KALMAN FILTERS WITH BAYESIAN QUADRATIC GAME FUSION IN NETWORKS

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***Abstract*—Distributed filtering in network is a fundamental problem in the field of network signal processing. Each node estimates or tracks some unknown state relying on the private observation and the fusion information from the network. Network fusion is generally a way of interaction over network, by which nodes can learn from each other and make decision mutually. Unlike conventional methods, we construct a distributed filter using Bayesian network game as a fusion tool, where all the nodes exchange their best strategies instead of exchanging local estimators. The proposed algorithm is a coalition of signal processing and game theory in network, which can be extended to more general signal processing and decision making models.**

***Index Terms*—distributed filtering, network signal processing, network game, Bayesian game**

I. Introduction

Distributed filtering in network has been an appealing research point in the field of network signal processing [1]–[4] recently. The basic application deals with nodes in a network collaboratively tracking the real trajectory of an unknown state with limited information and interaction capability. A typical distributed network filter iterates with two steps, local innovation and network fusion [4]–[6]. For a specific node, the local innovation executes estimation update using new private observations, which can be configured with traditional signal processing method such as RLS, RLMMSE and Kalman filter [7] independent of network topology. The network fusion updates self estimation using fused information from other nodes. Existing network fusion strategies include incremental [8], [9], consensus [4], [6], [10]–[14], diffusion [15], [16] and network game [5]. All these strategies mingle and process the network information locally with commutative messages such as observations, estimations, auxiliary variables or actions.

We focus on the widely used Kalman filter (KF) in a network structure in this paper. A distributed Kalman filter builds in some network fusion algorithms, like consensus [6], [10]–[12] and diffusion [16], [17]. All of these methods can be seen as *parameter-based* fusion, where the information exchanged over network usually are local estimations on a shared state variable.

In contrast, the Bayesian network game is powerful to analyze the interaction of strategies among players in a network with incomplete information. Ceyhun Eksin *et al.* [18] proposed an analytical solution to a Bayesian network game of quadratic utilities (BQNG), which is a seminal work incorporating network game into signal processing in network. The optimal actions, derived by best strategies, incorporate with all available historical information and help nodes refine their estimations on a static unknown state. Exchanging strategies, rather than parameters, can be seen as a new type of *strategy-based* fusion way.

In [5], we proposed a recursive distributed filter for multiple observations based on BQNG fusion. However, it only deals with the estimation problem on a static state. In order to track a time-varying object, we construct a distributed KF extending our previous work, namely the BQNG KF.

The paper is organized as follows. Sec.II describes the system model of the BQNG KF. Sec.III provides the algorithm inference of the proposed filter in detail, while Sec.IV exhibits its numerical performance. The conclusion ends up in Sec.V.

II. System Model

Consider a network tracks on an unknown state θ_t in a distributed manner. The network is represented by an undirected connected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ with its node set as $\mathcal{V} = \{1, 2, ..., N\}$ and its edge set as $\mathcal{E}$. The unknown state θ_t varies with time

$$\theta_t = f_t\theta_{t\text{-}1} + w_{\theta,t}, \tag{1}$$

where f_t is the transition factor and $w_{\theta,t} \sim \mathcal{N}(0, \sigma^2_{\theta,t})$ is the transition noise. Node i receives an observation on θ_t as

$$s_{i,t} = h_{i,t}\theta_t + n_{i,t}, \quad \forall i \in \mathcal{V}, \tag{2}$$

where $h_{i,t}$ is the observing factor and $n_{i,t} \sim \mathcal{N}(0, \sigma^2_{i,t})$ is the measurement noise independent among nodes. $s_{i,t}$ is private for node i and can not be shared with other nodes.

The fusion information comes from a network game, in which each node performs an action following a specific strategy with incomplete information. Use $a_{i,t}$ to denote the action of node i at t, which is supposed to carry the fusion information. Node i is granted a quadratic utility as

$$u_{i,t}(a_{i,t}, a_{j,t}, \theta_t) = -\frac{1}{2}a^2_{i,t} + \sum_{j\in\mathcal{V}\backslash i} \beta_{ij,t}a_{i,t}a_{j,t} + \delta a_{i,t}\theta_t, \tag{3}$$

where $\beta_{ij,t} \in \mathbb{R}$ evaluates the reciprocal effect between i and $j, j \in \mathcal{V}\backslash i$ while $\delta \in \mathbb{R}$ measures the influence of θ_t. Node i is only allowed to communicate with its neighbors, so $\beta_{ij,t} = 0, \forall j \notin \mathcal{N}(i)$ where $\mathcal{N}(i)$ represents the neighbor set of i. The quadratic utility ensures the best solution is exclusive as long as existed, which is perfectly suitable here since an intelligent node always prefers an unique action for the highest reward. The quadratic utility has been widely used in applications like cognitive radio [19], [20] and resource allocation [21]. The equilibria analysis of the game can refer to [22]–[24].

However, solving (3) requires complete knowledge of $\{a_{j,t}\}_{j\in\mathcal{V}\backslash i}$ and θ_t, which is infeasible with a noisy observation model in a network. Node i has to reason about them by history at t, i.e., the private observation $s_{i,t}$ and the strategies of neighbors from t-1. That is why we need the Bayesian network game. Let $\varphi_{i,t}$ be the strategy of i in the BQNG, which explains a mapping from the history $h_{i,t}$ to $a_{i,t}$

$$\varphi_{i,t} : h_{i,t} \mapsto a_{i,t}. \tag{4}$$

Fig. 1 illustrates $h_{i,t}$ of node i, which can be expressed as

$$h_{i,t} = \{h_{i,t\text{-}1}, s_{i,t}, \{\varphi_{j,t\text{-}1}\}_{j\in\mathcal{N}(i)}\}. \tag{5}$$

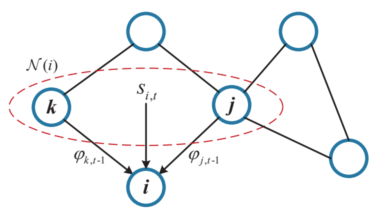


Fig. 1. History of Node i.

Use $\mathbb{E}_{i,t}[\cdot]$ as a compact notation of i's conditional belief on $h_{i,t}$. Take the expectation to (3) and

$$\begin{aligned} &u_{i,t}(a_{i,t}, \mathbb{E}_{i,t}[a_{j,t}], \mathbb{E}_{i,t}[\theta_t]) \\ &= -\frac{1}{2}a_{i,t}^2 + \sum_{j\in\mathcal{V}\backslash i} \beta_{ij,t} a_{i,t}\mathbb{E}_{i,t}[a_{j,t}] + \delta a_{i,t}\mathbb{E}_{i,t}[\theta_t], \end{aligned} \tag{6}$$

which is strictly concave with respect to $a_{i,t}$. The optimal action $a_{i,t}^*$ can be obtained by maximizing utility in (6) as

$$a_{i,t}^* = \arg\max_{a_{i,t}\in\mathbb{R}} u_{i,t}(a_{i,t}, \mathbb{E}_{i,t}[a_{j,t}], \mathbb{E}_{i,t}[\theta_t]). \tag{7}$$

The whole network may reach an equilibrium provided that all nodes play by their optimal actions.

In BQNG, node i needs to build up beliefs $\mathbb{E}_{i,t}[\mathbf{s}_t]$ and $\mathbb{E}_{i,t}[\theta_t]$ on the unknown parameters $\mathbf{s}_t$ and θ_t, where $\mathbf{s}_t = [s_{1,t}, s_{2,t}..., s_{N,t}]^{\mathrm{T}}$ is the global observing vector. Let $\tilde{}$ denote prior beliefs and $\hat{}$ denote posterior beliefs. Suppose the prior beliefs $\tilde{\mathbb{E}}_{i,t}[\mathbf{s}_t]$ and $\tilde{\mathbb{E}}_{i,t}[\theta_t]$ are linear Gaussian models, expressed as

$$\begin{aligned} \tilde{\mathbb{E}}_{i,t}[\mathbf{s}_t] &\sim \mathcal{N}(\tilde{L}_{i,t}\mathbf{s}_t, \tilde{M}_{ss,i,t}), \\ \tilde{\mathbb{E}}_{i,t}[\theta_t] &\sim \mathcal{N}(\tilde{k}_{i,t}^{\mathrm{T}}\mathbf{s}_t, \tilde{M}_{\theta\theta,i,t}), \end{aligned} \tag{8}$$

where $\tilde{L}_{i,t} \in \mathbb{R}^{N\times N}$, $\tilde{k}_{i,t} \in \mathbb{R}^N$, $\tilde{M}_{ss,i,t} \in \mathbb{R}^{N\times N}$ and $\tilde{M}_{\theta\theta,i,t} \in \mathbb{R}^{N\times N}$. Proved in [18], the solution to (7) in linear Gaussian model is of a form as

$$a_{i,t} = v_{i,t}\tilde{\mathbb{E}}_{i,t}[\mathbf{s}_t], \tag{9}$$

where $v_{i,t} \in \mathbb{R}^N$ is the crucial to the best strategy $\varphi_{i,t}$.

III. Algorithm

To take full advantage of all available information, the BQNG KF is designed in two layers as Fig. 2.

The filter layer updates estimation based on available information, including $s_{i,t}$ as private observation and $a_{i,t}$ as fusion information. The game layer accomplishes the information

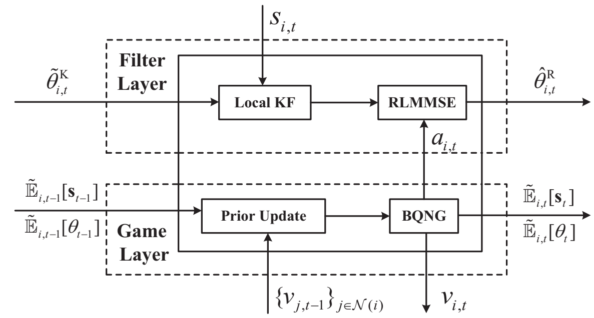


Fig. 2. Structure of the BQNG KF.

fusion among nodes by the Bayesian network game and generates $a_{i,t}$ as one-way feedback to the filter layer.

In our algorithm, all the nodes observe and play alternatingly at the same rate, which suggests a big difference comparing with the model in [18].

A. Filter Layer

This layer engages private observation $s_{i,t}$ and fusion information $a_{i,t}$ together to track the unknown state θ_t. The parameter structure is illustrated in Fig. 3.

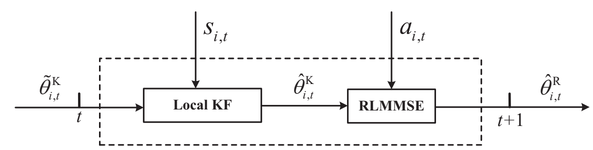


Fig. 3. Structure of the filter layer.

For θ_t in (1), the Local KF deals with local update using $s_{i,t}$ with two steps, the KF prediction and the KF update. Use $\breve{\theta}_{i,t}^{\mathrm{K}}$ to denote the prediction on θ_t based on $\tilde{\theta}_{i,t}^{\mathrm{K}}$, which carries information from last moment that $\tilde{\theta}_{i,t}^{\mathrm{K}} = \hat{\theta}_{i,t\text{-}1}^{\mathrm{R}}$. The KF prediction is

$$\breve{\theta}_{i,t}^{\mathrm{K}} = f_t\tilde{\theta}_{i,t}^{\mathrm{K}}, \tag{10}$$

$$\breve{Q}_{i,t}^{\mathrm{K}} = f_t\hat{Q}_{i,t\text{-}1}^{\mathrm{R}}f_t^{\mathrm{T}} + \sigma_{\theta,t}^2. \tag{11}$$

The KF update is equivalent to be a RLMMSE that

$$\hat{\theta}_{i,t}^{\mathrm{K}} = \breve{\theta}_{i,t}^{\mathrm{K}} + K_{i,t}^{\mathrm{K}}(s_{i,t} - \breve{\theta}_{i,t}^{\mathrm{K}}), \tag{12}$$

$$\hat{Q}_{i,t}^{\mathrm{K}} = (I - K_{i,t}^{\mathrm{K}}h_{i,t}^{\mathrm{T}})\breve{Q}_{i,t}^{\mathrm{K}}, \tag{13}$$

where $K_{i,t}^{\mathrm{K}} = \breve{Q}_{i,t}^{\mathrm{K}}h_{i,t}(\sigma_{i,t}^2 + h_{i,t}^{\mathrm{T}}\breve{Q}_{i,t}^{\mathrm{K}}h_{i,t})$ is the KF gain matrix.

Afterwards, the filter layer implements a RLMMSE to do update from $\hat{\theta}_{i,t}^{\mathrm{K}}$ to $\hat{\theta}_{i,t}^{\mathrm{R}}$ using $a_{i,t}$. By the analysis in [5], $a_{i,t}$ has the following form

$$a_{i,t} = H_{i,t}^{\mathrm{G}}\theta_t + n_{i,t}^{\mathrm{G}}, \tag{14}$$

where $n_{i,t}^{\mathrm{G}} \sim \mathcal{N}(0, (\sigma_{i,t}^{\mathrm{G}})^2)$. Equation (14) can be regarded as a linear measure on θ_t and the update follows

$$\hat{\theta}_{i,t}^{\mathrm{R}} = \hat{\theta}_{i,t}^{\mathrm{K}} + K_{i,t}^{\mathrm{R}}(a_{i,t} - H_{i,t}^{\mathrm{G}}\hat{\theta}_{i,t}^{\mathrm{K}}), \tag{15}$$

$$\hat{Q}_{i,t}^{\mathrm{R}} = (I - K_{i,t}^{\mathrm{R}}(H_{i,t}^{\mathrm{G}})^{\mathrm{T}})\hat{Q}_{i,t}^{\mathrm{K}}, \tag{16}$$

where $K_{i,t}^{\mathrm{R}} = \hat{Q}_{i,t}^{\mathrm{K}}H_{i,t}^{\mathrm{G}}((\sigma_{i,t}^{\mathrm{G}})^2 + (H_{i,t}^{\mathrm{G}})^{\mathrm{T}}\hat{Q}_{i,t}^{\mathrm{K}}H_{i,t}^{\mathrm{G}})$.

The estimation remains unbiased provided that the prior estimation $\tilde{\theta}_{i,0}^{\mathrm{K}}$ is unbiased since the KF and the RLMMSE are both unbiased filters.

B. Game Layer

This layer calculates the fusion result $a_{i,t}$ by BQNG and constructs a linear Gaussian measure function in the form of (14). Please refer to [18] to find the basic theory of the BQNG in detail. Briefly speaking, each node of the network builds estimators in (8) locally and acts in the form of (9) to maximize self utility in (6) through a BQNG. Fig. 4 diagrams the parameter setting of the game layer.

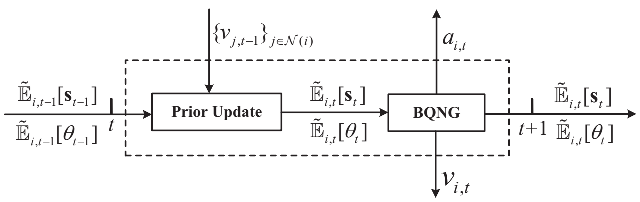


Fig. 4. Structure of the game layer.

The Prior Update renewals beliefs $\tilde{\mathbb{E}}_{i,t}[\mathbf{s}_t]$ and $\tilde{\mathbb{E}}_{i,t}[\theta_t]$ by reasoning from t-1 using $\{v_{j,t\text{-}1}\}_{j\in\mathcal{N}(i)}$. These beliefs are used for calculating the action in the BQNG as illustrated in Fig. 4. The Prior Update remains the structure of the beliefs as (8), which is proved in **Theorem1**. This structure is crucial to solve the BQNG analytically, as indicated later in (27).

Equation (8) and (9) indicates $a_{i,t} = v_{i,t}\tilde{L}_{i,t}\mathbf{s}_t$. Since $\tilde{L}_{i,t}$ is local belief and $\mathbf{s}_t$ is not available as a global vector, the real strategy message that could be transmitted among nodes is $v_{i,t}$, which means $\{\varphi_{j,t\text{-}1}\}_{j\in\mathcal{N}(i)} = \{v_{j,t\text{-}1}\}_{j\in\mathcal{N}(i)}$.

By *Lemma 2* in [18], we construct observing matrix $\mathbf{H}^{\mathrm{S}}_{i,t} = [L^{\mathrm{T}}_{j_1,t\text{-}1}v_{j_1,t\text{-}1},...,L^{\mathrm{T}}_{j_{d_i},t\text{-}1}v_{j_{d_i},t\text{-}1}] \in \mathbb{R}^{N\times d_i}$ and the update of the posterior beliefs on $\mathbf{s}_{t\text{-}1}$ and $\theta_{t\text{-}1}$ acts as

$$\begin{array}{rcl} G_{\theta,i,t} &=& \tilde{M}_{\theta s,i,t\text{-}1}\mathbf{H}^{\mathrm{S}}_{i,t}((\mathbf{H}^{\mathrm{S}}_{i,t})^{\mathrm{T}}\tilde{M}_{ss,i,t\text{-}1}\mathbf{H}^{\mathrm{S}}_{i,t})^{\text{-}1}, \\ G_{\mathbf{s},i,t} &=& \tilde{M}_{ss,i,t\text{-}1}\mathbf{H}^{\mathrm{S}}_{i,t}((\mathbf{H}^{\mathrm{S}}_{i,t})^{\mathrm{T}}\tilde{M}_{ss,i,t\text{-}1}\mathbf{H}^{\mathrm{S}}_{i,t})^{\text{-}1}, \\ \hat{k}^{\mathrm{T}}_{i,t\text{-}1} &=& \tilde{k}^{\mathrm{T}}_{i,t\text{-}1} + G_{\theta,i,t}((\mathbf{H}^{\mathrm{S}}_{i,t})^{\mathrm{T}} - (\mathbf{H}^{\mathrm{S}}_{i,t})^{\mathrm{T}}\tilde{k}_{i,t\text{-}1}), \\ \hat{L}_{i,t\text{-}1} &=& \tilde{L}_{i,t\text{-}1} + G_{\theta,i,t}((\mathbf{H}^{\mathrm{S}}_{i,t})^{\mathrm{T}} - (\mathbf{H}^{\mathrm{S}}_{i,t})^{\mathrm{T}}\tilde{L}_{i,t\text{-}1}), \\ \hat{M}_{\theta\theta,i,t\text{-}1} &=& \tilde{M}_{\theta\theta,i,t\text{-}1} - G_{\theta,i,t}(\mathbf{H}^{\mathrm{S}}_{i,t})^{\mathrm{T}}\tilde{M}_{s\theta,i,t\text{-}1}, \\ \hat{M}_{ss,i,t\text{-}1} &=& \tilde{M}_{ss,i,t\text{-}1} - G_{s,i,t}(\mathbf{H}^{\mathrm{S}}_{i,t})^{\mathrm{T}}\tilde{M}_{ss,i,t\text{-}1}, \\ \hat{M}_{\theta s,i,t\text{-}1} &=& \tilde{M}_{\theta s,i,t\text{-}1} - G_{\theta,i,t}(\mathbf{H}^{\mathrm{S}}_{i,t})^{\mathrm{T}}\tilde{M}_{ss,i,t\text{-}1}. \end{array} \tag{17}$$

Then we get the posterior beliefs on $\mathbf{s}_{t\text{-}1}$ and $\theta_{t\text{-}1}$

$$\begin{array}{l} \hat{\mathbb{E}}_{i,t}[\mathbf{s}_{t\text{-}1}] \sim \mathcal{N}(\hat{L}_{i,t\text{-}1}\mathbf{s}_{t\text{-}1}, \hat{M}_{ss,i,t\text{-}1}), \\ \hat{\mathbb{E}}_{i,t}[\theta_{t\text{-}1}] \sim \mathcal{N}(\hat{k}^{T}_{i,t\text{-}1}\mathbf{s}_{t\text{-}1}, \hat{M}_{\theta\theta,i,t\text{-}1}). \end{array} \tag{18}$$

Equation (1) indicates the relationship between θ_t and $\theta_{t\text{-}1}$ as

$$\theta_{t\text{-}1} = f_t^{\text{-}1}\theta_t + f_t^{\text{-}1}w_{\theta,t}, \tag{19}$$

which implies an unbiased backward inference from t to t-1. The global observing vector $\mathbf{s}_t$ has the form

$$\mathbf{s}_t = \mathbf{H}_t\mathbf{1}\theta_t + \mathbf{n}_t, \tag{20}$$

where $\mathbf{H}_t$ is a diagonal matrix with $\mathbf{H}_t(i,i) = h_{i,t}$ and $\mathbf{n}_t \sim \mathcal{N}(\mathbf{0}, \Sigma_t^2)$ with $\Sigma_t^2(i,i) = \sigma_{i,t}^2$ and 0 for else. Substitute (19) into $\mathbf{s}_{t\text{-}1} = \mathbf{H}_{t\text{-}1}\mathbf{1}\theta_{t\text{-}1} + \mathbf{n}_{t\text{-}1}$ we get

$$\mathbf{s}_{t\text{-}1} = \mathbf{H}_{t\text{-}1}\mathbf{1}(f_t^{\text{-}1}\theta_t + f_t^{\text{-}1}w_{\theta,t}) + \mathbf{n}_{t\text{-}1}. \tag{21}$$

Then implement elimination to (20) and (21)

$$\mathbf{s}_t = \mathbf{H}_t f_t \mathbf{H}^{\text{-}1}_{t\text{-}1}\mathbf{s}_{t\text{-}1} + \mathbf{H}_t\mathbf{1}w_{\theta,t} + \mathbf{H}_t f_t\mathbf{H}^{\text{-}1}_{t\text{-}1}\mathbf{n}_{t\text{-}1} + \mathbf{n}_t, \tag{22}$$

which is the transition from $\mathbf{s}_{t\text{-}1}$ to $\mathbf{s}_t$. Equation (22) declares that $\mathbf{s}_t$ could be regarded as a linear observation of $\mathbf{s}_{t\text{-}1}$ added with a zero-mean Gaussian noise. Define $\mathbf{H}_{\mathbf{s},t}$ and $\mathbf{w}_{\mathbf{s},t}$ for simplicity and

$$\mathbf{s}_t = \mathbf{H}_{\mathbf{s},t}\mathbf{s}_{t\text{-}1} + \mathbf{w}_{\mathbf{s},t}, \tag{23}$$

where the observing matrix $\mathbf{H}_{\mathbf{s},t} = \mathbf{H}_t f_t \mathbf{H}^{\text{-}1}_{t\text{-}1}$ and the noise $\mathbf{w}_{\mathbf{s},t} = \mathbf{H}_t\mathbf{1}w_{\theta,t} + \mathbf{H}_t f_t\mathbf{H}^{\text{-}1}_{t\text{-}1}\mathbf{n}_{t\text{-}1} + \mathbf{n}_t \sim \mathcal{N}(\mathbf{0}, \Sigma^2_{\mathbf{s},t})$. From (23) $\mathbf{s}_{t\text{-}1}$ can be expressed as

$$\mathbf{s}_{t\text{-}1} = \mathbf{H}^{\text{-}1}_{\mathbf{s},t}\mathbf{s}_t + \mathbf{H}^{\text{-}1}_{\mathbf{s},t}\mathbf{w}_{\mathbf{s},t}. \tag{24}$$

Substitute this into (18) and

$$\begin{array}{l} \tilde{\mathbb{E}}_{i,t}[\mathbf{s}_t] \sim \mathcal{N}(\tilde{L}_{i,t}\mathbf{s}_t, \tilde{M}_{ss,i,t}), \\ \tilde{\mathbb{E}}_{i,t}[\theta_t] \sim \mathcal{N}(\tilde{k}^{\mathrm{T}}_{i,t}\mathbf{s}_t, \tilde{M}_{\theta\theta,i,t}), \end{array} \tag{25}$$

where $\tilde{L}_{i,t} = \hat{L}_{i,t\text{-}1}\mathbf{H}^{\text{-}1}_{\mathbf{s},t}$, $\tilde{k}^{\mathrm{T}}_{i,t} = \hat{k}^{\mathrm{T}}_{i,t\text{-}1}\mathbf{H}^{\text{-}1}_{\mathbf{s},t}$, $\tilde{M}_{ss,i,t} = \hat{M}_{ss,i,t\text{-}1} + \hat{L}_{i,t\text{-}1}\mathbf{H}^{\text{-}1}_{\mathbf{s},t}\Sigma^2_{\mathbf{s},t}\hat{L}^{\mathrm{T}}_{i,t\text{-}1}$ and $\tilde{M}_{\theta\theta,i,t} = \hat{M}_{\theta\theta,i,t\text{-}1} + \hat{k}^{\mathrm{T}}_{i,t\text{-}1}\mathbf{H}^{\text{-}1}_{\mathbf{s},t}\Sigma^2_{\mathbf{s},t}\hat{k}_{i,t\text{-}1}$. The Prior Update is done with the structure in (8) holds.

As for the BQNG, node i can use beliefs in (25) to calculate its best action $a_{i,t}$ as a feedback to the filter layer and output its best strategy message $v_{i,t}$ to its neighbors for the Prior Update in the next iteration. By *Lemma* 1 in [18], we construct matrix $\mathbf{L}_t \in \mathbb{R}^{N^2\times N^2}$, vector $\mathbf{k}_t \in \mathbb{R}^{N^2}$ and $\mathbf{v}_t \in \mathbb{R}^{N^2}$ as below

$$\left\{\begin{array}{rcl} \mathbf{L}_t &=& \begin{bmatrix} \tilde{L}^{\mathrm{T}}_{1,t} & \text{-}\beta_{12,t}\tilde{L}^{\mathrm{T}}_{1,t}\tilde{L}^{\mathrm{T}}_{2,t} & \cdots & \text{-}\beta_{1N,t}\tilde{L}^{\mathrm{T}}_{1,t}\tilde{L}^{\mathrm{T}}_{N,t} \\ \text{-}\beta_{21,t}\tilde{L}^{\mathrm{T}}_{2,t}\tilde{L}^{\mathrm{T}}_{1,t} & \tilde{L}^{\mathrm{T}}_{2,t} & \cdots & \text{-}\beta_{2N,t}\tilde{L}^{\mathrm{T}}_{2,t}\tilde{L}^{\mathrm{T}}_{N,t} \\ \vdots & \cdots & \ddots & \vdots \\ \text{-}\beta_{N1,t}\tilde{L}^{\mathrm{T}}_{N,t}\tilde{L}^{\mathrm{T}}_{1,t} & \text{-}\beta_{N2,t}\tilde{L}^{\mathrm{T}}_{N,t}\tilde{L}^{\mathrm{T}}_{2,t} & \cdots & \tilde{L}^{\mathrm{T}}_{N,t} \end{bmatrix}, \\ \mathbf{k}_t &=& [\tilde{k}^{\mathrm{T}}_{1,t}, \tilde{k}^{\mathrm{T}}_{2,t}, \ldots, \tilde{k}^{\mathrm{T}}_{N,t}]^{\mathrm{T}}, \\ \mathbf{v}_t &=& [v^{\mathrm{T}}_{1,t}, v^{\mathrm{T}}_{2,t}, \ldots, v^{\mathrm{T}}_{N,t}]^{\mathrm{T}}. \end{array}\right. \tag{26}$$

Node i can locally calculate strategy coefficients $\mathbf{v}_t$ by solving the following equilibrium equations

$$\mathbf{L}_t\mathbf{v}_t = \delta\mathbf{k}_t. \tag{27}$$

The action of node i should be

$$a_{i,t} = v_{i,t}\tilde{L}_{i,t}\mathbf{s}_t. \tag{28}$$

In fact, the aim of this game is to figure out the strategy $v_{i,t}$ *for information fusion. The action of node* i *depends on its strategy and its cognition on* $\mathbf{s}_t$*, which may be different among nodes*. We just need to determine a cognitive manner on $\mathbf{s}_t$ to calculate an attainable $a_{i,t}$ for the filter layer. For example, we can simply assume that the cognition on $\mathbf{s}_t$ to be $\mathbf{1}s_{i,t}$ for each node i, etc.

Theorem 1. *In the game layer, structure in* (8) *always hold with initialization*

$$\begin{array}{l} \tilde{\mathbb{E}}_{i,0}[\mathbf{s}_0] \sim \mathcal{N}(\tilde{L}_{i,0}\mathbf{s}_0, \tilde{M}_{ss,i,0}), \\ \tilde{\mathbb{E}}_{i,0}[\theta_0] \sim \mathcal{N}(\tilde{k}^{\mathrm{T}}_{i,0}\mathbf{s}_0, \tilde{M}_{\theta\theta,i,0}). \end{array} \tag{29}$$

Proof. We prove by induction. At $t = 0$, the prior estimation on $\mathbf{s}_0$ and θ_0 in (29) is linear Gaussian variables. Suppose at t-1, $\tilde{\mathbb{E}}_{i,t\text{-}1}[\mathbf{s}_{t\text{-}1}]$ and $\tilde{\mathbb{E}}_{i,t\text{-}1}[\theta_{t\text{-}1}]$ satisfy linear Gaussian assumptions, we can verify the Gaussian form of these estimators remains in t by (25). The proof is done and it enables the filter to work in a recursive way. □

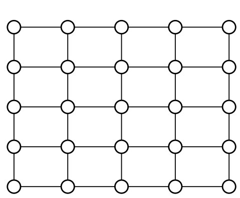
IV. Numerical Results

For specific analysis, the game needs to be initialized in a reasonable way. The utility parameters in (6) are set up as

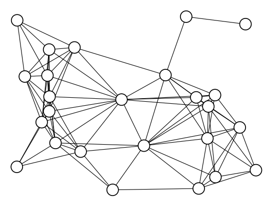
$$\sum_{j\in\mathcal{V}\setminus i} |\beta_{ij,0}| = 0.5, \delta = 0.5, \tag{30}$$

which ensures the existence of equilibrium solution according to *Remark 3* in [18]. The estimators of the game are initialized as (29), where $\tilde{L}_{i,0} = \mathbf{1}\mathbf{e}_i^{\mathrm{T}}$, $\tilde{k}_{i,0}^T = \mathbf{e}_i^{\mathrm{T}}$, $\tilde{M}_{\theta\theta,i,0} = \sigma_{i,0}^2$, $\tilde{M}_{\theta s,i,0} = \sigma_{i,0}^2\overline{\mathbf{e}_i}^{\mathrm{T}}$ and $\tilde{M}_{ss,i,0} = \sigma_{i,0}^2(\mathbf{1}\overline{\mathbf{e}_i}^{\mathrm{T}} + \overline{\mathbf{e}_i}\overline{\mathbf{e}_i}^{\mathrm{T}})$. $\mathbf{e}_i \in \mathbb{R}^n$ is the vector where 1 in i'th column and 0 otherwise, and $\overline{\mathbf{e}_i} = \mathbf{1} - \mathbf{e}_i$.

A regular network with grid structure and a random network are considered, respectively. Both networks are generated with 25 nodes as Fig. 5 showed.

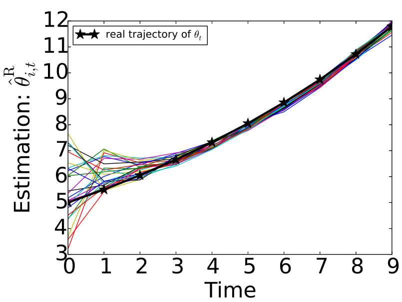


(a) The regular network.

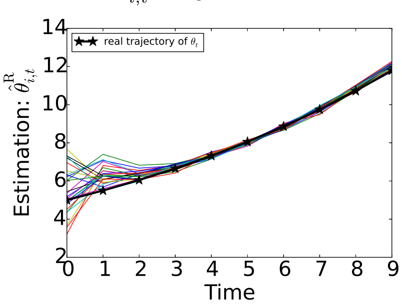


(b) The random network.

Fig. 5. The networks with 25 nodes.



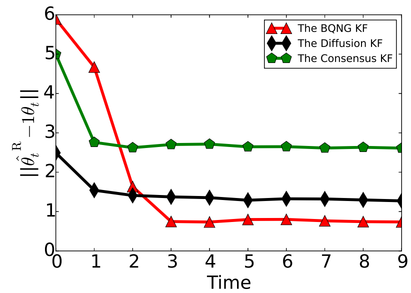
(a) $\hat{\theta}^{\mathrm{R}}_{i,t}$ in regular network.



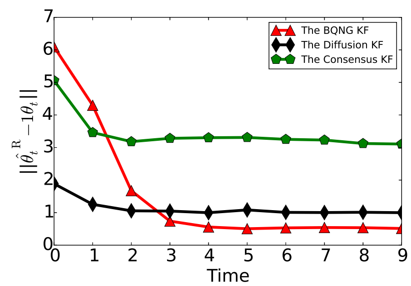
(b) $\hat{\theta}^{\mathrm{R}}_{i,t}$ in random network.

Fig. 6. The BQNG KF in networks.

At beginning, $\theta_0 = 5$, $f_t = 1.1, \sigma_{\theta,t}^2 = 0.3$. For the observing function in (2), $h_{i,t} = 1, \sigma_{i,t}^2 = 1, \forall i \in \mathcal{V}$. Plots in Fig. 6 exhibit the performance of the BQNG KF in both networks, respectively. The y-axis labels the estimation output $\hat{\theta}^{\mathrm{R}}_{i,t}$ while the x-axis labels the time slice. The Black-star-line marks the real trajectory of θ_t in both figures while the ordinary curves display the performances of all nodes in network, which could successfully track the change of θ_t with relatively high accuracy after a few iterations.



(a) $\| \hat{\theta}^{\mathrm{R}}_t - \mathbf{1}\theta_t \|$ in regular network.



(b) $\| \hat{\theta}^{\mathrm{R}}_t - \mathbf{1}\theta_t \|$ in random network.

Fig. 7. The estimation error.

To be more persuasive, we simulate the Diffusion KF [15] and the Consensus KF [6] under the same condition in both networks. Define $\hat{\theta}^{\mathrm{R}}_t = [\hat{\theta}^{\mathrm{R}}_{1,t}, ..., \hat{\theta}^{\mathrm{R}}_{N,t}]^{\mathrm{T}}$ as the estimation vector at t. We use the modulus of the estimation error vector to measure the global estimation performance of the network at t, expressed as $\| \hat{\theta}^{\mathrm{R}}_t - \mathbf{1}\theta_t \|$. Fig. 7 compares the estimation errors among the BQNG KF, the Diffusion KF and the Consensus KF in both networks. Each curve is the average of 100 sets of observations. We can find that the BQNG KF performs no worse than the others. The Diffusion KF and the Consensus KF perform better at beginning since the they use neighbor's information at first iteration while the BQNG KF does not. As time goes, the BQNG KF shows its ability to fusion the information in network more efficiently. The BQNG KF proves its ability in estimation and its robustness against diversiform topologies.

V. Conclusion

We have presented a distributed network filter integrating the KF and the BQNG together consisting of two layers. The filter layer is the skeleton that deals with estimation update while the game layer is configured as a network fusion tool that provides one-way feedback to the filter layer. The BQNG KF is a novel way to utilize the network information, and can be applied to more signal processing and decision making problems in network.

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