PARAMETRIC ANALOG MAPPINGS FOR CORRELATED GAUSSIAN SOURCES OVER AWGN CHANNELS

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ABSTRACT

We address the transmission of bivariate Gaussian sources using analog Joint Source Channel Coding (JSCC). The analog mappings are specifically designed to exploit the correlation between the source symbols. A parametric mapping based on sinusoidal functions is proposed and its performance is compared to that of the optimal non parametric mappings and other applicable analog JSCC mappings, and also to the theoretical bound. The obtained results show that the performance of the parametric mapping closely approaches the optimal performance, and it is very similar to that of the non parametric one.

Index Terms— Analog Joint Source Channel Coding, Parametric Mappings, Correlated Souces.

1. INTRODUCTION

The transmission of independent sources is a common assumption in signal processing for wireless communications. Nevertheless, this premise is not adequate for modeling multitude of practical situations where the information to be transmitted is correlated in some way. In this work, we address the transmission of correlated information using analog Joint Source Channel Coding (JSCC).

Analog JSCC techniques are based on the joint optimization of the source and channel encoding. This strategy has been shown to approach the optimal cost-distortion tradeoff, specially for the compression of independent analog sources in a single-user scenario [1, 2, 3]. In this case, the discrete-time continuous-amplitude symbols are directly encoded using analog mappings based on parametric space-filling curves [4, 5]. Analog JSCC presents some advantages with respect to digital systems based on the source-channel separation [6]. The complexity and delay are drastically reduced because the processing is at symbol level. In addition, analog JSCC schemes do not saturate at high SNR region (quantization effect) and present graceful degradation for imperfect channel information. Moreover, they can be adapted in time-varying channels without a complete redesign of the system.

Previous works have addressed the design of analog JSCC mappings for correlated sources, specially for MAC communications [7, 8, 9]. For AWGN channels, non parametric encoders based on Power Constrained Channel Optimized Vector Quantizers have been proposed in [10]. In general, non parametric analog JSCC mappings for different scenarion can be obtained by following the iterative procedure proposed in [8, 11] or using deterministic annealing [12]. The main advantage of these methods is that they can find near optimal solutions given a prior distribution of the source symbols, but they can also fall in local minimum and are rather complex.

Approximations for these general encoders by parametric curves largely simplifies the design of the analog JSCC systems because any point of the source space can directly be mapped to the corresponding channel symbols (and viceversa) by using the equation defining the parametric curve. In this work, based on previous results for non-parametric encoding of correlated Gaussian sources, a parametric version is proposed and is shown to achieve near optimal performance for the Signal-to-Noise Rate (SNR) levels of interest.

2. SYSTEM MODEL

Let us consider the transmission of bivariate Gaussian source symbols $\mathbf{s} = [s_1, s_2]^T$ with zero mean and covariance matrix

$$\mathbf{C_s} = E\left[\mathbf{s}\mathbf{s}^H\right] = \begin{bmatrix} \sigma_s^2 & \rho\sigma_s^2 \\ \rho\sigma_s^2 & \sigma_s^2 \end{bmatrix}$$

where ρ is the correlation factor. The probability density function (pdf) of $\mathbf{s}$ is therefore given by

$$p_{\mathrm{s}}(\mathbf{s}) = \frac{1}{\sqrt{(2\pi)^2 \det\{\mathbf{C_s}\}}} \exp\left(-\frac{1}{2}\mathbf{s}^T\mathbf{C_s}^{-1}\mathbf{s}\right). \quad (1)$$

We explore the utilization of non linear analog mappings to directly transform two correlated source symbols $\mathbf{s} = [s_1, s_2]^T$ into one channel symbol x, that is transmitted over a noisy channel. Analog mappings are mathematically defined as a function $g : \mathbb{R}^2 \to \mathbb{R}$ such that $x = g(\mathbf{s})$. The analog encoder is power normalized to guarantee that $\mathbb{E}[\|g(\mathbf{s})\|^2] = 1$.

This work has been funded by Office of Naval Research of United States and Ministerio de Economía y Competitividad of Spain under grants N62909-15-1-2014 and TEC2013-47141-C4-1-R.

In this work, we focus on analog JSCC mappings based on geometric curves that fill the source space efficiently.

The channel distortion is modeled as Additive White Gaussian Noise (AWGN). Thus, the received signal can be expressed as

$$y = x + n, \tag{2}$$

where $n \sim \mathcal{N}(0, \sigma_n^2)$ is a real-valued Gaussian random variable. At reception, an estimate of the source symbols $\hat{\mathbf{s}} = [\hat{s}_1, \hat{s}_2]^T$ is calculated from the received symbol y by using an appropriate analog decoder. Thus, the decoder will be a function $h : \mathbb{R} \to \mathbb{R}^2$ such that $\hat{\mathbf{s}} = h(y)$.

In the next sections, the optimal non parametric mappings are first obtained for this scenario. Then, a parametric version of these mappings is proposed as a more practical alternative.

3. NON PARAMETRIC ANALOG MAPPINGS

The objective of analog communications is to transmit the source information minimizing the distortion between the source symbols and the estimates at the decoder output. In this work, Minimum Square Error (MSE) will be used as distortion measure. In this case, the optimal analog JSCC mappings and the corresponding decoder can be obtained by using the iterative algorithm presented in [11]. This algorithm is based on an alternating optimization between the analog encoder and the decoder to decrease the MSE cost function successively at each iteration.

The analog decoder can be calculated directly given the encoder $g(\cdot)$ according to the Minimum Mean Square Error (MMSE) criterion, i.e.

$$h(y) = \mathbb{E}[s|y] = \frac{\int \mathbf{s}\, p_s(\mathbf{s})\, p_n(y - g(\mathbf{s}))\, \mathrm{d}\mathbf{s}}{\int p_s(\mathbf{s})\, p_n(y - g(\mathbf{s}))\, \mathrm{d}\mathbf{s}}, \tag{3}$$

where $p_n(x) = (2\pi\sigma_n^2)^{-1/2} \exp\left(-x^2/\sigma_n^2\right)$ represents the Gaussian noise distribution.

Unlike the decoder, the optimal encoder cannot be explicitly calculated from the decoder. However, [11] proposes an steepest descent algorithm where the gradient of the cost function, subject to a power constraint, is given by

$$\nabla J[g] = \lambda p_s(\mathbf{s}) g(\mathbf{s}) - \int h'(g(\mathbf{s}) + n)[\mathbf{s} - h(g(\mathbf{s}) + n)] p_n(n) p_s(\mathbf{s}) \mathrm{d}n, \tag{4}$$

where λ is a Lagrange multiplier to satisfy the transmit power constraint $P[g] = \int g^2(\mathbf{s}) p_s(\mathbf{s}) \mathrm{d}\mathbf{s} \leq P$. This gradient enables us to iteratively update the encoder by using the decoder previously obtained.

As observed, (3) and (4) require the numerical calculation of involved integrals. Source and channel spaces are hence discretized and the analog encoder-decoder duple is specifically defined for the considered points. A mapping table needs to be stored at the transmitter to encode the source symbols, while the corresponding table for the decoder is at the receiver. Notice that the performance of the resulting mappings largely depends on the resolution of this discretization step, which in turn determines the table sizes. Indeed, the optimal encoder-decoder duple is different for each noise variance. Thus, we have to calculate the mapping table and the corresponding decoder for each potential value of the noise variance and store those tables at the transmitter and receiver. On the other hand, the optimization problem is non convex and the algorithm can converge to local minima. The election of a good initial conditions and the use of optimization techniques like noisy channel relaxation [13] can mitigate this problem and avoid poor local minima.

4. PARAMETRIC ANALOG MAPPINGS

Parametric mappings are advantageous with respect to the non parametric ones because they allow to lower the computational complexity of the general MMSE decoding. Also, the mapping can be updated depending on the signal properties and the channel conditions by adapting their parameters.

The shape of the non parametric mappings obtained in the previous section suggests that a linear mapping of the source symbols is optimal for low SNRs. However, the shape of these mappings resembles a sinusoidal function in the medium and high SNR region. For example, Figure 1 shows the non parametric mapping obtained for SNR = 25 dB. In this figure, the red points define the curve employed to fill the bi-dimensional source space, and the blue lines represents the mapping from the source symbols to the corresponding channel symbols. Similar conclusions were obtained in [10] for the transmission of multivariate Gaussian sources by using Power Constrained Channel Optimized Vector Quantizer (PCCOVQ). Notice that these encoders can be interpreted as a discrete version of the optimal analog JSCC mappings. These results also corroborate the theoretical analysis of [7] for the considered ρ values.

After examining the non parametric mappings obtained for different SNRs, we propose to use this parametric curve:

$$\mathbf{K}(t) = \mathbf{U\Sigma} \begin{bmatrix} t - \frac{1}{2\alpha}\sin(\alpha t) \\ \Delta \sin(\alpha t) \end{bmatrix}, \tag{5}$$

where $\mathbf{K}(t)$ represents the point into the bi-dimensional space corresponding to the parameter t on the curve, and $\mathbf{C}_s = \mathbf{U}^H \mathbf{\Sigma} \mathbf{U}$ is the eigendecomposition of $\mathbf{C}_s$, with $\mathbf{U}$ the matrix with the eigenvectors of $\mathbf{C}_s$ as columns, and $\mathbf{\Sigma} = \mathrm{diag}\{\lambda_1, \lambda_2\}$, $\lambda_1 > \lambda_2$, the eigenvalues of $\mathbf{C}_s$. The parameters α and Δ represent the frequency and the amplitude of the sinusoidal function, respectively, hence defining the shape of the proposed mapping. The optimal values for these parameters depend on the source correlation and SNR values. An adequate optimization of α and Δ is important to closely approach the optimal cost-distortion tradeoff.

The optimization step can be implemented by a exhaustive search on the parameter space to jointly determine the optimal

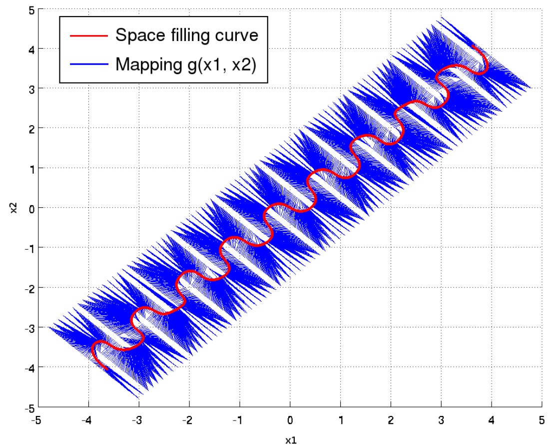

Fig. 1. Non parametric mapping obtained for SNR = 25 dB .

values for α and Δ. Although computationally costly, the optimal parameter values can be calculated offline and then stored into a lookup table. The optimization procedure can be carried out efficiently by fitting the optimal non parametric mappings. We have applied non-linear least squares fitting to the non-parametric curves by means of a gradient algorithm subject to a power constraint to obtain the optimal values of α and Δ for a range of SNR between 0 dB and 30 dB [14]. These values are presented in Table 1 and Table 2 for correlation factor $\rho = 0.9$ and $\rho = 0.75$, respectively.

Equation (5) defines the parametric curve that maps points from the channel space to the source space, but we also need to specify how the points on the source space are mapped to the parametric curve. The classical solution consists in mapping each source point to the closest point on the curve in terms of the Euclidean distance, i.e.

$$x = M(\mathbf{s}) = \arg\min_t \ \|\mathbf{s} - \mathbf{K}(t)\|^2. \quad (6)$$

This mapping operation based on the Euclidean distance has a low computational complexity, but it does not exploit the information about the channel noise. However, this information can be partially integrated into the analog encoder with

$$x = N(\mathbf{s}) = \arg\min_t \int_{-\infty}^{\infty} \|\mathbf{s} - \mathbf{K}(u)\|^2 p_n(u-t)\mathrm{d}u \quad (7)$$

which simplifies to the minimum distance encoder in (6) by assuming that the channel noise is zero, i.e. that the noise distribution is $p_n(x) = \delta(x)$.

An estimate of the source symbols is computed at the receiver by the analog decoder from the noisy received symbols. The optimal decoder is given by (3) since it minimizes the distortion (MSE) between the source and the estimated symbols. Recall that the application of MMSE decoding requires the discretization of the source and channel spaces to numerically calculate the corresponding integrals.

SNR (dB)	0	5	10	15	20	25	30
α	1.95	3.02	4.63	5.11	6.9	8.2	10.39
Δ	0.008	0.03	0.14	0.16	0.93	1.2	1.44

Table 1. Optimal values of α and Δ for 2:1 analog JSCC of bivariate Gaussian sources with rho = 0.9.

SNR (dB)	0	5	10	15	20	25	30
α	2.0	2.2	3.79	4.66	5.01	6.52	8.52
Δ	0..001	0.001	0.14	0.83	1.24	1.41	1.55

Table 2. Optimal values of α and Δ for 2:1 analog JSCC of bivariate Gaussian sources with rho = 0.75.

For the case of independent sources, a two-stage receiver based on the concatenation of a linear MMSE filter and a Maximum Likelihood (ML) decoder was proposed in [15]. This approach provides similar performance to the optimal MMSE decoder with minimum complexity and delay. In this work, we apply this strategy for correlated sources, and therefore the source symbol estimates are calculated as

$$\hat{\mathbf{s}} = h(y) = \mathbf{K}\left(\frac{y}{1+\sigma_n^2}\right). \quad (8)$$

5. RESULTS

In this section we present the results of the computer experiments carried out to evaluate the performance of the parametric and non parametric mappings described in the previous sections. The obtained results are compared to the corresponding optimal bounds for the considered scenarios and to the performance of other suitable analog mappings.

The performance of analog communications is measured in terms of the Signal-to-Distortion Rate (SDR) with respect to the SNR. The SDR is defined as

$$\mathrm{SDR(dB)} = 10\log_{10}(\sigma_s^2/\mathrm{MSE}),$$

where σ_s^2 is the source variance and the term $\mathrm{MSE} = 1/2\sum_{i=1}^{2} E[\|\hat{s}_i - s_i\|^2]$ represents the MSE between the source and the estimated symbols.

The optimal performance for analog communications is determined by the optimal cost-distortion tradeoff curve. In the literature, this upper bound is referred to as the Optimum Performance Theoretically Attainable (OPTA) and it is calculated by equating the rate distortion of the source $R(D)$ and the channel capacity, $C(\sigma_n^2)$. Assuming normalized channel symbols, the capacity of an AWGN channel is given by

$$C = \frac{1}{2}\log\left(1 + \frac{1}{\sigma_n^2}\right). \quad (9)$$

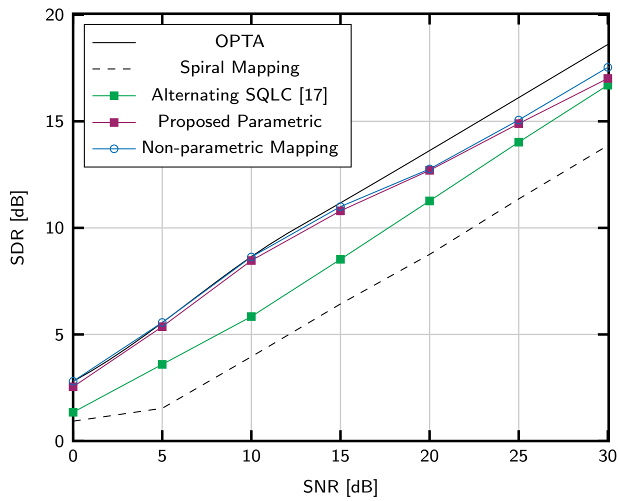

Fig. 2. SDR for different mappings with $\rho = 0.9$.

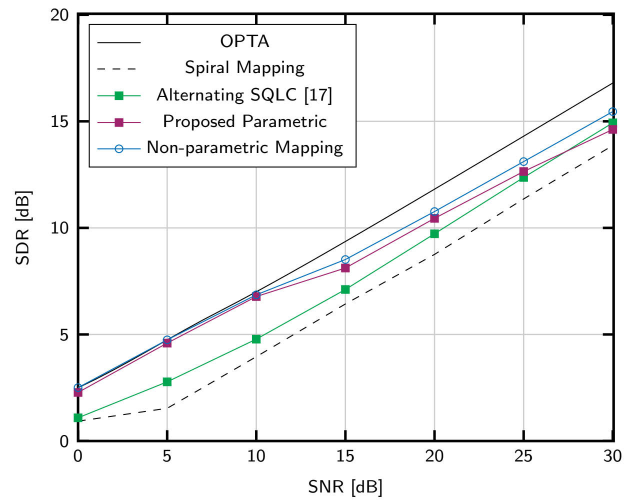

Fig. 3. SDR for different mappings with $\rho = 0.75$.

For multivariate Gaussian sources and the MSE as the distortion criterion, the rate distortion function can be represented parametrically as [16]

$$D(\theta) = \frac{1}{M}\sum_{i=1}^{M} \min[\theta, \lambda_i],$$

$$R(\theta) = \frac{1}{M}\sum_{i=1}^{M} \max\left[0, \frac{1}{2}\log\left(\frac{\lambda_i}{\theta}\right)\right], \qquad (10)$$

where $D(\theta)$ is the distortion function, λ_i represents the eigenvalues of the covariance matrix $\mathbf{C}_s$ and M is given by the number of eigenvalues larger than zero.

By equating (9) and (10), we obtain the following expression for the OPTA bound

$$\text{OPTA} = \begin{cases} \frac{2\sigma_n^2+2}{2\sigma_n^2+(1-\rho)} & \text{SNR} < \frac{2\rho}{1-\rho} \\ \sqrt{\frac{\sigma_n^2+1}{\sigma_n^2(1-\rho^2)}} & \text{SNR} \geq \frac{2\rho}{1-\rho}. \end{cases} \qquad (11)$$

In the computer experiments, we consider the transmission of source symbols generated from a bivariate Gaussian distribution with zero mean and covariance matrix $\mathbf{C}_s$. Each pair of source symbols is 2:1 compressed by using (7), and the resulting channel symbol is transmitted over an AWGN channel with noise variance σ_n^2. At reception, the observed noisy symbol is decoded with the two-stages receiver and the distortion with respect to the original symbols is measured.

Figure 2 and Figure 3 show the performance curves for both the proposed parametric mapping with the encoder defined in (6), and the optimal non parametric mapping with $\rho = 0.9$ and $\rho = 0.75$, respectively. The OPTA calculated according to (11) is also plotted in the figures to represent the theoretical optimal performance. In order to complete the analysis, we also include the performance achieved for other two parametric analog JSCC mappings that can be applied for the considered communication model. The first of them is the Archimedean spiral [1, 2], traditionally employed for 2:1 compression of independent sources. In this case, the source samples are first decorrelated and then encoded with this mapping. The second one is a variant of the Scalar Quantizer Linear Coder (SQLC) [9] that is referred to as alternating sign SQLC and was proposed in [17] for bivariate Gaussian over Broadcast Channels (BC). This BC mapping defines a projection matrix $\mathbf{H}$, which has been assumed to be $\mathbf{H} = \mathbf{U}\mathbf{\Sigma}$.

As observed in both figures, the performance of the proposed parametric mapping closely approaches the OPTA, and significantly outperforms the other two parametric mappings for all range of SNRs. Notice that the proposed mapping and the alternating sign SQLC present very similar performance for SNRs around 30 dB. In addition, the performance of the parametric and non parametric mappings is practically the same for low SNRs (SNR $<$ 20 dB), but a slight difference can be appreciated in the high SNR region. This is more noticeable as the correlation decreases ($\rho = 0.75$). This gap is because the shape of the optimal mappings in that region can no longer be represented strictly by (5), and the use of a suboptimal decoder. The worst performance corresponds to the use of the Archimedean spiral since this mapping is not able to exploit the correlation between the source symbols.

6. CONCLUSION

In this work, we have studied the application of analog JSCC mappings for the transmission of correlated Gaussian sources over AWGN channels. We have proposed a parametric curve based on sinusoidal functions as approximation to the optimal non parametric mappings, whose high complexity and limited flexibility makes their practical implementation rather difficult. The proposed analog JSCC mapping closely approaches the optimal performance with lower complexity and delay.

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