclusion, the proposed algorithm provides a very good tradeoff between complexity and performance.

6. REFERENCES

[1] IEEE P802.11ac, *Specification Framework for TGac. IEEE 802.11-09/0992r21*, Jan 2011.

[2] http://www.3gpp.org/ftp/Information/WORKPLAN/DescriptionReleases 10 description 20100924.zip, *Overview of 3GPP Release 10 V0.0.8 (2010-09) [Online]*.

[3] E. Bigilieri, R. Calderbank, A. Constantinides, and A. Goldsmith, *MIMO Wireless Communications*, Cambridge Univ. Press, 2010.

[4] F. Rusek, D. Persson, B.K. Lau, E. G. Larsson, T.L. Marzetta, O. Edfors, and F. Tufvesson, "Scaling up MIMO: Opportunities and challenges with very large arrays," *IEEE Signal Proc. Magazine*, vol. 30, no. 1, pp. 40–60, Jan. 2013.

[5] C. Studer, A. Burg, and H. Bölcskei, "Soft-output sphere decoding: algorithms and VLSI implementation," *IEEE J. Sel. Areas Commun.*, vol. 26, no. 2, pp. 290–300, 2008.

[6] R. Wang and G. Giannalis, "Approaching MIMO channel capacity with reduced-complexity soft sphere decoding," *IEEE Wireless Communications and Networking Conf.*, vol. 44, pp. 1620–1625, 2004.

[7] L.G. Barbero, T. Ratnarajah, and C. Cowan, "A low-complexity soft-MIMO detector based on the fixed-complexity sphere decoder," in *Acoustics, Speech and Signal Processing, 2008. ICASSP 2008. IEEE International Conference on*, April 2008, pp. 2669–2672.

[8] B.M. Hochwald and S. ten Brink, "Achieving Near-Capacity on a Multiple-Antenna Channel," *IEEE Trans. Commun.*, vol. 51, pp. 389–399, 2003.

[9] X. Wu and J. Thompson, "A fixed-complexity soft-MIMO detector via parallel candidate adding scheme and its FPGA implementation," *Commun. Lett.*, vol. 15, no. 2, pp. 241–243, Feb. 2011.

[10] E. G. Larsson and J. Jaldén, "Fixed-complexity soft MIMO detection via partial marginalization," *IEEE Trans. on Signal Proccesing*, vol. 56, no. 1, pp. 3397–3407, Aug. 2008.

[11] D. Persson and E. G. Larsson, "Partial Marginalization soft MIMO detection with higher order constellations," *IEEE Trans. on Signal Proccesing*, vol. 59, no. 1, pp. 453–458, Jan. 2011.

[12] M. Cîrkić and E. G. Larsson, "SUMIS: Near-optimal soft-in soft-out MIMO detection with low and fixed complexity," *Signal Processing, IEEE Transactions on*, vol. 62, no. 12, pp. 3084–3097, June 2014.

[13] A. Björck, "Numerical methods for least squares problems," *Philadelphia: SIAM*, 1996.

[14] M. Stojnic, H. Vikalo, and B. Hassibi, "Speeding up the Sphere Decoder with H^∞ and SDP inspired lower bounds," *IEEE Transactions on Signal Processing*, vol. 56, no. 2, pp. 712–726, 2008.

[15] V.M. Garcia-Molla, A. Vidal, A. Gonzalez, and S. Roger, "Maximum likelihood detection through sphere decoding combined with box optimization," *Signal Processing, Elsevier*, vol. 98, pp. 287–294, 2014.