A NEW LOW-RANK SOLUTION RESULT FOR A SEMIDEFINITE PROGRAM PROBLEM SUBCLASS WITH APPLICATIONS TO TRANSMIT BEAMFORMING OPTIMIZATION

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ABSTRACT

This paper considers a special subclass of separable semidefinite programs (SDPs), with the goal of identifying certain conditions under which the SDP has a low-rank solution. We prove that when the data matrices of the SDP satisfy certain matrix inequalities, the SDP has a low-rank solution. Moreover, the rank of this solution is related to parts of the data matrices only, irrespective of any other factors such as the number of constraints. This is quite different from the well-known Shapiro-Barvinok-Pataki rank reduction result, where the rank of the SDP solution relies on the number of constraints. The usefulness of our result is demonstrated through advanced beamforming applications in simultaneous wireless information and power transfer (SWIPT) and physical-layer security, for which rank-one optimal solutions can be easily identified by checking our derived matrix inequality conditions.

***Index Terms*—** Quadratically constrained quadratic program, semidefinite relaxation, SWIPT, Physical-layer security

1. INTRODUCTION

Quadratically constrained quadratic program (QCQP) is an important class of optimization problems, as evidenced by its wide scope of applications in signal processing, communications and other areas [1, 2]. The QCQP problem class is nonconvex, and in fact, NP-hard in general. As a result, efficient approximations for QCQP are usually sought, and a widely used approximation technique is semidefinite relaxation (SDR) [1]. The principle of SDR is to first reformulate the QCQP as a rank-one-constrained matrix optimization problem, and then relax the problem to a convex semidefinite program (SDP) by ignoring the rank constraints. In general, SDR is not tight; that is, given an arbitrary QCQP problem instance, there may be a gap between the optimal values of the QCQP and its corresponding rank-relaxed SDP. For such instances, the SDP does not admit a rank-one solution, and some specific approximation algorithm, such as the randomization algorithm, is generally used to generate a feasible (and suboptimal) QCQP solution from the (higher-rank) SDP solution.

There are however some QCQP problem subclass where SDR is provably tight [3–5] and we can retrieve a rank-one optimal solution from the (rank-relaxed) SDP. Examples of these special QCQPs include the trust region subproblem [3] and its generalizations [4], and complex-valued QCQPs with two inequality constraints or less [5]. A more general result that covers the above results is the Shapiro-Barvinok-Pataki (SBP) rank-reduction result [6] (also [1] for a review). In essence, the SBP result studies when a low-rank solution to the SDP exists, and how such a low-rank solution can be retrieved. To be specific, the SBP result considers a real-valued SDP with m constraints (both inequality and equality), and shows that under some rather mild assumptions there exists an optimal SDP solution whose rank, denoted by r, satisfies $r(r+1)/2 \le m$. The SBP result for the complex-valued separable SDP case was also considered in [7]. Simply speaking, the implication of the SBP result and its extensions is that if the QCQP does not have too many constraints, then a rank-one or low-rank SDP solution can be obtained.

In this paper we consider a different SDP problem subclass. To describe it, let $\mathbb{H}^n$ be the set of $n \times n$ complex Hermitian matrices. Also, let $\boldsymbol{X}_1, \ldots, \boldsymbol{X}_{m+1} \in \mathbb{H}^n$ be a set of optimization variables. We consider a complex-valued separable SDP as follows:

$$\min_{\{\boldsymbol{X}_i\}} \sum_{i=1}^{m+1} \mathrm{Tr}(\boldsymbol{C}_i \boldsymbol{X}_i) \tag{1a}$$

$$\text{s.t. } \mathrm{Tr}(\boldsymbol{A}_{ii}\boldsymbol{X}_i) \ge \sum_{\substack{j=1,\\ j\neq i}}^{m+1} \mathrm{Tr}(\boldsymbol{A}_{ij}\boldsymbol{X}_j) + b_i,\ i = 1, \ldots, m \tag{1b}$$

$$\sum_{j=1}^{m+1} \mathrm{Tr}(\boldsymbol{F}_{ij}\boldsymbol{X}_j) \le d_i,\ i = 1, \ldots, p, \tag{1c}$$

$$\boldsymbol{X}_1, \ldots, \boldsymbol{X}_{m+1} \succeq \boldsymbol{0}, \tag{1d}$$

where $\boldsymbol{A}_{ij}, \boldsymbol{C}_i, \boldsymbol{F}_{ij} \in \mathbb{H}^n$ for all i, j, and $b_i, d_i \in \mathbb{R}$ for all i.

Following the spirit of SBP rank reduction, our aim is to prove when problem (1) has a low-rank solution with respect to (w.r.t.) $\boldsymbol{X}_1, \ldots, \boldsymbol{X}_m$. Note that we do not care about the rank of $\boldsymbol{X}_{m+1}$; the reason is application-driven and will become clear as we discuss the applications later. Our main result can roughly be summarized as follows: *When the* $\{\boldsymbol{A}_{ij}, \boldsymbol{F}_{ij}, \boldsymbol{C}_i\}$ *satisfies certain matrix inequality relationships, there exists a low-rank solution* $(\boldsymbol{X}_1, \ldots, \boldsymbol{X}_m)$ *to the SDP in* (1) *and there is an efficient way to compute such a solution.* Notice that unlike the SBP rank-reduction result, our result does not depend on the number of constraints. The detailed descriptions of the result and its proof will be provided in the next section. The applications to transmit beamforming optimization will also be discussed.

2. MAIN RESULT

To proceed, let $f^{\star}$ denote the optimal objective value of problem (1). Instead of dealing with problem (1) directly, we consider the follow-

This work was supported in part by the National Natural Science Foundation of China under Grant 61401073, and in part by a Direct Grant of The Chinese University of Hong Kong (Project ID: 4055009).

ing problem:

$$\min_{\{\boldsymbol{X}_i\}} \sum_{i=1}^{m} \mathrm{Tr}(\boldsymbol{X}_i) \tag{2a}$$

$$\text{s.t. } \mathrm{Tr}(\boldsymbol{A}_{ii}\boldsymbol{X}_i) \geq \sum_{j\neq i} \mathrm{Tr}(\boldsymbol{A}_{ij}\boldsymbol{X}_j) + b_i, i = 1,\ldots,m, \tag{2b}$$

$$\sum_{j=1}^{m+1} \mathrm{Tr}(\boldsymbol{F}_{ij}\boldsymbol{X}_j) \leq d_i,\ i = 1,\ldots,p, \tag{2c}$$

$$\sum_{i=1}^{m+1} \mathrm{Tr}(\boldsymbol{C}_i\boldsymbol{X}_i) \leq f^\star, \tag{2d}$$

$$\boldsymbol{X}_1,\ldots,\boldsymbol{X}_{m+1} \succeq \boldsymbol{0}. \tag{2e}$$

In particular, note the constraint in (2d). Under this constraint, any feasible solution of problem (2) is also an optimal solution to problem (1). Thus, the optimal solution to problem (2) must be optimal to problem (1), too. Our interest now lies in studying sufficient conditions under which problem (2) admits rank-constrained solutions.

Theorem 1 *Consider problem* (2) *under the following conditions:*

i) $\boldsymbol{C}_i \succeq \boldsymbol{C}_{m+1}$ *for all* $i \in \{1,\ldots,m\}$*;*

ii) $\boldsymbol{A}_{j,i} \succeq \boldsymbol{A}_{j,m+1}$ *for all* $i, j \in \{1,\ldots,m\}$*,* $i \neq j$*;*

iii) $\boldsymbol{F}_{j,i} \succeq \boldsymbol{F}_{j,m+1}$ *for all* $i \in \{1,\ldots,m\}$*,* $j \in \{1,\ldots,p\}$*;*

iv) $\boldsymbol{A}_{ii} + \boldsymbol{A}_{i,m+1} \succeq \boldsymbol{0}$ *and* $\mathrm{rank}(\boldsymbol{A}_{ii} + \boldsymbol{A}_{i,m+1}) = r_i$*, for all* $i \in \{1,\ldots,m\}$*.*

Also, suppose that problem (2) *and its dual both have optimal solutions, and that problem* (2) *attains zero duality gap. Then, any optimal solution* $(\boldsymbol{X}_1^\star,\ldots,\boldsymbol{X}_{m+1}^\star)$ *to problem* (2) *must satisfy*

$$\mathrm{rank}(\boldsymbol{X}_i^\star) \leq r_i,\ i = 1,\ldots,m.$$

Proof of Theorem 1: For notational convenience, in this proof we simply denote $(\boldsymbol{X}_1,\ldots,\boldsymbol{X}_{m+1})$ as the optimal solution to problem (2). Since strong duality holds and problem (2) and its dual problem have optimal solutions, any optimal solution to problem (2) must satisfy the Karush-Kuhn-Tucker (KKT) conditions. Let us write down some of the KKT conditions that will lead to the desired result (the proof of the KKT conditions is omitted for conciseness):

$$\boldsymbol{Z}_i = \boldsymbol{I} + \zeta\boldsymbol{C}_i + \textstyle\sum_{\substack{j=1,\\ j\neq i}}^{m} \lambda_j\boldsymbol{A}_{ji} + \sum_{j=1}^{p} \nu_j\boldsymbol{F}_{ji} - \lambda_i\boldsymbol{A}_{ii}, \quad i = 1,\ldots,m, \tag{3a}$$

$$\boldsymbol{Z}_{m+1} = \zeta\boldsymbol{C}_{m+1} + \textstyle\sum_{j=1}^{m} \lambda_j\boldsymbol{A}_{j,m+1} + \sum_{j=1}^{p} \nu_j\boldsymbol{F}_{j,m+1}, \tag{3b}$$

$$\boldsymbol{X}_i\boldsymbol{Z}_i = \boldsymbol{0},\ i = 1,\ldots,m+1, \tag{3c}$$

$$\boldsymbol{X}_i \succeq \boldsymbol{0},\ \boldsymbol{Z}_i \succeq \boldsymbol{0},\ i = 1,\ldots,m+1, \tag{3d}$$

$$\boldsymbol{\lambda} \succeq \boldsymbol{0},\ \ \boldsymbol{\nu} \succeq \boldsymbol{0},\ \ \zeta \geq 0, \tag{3e}$$

where $\boldsymbol{\lambda}, \boldsymbol{\nu}, \zeta$ and $\{\boldsymbol{Z}_i\}_i$ are the dual variables of the constraints in (2b), (2c), (2d) and (2e), respectively. From (3a)-(3b), we can write

$$\boldsymbol{Z}_i = \boldsymbol{\Sigma}_i - \lambda_i(\boldsymbol{A}_{ii} + \boldsymbol{A}_{i,m+1}), \tag{4}$$

for $i = 1,\ldots,m$, where

$$\boldsymbol{\Sigma}_i = \boldsymbol{I} + \zeta(\boldsymbol{C}_i - \boldsymbol{C}_{m+1}) + \textstyle\sum_{\substack{j=1,\\ j\neq i}}^{m} \lambda_j(\boldsymbol{A}_{ji} - \boldsymbol{A}_{j,m+1}) + \sum_{j=1}^{p} \nu_j(\boldsymbol{F}_{ji} - \boldsymbol{F}_{j,m+1}) + \boldsymbol{Z}_{m+1}.$$

It can be verified that under the conditions in Theorem 1, every $\boldsymbol{\Sigma}_i$ is positive definite. Subsequently, we can pre-multiply both sides of (4) to yield

$$\boldsymbol{X}_i\boldsymbol{\Sigma}_i = \lambda_i\boldsymbol{X}_i(\boldsymbol{A}_{ii} + \boldsymbol{A}_{i,m+1}). \tag{5}$$

Since $\boldsymbol{\Sigma}_i$ is positive definite, we have $\mathrm{rank}(\boldsymbol{X}_i) = \mathrm{rank}(\boldsymbol{X}_i\boldsymbol{\Sigma}_i)$, which together with (5) implies $\mathrm{rank}(\boldsymbol{X}_i) = \mathrm{rank}(\lambda_i\boldsymbol{X}_i(\boldsymbol{A}_{ii} + \boldsymbol{A}_{i,m+1})) \leq \mathrm{rank}(\boldsymbol{A}_{ii} + \boldsymbol{A}_{i,m+1}) = r_i$. Thus, we obtain the desired result in Theorem 1. ■

Theorem 1 is vital in identifying an SDP problem subclass for which the solution ranks can be small. It should be noted that the low-rank result in Theorem 1 requires some special structures with the data matrices $\{\boldsymbol{A}_{ij}, \boldsymbol{F}_{ij}, \boldsymbol{C}_i\}$, but does not depend on other factors such as the number of constraints p in (2c); this is unlike the SBP rank-reduction result and its extensions [6,7], in which the rank conditions on $\boldsymbol{X}_i$ depend on the total number of constraints. Theorem 1 can be further interpreted as a rank-one solution result as follows.

Corollary 1 *Consider problem* (1) *under the conditions in Theorem 1, with the following two extra conditions:* $r_i = 1$ *for all* i*, and that any optimal solution* $(\boldsymbol{X}_1^\star,\ldots,\boldsymbol{X}_{m+1}^\star)$ *to problem* (1) *has* $\boldsymbol{X}_i^\star \neq \boldsymbol{0}$ *for* $i = 1,\ldots,m$*. Then, there exists an optimal solution to problem* (1) *whose ranks satisfy*

$$\mathrm{rank}(\boldsymbol{X}_i^\star) = 1,\ i = 1,\ldots,m.$$

Such an optimal solution can be obtained by solving two SDPs, namely, by first solving problem (1) *to obtain* $f^\star$*, and then solving problem* (2) *to obtain* $(\boldsymbol{X}_1^\star,\ldots,\boldsymbol{X}_{m+1}^\star)$*.*

As will be discussed in the next section, there are a number of transmit beamforming problems that fall within the scope of Corollary 1.

Remark 1 We can make Theorem 1 slightly more general by replacing condition iv) in Theorem 1 with another condition. Specifically, let $\boldsymbol{A}_{i,m+1} = \boldsymbol{A}_{i,m+1}^+ - \boldsymbol{A}_{i,m+1}^-$ for some $\boldsymbol{A}_{i,m+1}^+, \boldsymbol{A}_{i,m+1}^- \succeq \boldsymbol{0}$. It can be shown that the same result in Theorem 1 still holds if we replace condition iv) with $\boldsymbol{A}_{ii} + \boldsymbol{A}_{i,m+1}^+ \succeq \boldsymbol{0}$ and $\mathrm{rank}(\boldsymbol{A}_{ii} + \boldsymbol{A}_{i,m+1}^+) = r_i$, for $i = 1,\ldots,m$.

3. APPLICATIONS

In this section, we demonstrate the usefulness of Theorem 1 by showcasing several applications that falls into the form of problem (1).

3.1. Unicast Beamforming with Per-Antenna Power Constraints

Let us start with a simple example, namely, standard multiuser MISO downlink unicast beamforming. In this scenario, a multi-antenna base station (BS) sends K independent messages to K single-antenna users by transmit beamforming. Let n be the number of transmit antennas, and $\boldsymbol{h}_k \in \mathbb{C}^n$ and $\boldsymbol{w}_k \in \mathbb{C}^n$ be the channel vector and beamformer of user k, $k = 1,\ldots,K$, respectively. Also, denote

$$\mathsf{SINR}_k(\{\boldsymbol{W}_l\}_l) = \frac{\boldsymbol{h}_k^H\boldsymbol{W}_k\boldsymbol{h}_k}{\sigma_k^2 + \sum_{i\neq k} \boldsymbol{h}_k^H\boldsymbol{W}_i\boldsymbol{h}_k} \tag{6}$$

to be the user-k's signal-to-interference-and-noise ratio (SINR) for a given transmit covariance set $\{\boldsymbol{W}_l\}_l$, where σ_k^2 is the noise variance at user k. Our problem of interest is a beamforming optimization problem with per-antenna power constraints (PAPCs):

$$\begin{aligned} \min_{\{\boldsymbol{w}_k\}_k} \quad & \textstyle\sum_{k=1}^K \|\boldsymbol{w}_k\|^2 \\ \text{s.t.} \quad & \mathsf{SINR}_k(\{\boldsymbol{w}_l \boldsymbol{w}_l^H\}_l) \geq \gamma_k,\ k=1,\ldots,K, \\ & \left[\textstyle\sum_{k=1}^K \boldsymbol{w}_k \boldsymbol{w}_k^H\right]_{ii} \leq P_i,\ i=1,\ldots,n, \end{aligned} \tag{7}$$

where $\gamma_k > 0$ is the SINR threshold for user k, and $P_i > 0$ is the peak power constraint at antenna i; see the literature such as [8] for descriptions on why PAPCs may be preferred in practice. By denoting $\boldsymbol{W}_k = \boldsymbol{w}_k \boldsymbol{w}_k^H$ and dropping the rank-one constraint with $\boldsymbol{W}_k$, we obtain the SDR of (7) as follows:

$$\min_{\{\boldsymbol{W}_k\}_k} \quad \textstyle\sum_{k=1}^K \mathrm{Tr}(\boldsymbol{W}_k) \tag{8a}$$

$$\text{s.t.} \quad \mathsf{SINR}_k(\{\boldsymbol{W}_l\}_l) \geq \gamma_k,\ k=1,\ldots,K, \tag{8b}$$

$$\left[\textstyle\sum_{k=1}^K \boldsymbol{W}_k\right]_{ii} \leq P_i,\ i=1,\ldots,n, \tag{8c}$$

$$\boldsymbol{W}_k \succeq \boldsymbol{0},\ k=1,\ldots,K. \tag{8d}$$

The above SDR can be shown to be tight using our low-rank SDP result in Corollary 1. To show this, observe that problem (8) can be equivalently written as

$$\min_{\{\boldsymbol{W}_k\}_{k=1}^{K+1}} \quad \textstyle\sum_{k=1}^K \mathrm{Tr}(\boldsymbol{W}_k) + 0.5\mathrm{Tr}(\boldsymbol{W}_{K+1}) \tag{9a}$$

$$\text{s.t. (8b)} - \text{(8d), and } \boldsymbol{W}_{K+1} \succeq \boldsymbol{0} \tag{9b}$$

where $\boldsymbol{W}_{K+1}$ is a redundant variable. In particular, it is easy to verify that an optimal solution to Problem (9) must have $\boldsymbol{W}_{K+1} = \boldsymbol{0}$. Problem (9) is an instance of problem (1) where $m = K$, $\boldsymbol{X}_i = \boldsymbol{W}_i, i = 1,\ldots,K+1, p = n$,

$$\begin{aligned} & \boldsymbol{C}_i = \boldsymbol{I},\ i=1,\ldots,m, \quad \boldsymbol{C}_{K+1} = 0.5\boldsymbol{I}, \\ & \boldsymbol{A}_{ii} = \tfrac{1}{\gamma_i}\boldsymbol{h}_i\boldsymbol{h}_i^H,\ \boldsymbol{A}_{ij} = \boldsymbol{h}_i\boldsymbol{h}_i^H, j=1,\ldots,m, j\neq i, \\ & \boldsymbol{A}_{i,m+1} = \boldsymbol{0},\ b_i = \sigma_i^2, \\ & \boldsymbol{F}_{ij} = \boldsymbol{e}_i\boldsymbol{e}_i^T,\ j=1,\ldots,m,\ i=1,\ldots,p, \\ & \boldsymbol{F}_{i,m+1} = \boldsymbol{0},\ d_i = P_i,\ i=1,\ldots,p; \end{aligned} \tag{10}$$

here $\boldsymbol{e}_i \in \mathbb{R}^p$ is a unit vector with the ith element being one. One can verify that (10) satisfy the conditions in Theorem 1 with $r_i = 1$. Also, owing to the SINR constraints in (8b), any feasible point of problem (9) must satisfy $\boldsymbol{W}_i \neq \boldsymbol{0}$ for $i = 1,\ldots,K$. It follows from Corollary 1 that problem (9) has a rank-one solution (under the mild assumption in Theorem 1). Hence, SDR is tight for the PAPC-constrained unicast beamforming problem.

3.2. Simultaneous Wireless Information and Power Transfer

Next, consider another scenario called simultaneous wireless information and power transfer (SWIPT). On one hand we provide the same unicast multiuser MISO dowlink beamforming as in Sec. 3.1, and on the other hand we serve additional L energy receivers (ERs) that aim at receiving energy from the BS. We adopt the energy-signal-aided transmit strategy suggested in [9], where the beamformed information is superimposed with a dedicated energy signal $\boldsymbol{z}(t) \in \mathbb{C}^n$. Herein, $\boldsymbol{z}(t)$ is randomly distributed with mean zero and covariance matrix $\boldsymbol{\Phi}$. The energy signal $\boldsymbol{z}(t)$ is assumed to be known by the information receivers (IRs), and can be canceled by the IRs. Thus, the received SINRs at the IRs after energy signal cancellation have the same form as (6). On the other hand, the received energy (normalized by unit time) at ER i may be formulated as [9]:

$$\mathsf{Q}_i(\{\boldsymbol{w}_k\boldsymbol{w}_k^H\}_k, \boldsymbol{\Phi}) = \xi_i \mathrm{Tr}\left(\boldsymbol{G}_i(\boldsymbol{\Phi} + \textstyle\sum_{k=1}^K \boldsymbol{w}_k\boldsymbol{w}_k^H)\right)$$

for $i = 1,\ldots,L$, where $0 < \xi_i \leq 1$ is the energy transfer efficiency of ER i, $\boldsymbol{G}_i = \boldsymbol{g}_i\boldsymbol{g}_i^H$ and $\boldsymbol{g}_i \in \mathbb{C}^n$ is the channel vector from the BS to ER i. Now, the SWIPT unicast downlink beamforming problem with energy signal cancellation may be formulated as follows:

$$\begin{aligned} \min_{\{\boldsymbol{w}_i\}_i, \boldsymbol{\Phi}\succeq\boldsymbol{0}} \quad & \sum_{k=1}^K \mathrm{Tr}(\boldsymbol{w}_k\boldsymbol{w}_k^H) + \mathrm{Tr}(\boldsymbol{\Phi}) \\ \text{s.t.} \quad & \mathsf{SINR}_k(\{\boldsymbol{w}_l\boldsymbol{w}_l^H\}_l) \geq \gamma_k,\ k=1,\ldots,K \\ & \mathsf{Q}_i(\{\boldsymbol{w}_l\boldsymbol{w}_l^H\}_l, \boldsymbol{\Phi}) \geq \eta_i,\ i=1,\ldots,L, \end{aligned} \tag{11}$$

where $\eta_i > 0$ is the minimum energy transfer requirement for ER i. Similar to (8), the SDR of (11) is given as follows.

$$\min_{\{\boldsymbol{W}_i\}, \boldsymbol{\Phi}} \quad \mathrm{Tr}(\bar{\boldsymbol{W}}) + \mathrm{Tr}(\boldsymbol{\Phi}) \tag{12a}$$

$$\text{s.t.} \quad \mathsf{SINR}_k(\{\boldsymbol{W}_l\}_l) \geq \gamma_k,\ k=1,\ldots,K, \tag{12b}$$

$$\mathsf{Q}_i(\{\boldsymbol{W}_l\}_l, \boldsymbol{\Phi}) \geq \eta_i,\ i=1,\ldots,L, \tag{12c}$$

$$\boldsymbol{W}_1,\ldots,\boldsymbol{W}_K, \boldsymbol{\Phi} \succeq \boldsymbol{0}, \tag{12d}$$

where for convenience we denote

$$\bar{\boldsymbol{W}} = \textstyle\sum_{i=1}^K \boldsymbol{W}_i.$$

Problem (12) can be rewritten as problem (1) via setting $m = K$, $\boldsymbol{X}_i = \boldsymbol{W}_i, i = 1,\ldots,K, \boldsymbol{X}_{m+1} = \boldsymbol{\Phi}, p = L$,

$$\begin{aligned} & \boldsymbol{A}_{ii} = \tfrac{1}{\gamma_i}\boldsymbol{h}_i\boldsymbol{h}_i^H,\ \boldsymbol{A}_{ij} = \boldsymbol{h}_i\boldsymbol{h}_i^H, j=1,\ldots,m, j\neq i, \\ & \boldsymbol{A}_{i,m+1} = \boldsymbol{0},\ b_i = \sigma_i^2,\ i=1,\ldots,m, \\ & \boldsymbol{C}_i = \boldsymbol{I},\ i=1,\ldots,m+1, \\ & \boldsymbol{F}_{ij} = -\boldsymbol{G}_i,\ \forall\, j,\ \ d_i = -\eta_i/\xi_i,\ i=1,\ldots,p. \end{aligned} \tag{13}$$

One can verify that the SWIPT problem satisfies the conditions in Theorem 1 with $r_i = 1$, and that any feasible point of problem (12) must satisfy $\boldsymbol{W}_i \neq \boldsymbol{0}$ for $i = 1,\ldots,K$. Hence, it follows from Corollary 1 that the SDR of the SWIPT problem is tight.

There are several SWIPT extensions for which the SDR tightness result provided by Corollary 1 still applies.

1) SWIPT without Energy Signal Cancellation: The above SWIPT formulation assumes that the IRs are able to cancel the energy signal from their received signals. Since performing the latter requires additional processing efforts, it may not be always possible to do so in practice. In the scenario where the IRs do not perform energy signal cancellation, the energy signal is seen as interference and the SINR expressions should be modified as

$$\mathsf{SINR}_k(\{\boldsymbol{W}_l\}_l, \boldsymbol{\Phi}) = \frac{\boldsymbol{h}_k^H\boldsymbol{W}_k\boldsymbol{h}_k}{\boldsymbol{h}_k^H(\sum_{i\neq k}\boldsymbol{W}_i + \boldsymbol{\Phi})\boldsymbol{h}_k + \sigma_k^2}. \tag{14}$$

The corresponding SDR of SWIPT problem in (12) can be expressed as problem (1) using the same settings as in (13), except that

$$\boldsymbol{A}_{i,m+1} = \boldsymbol{h}_i\boldsymbol{h}_i^H.$$

It can be verified that the conditions in Theorem 1 still hold. Hence, by Corollary 1, the SDR in this case is tight.

2) SWIPT with Correlation-Based CSIs of ERs: Accurate information of the ERs' channels $\boldsymbol{g}_i$ may not be always available. Suppose that we only have information of the channel correlation matrices $\mathbb{E}[\boldsymbol{g}_i\boldsymbol{g}_i^H]$; i.e., the so-called correlation-based CSI. For such a scenario, we may modify the SWIPT formulation by replacing the left-hand side of the energy transfer constraints in (12c) with long-term average transferred energies; i.e.,

$$\xi_i\mathbb{E}[\boldsymbol{g}_i^H(\bar{\boldsymbol{W}} + \boldsymbol{\Phi})\boldsymbol{g}_i].$$

Such a modified formulation can simply be accomplished by redefining $\boldsymbol{G}_i$ in (12c) as $\boldsymbol{G}_i = \mathbb{E}[\boldsymbol{g}_i \boldsymbol{g}_i^H]$. Again, it is easy to verify that the conditions in Theorem 1 are still satisfied. Hence, by Corollary 1, SDR is still tight.

3) SWIPT for Maximal Energy Transfer with PAPCs: Consider the SDR of an alternative SWIPT formulation, given as follows

$$\max_{\{\boldsymbol{W}_i\},\boldsymbol{\Phi}} \sum_{i=1}^{L} \xi_i \mathrm{Tr}\left(\boldsymbol{G}_i(\bar{\boldsymbol{W}} + \boldsymbol{\Phi})\right) \tag{15a}$$

$$\text{s.t. } \mathsf{SINR}_k(\{\boldsymbol{W}_l\}_l) \geq \gamma_k,\ k = 1, \ldots, K, \tag{15b}$$

$$\mathrm{Tr}(\bar{\boldsymbol{W}} + \boldsymbol{\Phi}) \leq P_{\mathsf{sum}}, \tag{15c}$$

$$\boldsymbol{W}_1, \ldots, \boldsymbol{W}_K, \boldsymbol{\Phi} \succeq \boldsymbol{0}, \tag{15d}$$

where $P_{\mathsf{sum}} > 0$ is the maximum allowable total transmission power of the BS, and $\mathsf{SINR}_k(\cdot)$ is defined in (6). As can be seen above, the design aim is to maximize the ERs' sum harvested energies, subject to the total transmission power constraint and IRs' SINR constraints. The tightness of the SDR in (15) has been studied in [9]. Here, let us consider a per-antenna power constrained version of problem (15):

$$\max_{\{\boldsymbol{W}_i\},\boldsymbol{\Phi}} \sum_{i=1}^{L} \xi_i \mathrm{Tr}\left(\boldsymbol{G}_i(\bar{\boldsymbol{W}} + \boldsymbol{\Phi})\right) \tag{16a}$$

$$\text{s.t. } \mathsf{SINR}_k(\{\boldsymbol{W}_l\}_l) \geq \gamma_k,\ k = 1, \ldots, K, \tag{16b}$$

$$[\bar{\boldsymbol{W}} + \boldsymbol{\Phi}]_{ii} \leq P_i,\ i = 1, \ldots, n, \tag{16c}$$

$$\boldsymbol{W}_1, \ldots, \boldsymbol{W}_K, \boldsymbol{\Phi} \succeq \boldsymbol{0}, \tag{16d}$$

where $P_i > 0$ is the maximum allowable transmission power of the ith antenna of the BS. The rank-one solution conditions, or SDR tightness, of problem (16) have not been studied in [9]. However, since problem (16) satisfies the conditions in Theorem 1 (same $\boldsymbol{A}_{ij}$ as in (13), $p = n$, $\boldsymbol{F}_{ij} = \boldsymbol{e}_i \boldsymbol{e}_i^T \forall\, j$, $d_i = P_i$, $\boldsymbol{C}_i = -\sum_{j=1}^{L} \xi_j \boldsymbol{G}_j$ for all i), we immediately declare that the SDR problem (16) is tight.

3.3. Physical-Layer Security

Consider again the MISO downlink unicast beamforming model in Sec. 3.1, and suppose that there are L single-antenna eavesdroppers, or Eves for short, that intend to overhear the K messages transmitted by the BS. To make the transmission secure, the BS adopts an artificial noise (AN)-aided unicast beamforming scheme where AN is superimposed in the beamformed signals as a means to interfere Eves' receptions; readers are referred to [10, 11] for a detailed description of AN-aided secure beamforming. Suppose that AN is randomly distributed with mean zero and covariance matrix $\boldsymbol{\Phi}$. Let $\boldsymbol{g}_i \in \mathbb{C}^n$ for $i = 1, \ldots, L$ be the channel vector of Eve i. Then, the SINR for the ith Eve to receive the kth message is given by

$$\mathsf{SINR}_{e,i}^{k}(\{\boldsymbol{w}_j \boldsymbol{w}_j^H\}_j, \boldsymbol{\Phi}) = \frac{\mathrm{Tr}(\boldsymbol{G}_i \boldsymbol{w}_k \boldsymbol{w}_k^H)}{\bar{\sigma}_i^2 + \mathrm{Tr}\left(\boldsymbol{G}_i(\sum_{l \neq k} \boldsymbol{w}_l \boldsymbol{w}_l^H + \boldsymbol{\Phi})\right)}$$

for $i = 1, \ldots, L$ and $k = 1, \ldots, K$, where $\boldsymbol{G}_i = \boldsymbol{g}_i \boldsymbol{g}_i^H$, $\bar{\sigma}_i^2$ is the ith Eve's receive noise variance. In addition, one can check that the received SINR at the ith legitimate user is exactly the same as (14). Now, the AN-aided physical-layer secure beamforming problem may be formulated as

$$\begin{aligned} \min_{\{\boldsymbol{w}_i\}_i, \boldsymbol{\Phi} \succeq \boldsymbol{0}}\ & \sum_{k=1}^{K} \mathrm{Tr}(\boldsymbol{w}_k \boldsymbol{w}_k^H) + \mathrm{Tr}(\boldsymbol{\Phi}) \\ \text{s.t. }\ & \mathsf{SINR}_k(\{\boldsymbol{w}_j \boldsymbol{w}_j^H\}_j, \boldsymbol{\Phi}) \geq \gamma_k,\ \forall\, k \in \mathcal{K}, \\ & \mathsf{SINR}_{e,i}^{k}(\{\boldsymbol{w}_j \boldsymbol{w}_j^H\}_j, \boldsymbol{\Phi}) \leq \eta_{ik},\ \forall\, (i,k) \in \mathcal{L} \times \mathcal{K}, \end{aligned} \tag{17}$$

where $\mathcal{K} = \{1, \ldots, K\}$, $\mathcal{L} = \{1, \ldots, L\}$, $\eta_{ik} \geq 0$ is the SINR threshold of the ith Eve for receiving the kth message. As seen, the goal of problem (17) is to use minimum transmit power to guarantee discriminative receive qualities between legitimate users and Eves. The corresponding SDR is given by

$$\min_{\{\boldsymbol{W}_i\}_i, \boldsymbol{\Phi}} \sum_{k=1}^{K} \mathrm{Tr}(\boldsymbol{W}_k) + \mathrm{Tr}(\boldsymbol{\Phi}) \tag{18a}$$

$$\text{s.t. } \mathsf{SINR}_k(\{\boldsymbol{W}_j\}_j, \boldsymbol{\Phi}) \geq \gamma_k,\ k = 1, \ldots, K, \tag{18b}$$

$$\mathsf{SINR}_{e,i}^{k}(\{\boldsymbol{W}_j\}_j, \boldsymbol{\Phi}) \leq \eta_{ik},\ \forall\, (i,k) \in \mathcal{L} \times \mathcal{K}, \tag{18c}$$

$$\boldsymbol{W}_1, \ldots, \boldsymbol{W}_K, \boldsymbol{\Phi} \succeq \boldsymbol{0}. \tag{18d}$$

Problem (18) has been shown to have rank-one solutions when there is only one legitimate user, i.e., $K = 1$ [10]. However, for general K, there is no result on the SDR tightness. We show below that problem (18) indeed has a rank-one solution w.r.t. $\boldsymbol{W}_1, \ldots, \boldsymbol{W}_K$. In particular, problem (18) can be written as problem (1) via setting $m = K$, $\boldsymbol{X}_i = \boldsymbol{W}_i$, $i = 1, \ldots, K$, $\boldsymbol{X}_{m+1} = \boldsymbol{\Phi}$, $p = KL$, and

$$\begin{aligned} & \boldsymbol{A}_{ii} = \tfrac{1}{\gamma_i} \boldsymbol{h}_i \boldsymbol{h}_i^H,\ \boldsymbol{A}_{ij} = \boldsymbol{h}_i \boldsymbol{h}_i^H, j = 1, \ldots, m, j \neq i, \\ & \boldsymbol{A}_{i,m+1} = \boldsymbol{h}_i \boldsymbol{h}_i^H,\ b_i = \sigma_i^2,\ i = 1, \ldots, m, \\ & \boldsymbol{C}_i = \boldsymbol{I},\ i = 1, \ldots, m+1, \\ & \boldsymbol{F}_{(l-1)K+k,j} = -\boldsymbol{G}_l,\ j \in \mathcal{K} \setminus k,\ \forall\, (l,k) \in \mathcal{L} \times \mathcal{K}, \\ & \boldsymbol{F}_{(l-1)K+k,k} = \boldsymbol{G}_l / \eta_{lk},\ \ \forall\, (l,k) \in \mathcal{L} \times \mathcal{K}, \\ & \boldsymbol{F}_{(l-1)K+k,m+1} = -\boldsymbol{G}_l,\ \ \forall\, (l,k) \in \mathcal{L} \times \mathcal{K}, \\ & d_{(l-1)K+k} = \bar{\sigma}_l^2,\ \forall\, (l,k) \in \mathcal{L} \times \mathcal{K}. \end{aligned} \tag{19}$$

One can verify that the conditions in Theorem 1 holds with $r_i = 1$, and that any feasible point of problem (18) must satisfy $\boldsymbol{W}_i \neq \boldsymbol{0}$ for $i = 1, \ldots, K$. Hence, it follows from Corollary 1 that the SDR of the physical-layer secure beamforming problem is tight.

Remark 2 Similar to the SWIPT problem, one can also consider correlation-based CSIs of Eves by replacing $\boldsymbol{G}_i = \boldsymbol{g}_i \boldsymbol{g}_i^H$ with $\boldsymbol{G}_i = \mathbb{E}[\boldsymbol{g}_i \boldsymbol{g}_i^H]$ in (18c). It can be easily verified that this does not change the settings in (19). Thus, the SDR (18) is still tight.

Remark 3 As another extension of the SDR (18), one can further add some shaping constraints [12] to fulfill some specific design requirement such as interference control. This can be done by adding the following constraints to (18):

$$\mathrm{Tr}\left(\boldsymbol{R}_i(\sum_{k=1}^{K} \boldsymbol{W}_k + \boldsymbol{\Phi})\right) \trianglerighteq_i \beta_i,\ i = 1, \ldots, J \tag{20}$$

where $\trianglerighteq_i \in \{=, \leq, \geq\}$, and $\boldsymbol{R}_i \in \mathbb{H}^n$ can be indefinite. Since for the ith shaping constraint, $\boldsymbol{W}_1, \ldots, \boldsymbol{W}_K$ and $\boldsymbol{\Phi}$ share the same $\boldsymbol{F}_{ij} = \boldsymbol{R}_i$ for all j, condition iii) in Theorem 1 always holds irrespective of the definiteness of $\boldsymbol{R}_i$. Consequently, Corollary 1 holds and we can again establish SDR tightness.

4. CONCLUSIONS

In this paper we studied a special class of SDP problems and derived a set of conditions under which the SDP is guaranteed to have a low-rank optimal solution. The result is important in identifying the tightness of SDR in some special cases. The derived conditions require the data matrices to satisfy certain matrix conditions, but have no restriction on the number of constraints as in the famous SBP rank-reduction result. We have showcased several advanced beamforming examples, such as SWIPT and physical-layer security, where we demonstrated how the derived conditions lead us to pinning down the tightness of SDR in those beamforming optimization problems.

5. REFERENCES

[1] Z.-Q. Luo, W.-K. Ma, A. M.-C. So, Y. Ye, and S. Zhang, "Semidefinite relaxation of quadratic optimization problems," *IEEE Signal Process. Mag.*, vol. 27, no. 3, pp. 20–34, May 2010.

[2] A. B. Gershman, N. D. Sidiropoulos, S. Shahbazpanahi, M. Bengtsson, and B. Ottersten, "Convex optimization-based beamforming," *IEEE Signal Process. Mag.*, vol. 27, no. 3, pp. 62–75, May 2010.

[3] C. Fortin and H. Wolkowicz, "The trust region subproblem and semidefinite programming," *Optim. Methods Softw.*, vol. 19, no. 1, pp. 41–67, 2004.

[4] J. Moré, "Generalizations of the trust region subproblem," *Optim. Methods Softw.*, vol. 2, no. 1, pp. 189–209, 1993.

[5] A. Beck and Y. Eldar, "Strong duality in nonconvex quadratic optimization with two quadratic constraints," *SIAM J. Optim.*, vol. 17, no. 3, pp. 844–860, 2006.

[6] G. Pataki, "On the rank of extreme matrices in semidefinite programs and the multiplicity of optimal eigenvalues," *Mathematics of Operations Research*, vol. 23, no. 2, pp. 339–358, 1998.

[7] Y. Huang and D. P. Palomar, "Rank-constrained separable semidefinite programming with applications to optimal beamforming," *IEEE Trans. Signal Process.*, vol. 58, no. 2, pp. 664–678, Feb. 2010.

[8] W. Yu and T. Lan, "Transmitter optimization for the multi-antenna downlink with per-antenna power constraints," *IEEE Trans. Signal Process.*, vol. 55, no. 6, pp. 2646–2660, June 2007.

[9] J. Xu, L. Liu, and R. Zhang, "Multiuser MISO beamforming for simultaneous wireless information and power transfer," *IEEE Trans. Signal Process.*, vol. 62, no. 18, pp. 4798–4810, Sept. 2014.

[10] W.-C. Liao, T.-H. Chang, W.-K. Ma, and C.-Y. Chi, "QoS-based transmit beamforming in the presence of eavesdroppers: An optimized artificial-noise-aided approach," *IEEE Trans. Signal Process.*, vol. 59, no. 3, pp. 1202–1216, Mar. 2011.

[11] Q. Li and W.-K. Ma, "Spatially selective artificial-noise aided transmit optimization for MISO multi-Eves secrecy rate maximization," *IEEE Trans. Signal Process.*, vol. 61, no. 10, pp. 2704–2717, May 2013.

[12] D. Hammarwall, M. Bengtsson, and B. Ottersten, "On downlink beamforming with indefinite shaping constraints," *IEEE Trans. Signal Process.*, vol. 54, pp. 3566–3580, Sep. 2006.