ALGEBRAIC SOLUTION FOR STATIONARY EMITTER GEOLOCATION BY A LEO SATELLITE USING DOPPLER FREQUENCY MEASUREMENTS

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ABSTRACT

This paper presents a new algebraic solution for the Doppler positioning problem where the position of a stationary emitter is estimated from Doppler frequency measurements collected by a single low-earth-orbit (LEO) satellite. The proposed algebraic solution can be used for effective initialization of more sophisticated iterative algorithms as it produces estimates sufficiently close to the actual position of the emitter to ensure convergence. It is computationally more efficient than existing initialization techniques, based on the point of closest approach, which require expensive nonlinear curve fitting procedures. The effectiveness of the proposed algebraic solution is demonstrated by way of numerical simulations.

Index Terms— Emitter geolocation, satellite navigation, search and rescue operations, Doppler-shifted frequency, algebraic solution

1. INTRODUCTION

Passive emitter geolocation using Doppler frequency measurements has been an area of substantial research interest for many years thanks to its wide-ranging applications in acoustic source localization, radar and sonar systems, and satellite positioning and navigation [1–22]. One of the most important applications of Doppler-based geolocation is found in the COSPAS-SARSAT distress alerting satellite system for search and rescue missions [10–22]. More than one million COSPAS-SARSAT distress beacons have been registered around the world. These distress beacons, operating at the international distress frequency of 406 MHz, can be activated anytime in an emergency situation. Emergency distress signals are relayed by satellites to ground tracking stations that compute the distress location and immediately inform the rescue authorities about the distress alert.

In the COSPAS-SARSAT system, the location of a stationary emitter, i.e., a distress beacon, is determined from Doppler frequency measurements collected by a single lowearth-orbit (LEO) satellite. Existing techniques for satellite Doppler positioning (see, e.g., [17–21] and the references therein) were developed based on numerical iterative leastsquares algorithms due to the nonlinearity of the Doppler equation. These iterative least-squares algorithms require an appropriate initialization to avoid divergence problems. Most initialization methods reported in the literature rely on calculations around the point of closest approach (PCA) [19-22]. The PCA, referred to as the satellite position with the closest distance to the emitter, can be determined from the inflection point of the observed Doppler curve. The emitter position can be subsequently computed from the slope of the Doppler curve at the PCA as well as the knowledge of satellite position and velocity. However, the main disadvantage of these initialization methods is that they require nonlinear curve fitting to observed Doppler points [16, 21], which is implemented through computationally-expensive high-order polynomial curve fitting. The large number of polynomial coefficients to be estimated increases the sensitivity of PCA estimates to Doppler measurement noise. The PCA can also be estimated from the data point with a zero Doppler-shift, but this approach requires high-accuracy frequency measurements (high SNR), as well as a high stability in transmitted frequency [16]. Another initialization approach was proposed in [17] based on a finite grid around the midpoint of satellite pass, in which different points in the grid are selected as the initial value for the iterative algorithm until convergence is achieved. However, this brute-force approach becomes computationally demanding as the size of the grid increases. Moreover, if the selected initial value is not close to the actual emitter position, the iterative algorithm may converge to a local optimum value and thus result in a wrong estimate of the emitter position.

In this paper we propose a new low-complexity algebraic solution for satellite Doppler positioning. The main advantage of the present work is that the proposed algorithm offers a closed-form solution which is computationally much cheaper than the existing PCA-based initialization techniques. More importantly, the proposed algorithm provides a good initial estimate of the emitter position, which is sufficiently close to the actual emitter position to enable convergence of the iterative algorithms. In a related work [8], a closed-form solution was developed for geolocating a radar emitter using Doppler frequency measurements obtained from a moving sensor platform. This approach can be adopted for satellite Doppler positioning if multiple satellite passes with different



Fig. 1. Satellite-emitter geometry in rectilinear motion model.

satellite tracks are available. Given that a LEO satellite takes around 100 minutes to complete an orbit around the earth, multiple satellite passes can result in very long waiting times, greatly delaying search and rescue operations. The long time interval between multiple satellite passes has been the main reason for studying scenarios with a single satellite pass (see e.g., [11–14, 17–20]). Motivated by this consideration, we aim to derive a closed-form solution of the emitter position from the Doppler measurements obtained from a single satellite pass only. Our algebraic solution, in contrast to the one in [8], can be used to initialize the iterative single-pass algorithms, e.g., in [17-20]. It can also be used to initialize iterative multiple-pass algorithms, e.g., [21], by averaging emitter position solutions obtained from each satellite pass. We observe from simulation results that the iterative maximumlikelihood algorithm initialized by the algebraic solution proposed in this paper closely achieves the Cramér-Rao lower bound (CRLB), thus demonstrating the effectiveness of the proposed algebraic algorithm for initialization purposes.

2. PROBLEM FORMULATION

In this paper the actual elliptical satellite orbit over the oblate spheroid-shaped Earth is replaced by the rectilinear model of the satellite motion proposed in [11, 12], where the satellite follows a uniform rectilinear motion at a fixed height over a flat Earth's surface as depicted in Fig. 1. Since the height of LEO satellites is relatively small compared to the Earth's radius, this model is an adequate approximation of the actual satellite orbit in the sense that our aim is to obtain a sufficiently good emitter position estimate to initialize iterative Doppler positioning algorithms. Moreover, this simple rectilinear model allows for an algebraic solution of the emitter position to be derived, leading to a computationally inexpensive estimator.

In the satellite-emitter geometry shown in Fig. 1, the satellite moves along the positive direction of the *x*-axis with a constant speed of V over the Earth's surface at a fixed height of h. The emitter position can be characterized by the PCA point x_{PCA} and the distance d from the emitter position to the satellite subtrack (the ground projection of the satellite orbit). The algebraic solution proposed in this paper aims to estimate x_{PCA} and d from N noisy Doppler-shifted frequency measurements \tilde{f}_k taken at satellite positions x_k at time instants $k \in \{1, ..., N\}$. The Doppler-shifted frequency measurement at time instant k is given by

$$\tilde{f}_k = f_k + n_k, \quad f_k = f_o \left(1 + \frac{V \cos \alpha_k}{c} \right)$$
(1)

where f_o is the frequency of the distress beacons, c is the speed of signal propagation, α_k is the relative angle between the x-axis and the vector pointing from the satellite to the emitter at time instant k as shown in Fig. 1, and $n_k \sim \mathcal{N}(0, \sigma^2)$ is the additive i.i.d. zero-mean Gaussian noise with variance σ^2 .

It is noted that, for the single-pass scenario, an ambiguity exists in the emitter position solution on either side of the satellite subtrack, i.e., one corresponding to the actual emitter position and the other to its mirror image [13, 14, 17, 18, 20]. Given an estimate of the PCA point and emitter-subtrack distance, the ambiguity can be resolved by taking into account the slight difference in the shapes of the two Doppler curves corresponding to the actual emitter position and its mirror image due to the rotation of the Earth [18–20]. In particular, the residual sets of the two solutions were shown to have different statistical properties and a quadratic-form test aiming to maximize the probability of a correct decision was proposed in [18]. In addition, the ambiguity can be resolved by using a subsequent pass of the satellite [17, 20] if the delay resulting from this is tolerable.

3. PROPOSED ALGEBRAIC SOLUTION

The proposed algebraic solution involves two separate steps of (i) estimating the PCA point x_{PCA} and (ii) estimating the emitter-subtrack distance d with details given below.

3.1. Estimating PCA point

From Fig. 1, we have the following trigonometric relationship:

$$\cos \alpha_k = \frac{x_{\rm PCA} - x_k}{l_k} = \frac{x_{\rm PCA} - x_k}{\sqrt{h^2 + d^2 + (x_{\rm PCA} - x_k)^2}} \quad (2)$$

where l_k denotes the emitter-satellite distance at time instant k. The distance l_k can be written in terms of α_1 and α_k based on the law of sines [8] (see Fig. 2). Applying the law of sines to the triangle formed by the emitter position and the satellite positions at time instants 1 and k as shown in Fig. 2 yields

$$\frac{x_k - x_1}{\sin \gamma_{k1}} = \frac{l_k}{\sin \alpha_1}.$$
(3)



Fig. 2. Triangle formed by emitter position and satellite positions at time instants 1 and k.

As a result, we have

$$l_{k} = \frac{(x_{k} - x_{1})\sin\alpha_{1}}{\sin\gamma_{k1}} = \frac{(x_{k} - x_{1})\sin\alpha_{1}}{\sin(\alpha_{k} - \alpha_{1})}.$$
 (4)

Substituting (4) into (2) gives

$$x_{\text{PCA}} = \frac{(x_k - x_1)\sin\alpha_1\cos\alpha_k}{\sin(\alpha_k - \alpha_1)} + x_k.$$
 (5)

Replacing the unknown angles α_1 and α_k in (5) by their estimates $\tilde{\alpha}_1$ and $\tilde{\alpha}_k$ yields

$$x_{\rm PCA} = \frac{(x_k - x_1)\sin\tilde{\alpha}_1\cos\tilde{\alpha}_k}{\sin(\tilde{\alpha}_k - \tilde{\alpha}_1)} + x_k + \eta_k \tag{6}$$

where η_k is the resulting pseudolinear noise term. Here the estimated angle $\tilde{\alpha}_k, k \in \{1, \dots, N\}$, is computed from the Doppler-shifted frequency measurement \tilde{f}_k in (1) as

$$\tilde{\alpha}_k = \cos^{-1}\left(\frac{c}{V}\left(\frac{\tilde{f}_k}{f_o} - 1\right)\right). \tag{7}$$

Stacking (6) for k = 2, ..., N and solving it for x_{PCA} in the least-squares sense yields an estimate \hat{x}_{PCA} :

$$\hat{x}_{\text{PCA}} = \frac{1}{N-1} \sum_{k=2}^{N} \left(\frac{(x_k - x_1) \sin \tilde{\alpha}_1 \cos \tilde{\alpha}_k}{\sin(\tilde{\alpha}_k - \tilde{\alpha}_1)} + x_k \right).$$
(8)

3.2. Estimating emitter-subtrack distance

Equation (2) can be rewritten as

$$d^{2} = (x_{\rm PCA} - x_{k})^{2} \tan^{2} \alpha_{k} - h^{2}.$$
 (9)

Replacing the unknown α_k and x_{PCA} with their estimates $\tilde{\alpha}_k$ and \hat{x}_{PCA} in (7) and (8), respectively, we obtain

$$d^{2} = (\hat{x}_{PCA} - x_{k})^{2} \tan^{2} \tilde{\alpha}_{k} - h^{2} + \delta_{k}, \quad k = 1, \dots, N$$
 (10)

where δ_k is a noise term accounting for the errors introduced by $\tilde{\alpha}_k$ and $\hat{x}_{_{\rm PCA}}$.

Stacking (10) for k = 1, ..., N and solving the resulting matrix equation for d^2 in the least-squares sense yields:

$$\hat{d}^2 = \left(\frac{1}{N}\sum_{k=1}^{N} (\hat{x}_{\rm PCA} - x_k)^2 \tan^2 \tilde{\alpha}_k\right) - h^2.$$
(11)

An estimate for the emitter-subtrack distance d is simply given by the positive square-root of \hat{d}^2 :

$$\hat{d} = \sqrt{\left(\frac{1}{N}\sum_{k=1}^{N} (\hat{x}_{\text{PCA}} - x_k)^2 \tan^2 \tilde{\alpha}_k\right) - h^2}.$$
 (12)

In (12), \hat{x}_{PCA} is only an estimate of the actual x_{PCA} . Therefore, the estimation noise in \hat{x}_{PCA} will be greatly amplified in (12) if the satellite data contains measurements taken at satellite positions close to the PCA point (where the values of $\tilde{\alpha}$ is around 90°) because $\tan \tilde{\alpha}_k \to \infty$ as $\tilde{\alpha}_k \to 90^\circ$. Thus, the estimation accuracy of \hat{d} will be significantly degraded unless (10) is properly weighted according to the variance of δ_k . To overcome this problem, we introduce a weight of $1/\tan^2 \tilde{\alpha}_k$ for each data point $k \in \{1, \ldots, N\}$ in (12), yielding a weighted estimate \hat{d} of d:

$$\hat{d} = \sqrt{\frac{\sum_{k=1}^{N} \left(\frac{1}{\tan^{2} \tilde{\alpha}_{k}} \left((\hat{x}_{\text{PCA}} - x_{k})^{2} \tan^{2} \tilde{\alpha}_{k} - h^{2} \right) \right)}{\sum_{k=1}^{N} \frac{1}{\tan^{2} \tilde{\alpha}_{k}}}} = \sqrt{\frac{\sum_{k=1}^{N} (\hat{x}_{\text{PCA}} - x_{k})^{2}}{\sum_{k=1}^{N} \cot^{2} \tilde{\alpha}_{k}}} - h^{2}}.$$
(13)

Equations (8) and (13) make up the proposed closed-form algebraic solution for the satellite Doppler positioning problem.

4. SIMULATION STUDIES

In this section we demonstrate the effectiveness of the proposed algebraic solution via a simulation example. In particular we compare the performance of the iterative maximum-likelihood (ML) algorithm initialized by the proposed solution with the CRLB to verify its good performance as an initialization algorithm. In the simulation, the following practical parameters are used: h = 850 km, V = 7.6 km/s, $f_o = 406$ MHz, $x_{\rm PCA} = 1500$ km, d = 850 km, and $c = 3 \times 10^8$ m/s. The LEO satellite collects Doppler-shifted frequency measurements 21 times at equally spaced points along the satellite track segment of 3000 km bounded by 0 km $\leq x_k \leq 3000$ km. For simplicity, the implementation of the iterative ML algorithm and the computation of the CRLB are carried out based on the satellite-emitter geometry model given in Section 2. The ML cost function is given by

$$J_{\rm ML}(\boldsymbol{\xi}) = \frac{1}{2} (\tilde{\mathbf{f}} - \mathbf{f}(\boldsymbol{\xi}))^T \mathbf{K}^{-1} (\tilde{\mathbf{f}} - \mathbf{f}(\boldsymbol{\xi}))$$
(14)

where $\boldsymbol{\xi} = [x_{\text{PCA}}, d]^T$, $\mathbf{f}(\boldsymbol{\xi}) = [f_1(\boldsymbol{\xi}), \dots, f_N(\boldsymbol{\xi})]^T$ is the vector of the Doppler-shifted frequencies as a function of $\boldsymbol{\xi}$, $\tilde{\mathbf{f}} = [\tilde{f}_1, \dots, \tilde{f}_N]^T$ is the vector of the corresponding noisy measurements, and $\mathbf{K} = \sigma^2 \mathbf{I}$ is the noise covariance matrix. The ML cost function in (14) is minimized via the following Gauss-Newton (GN) iterations [23]:

$$\hat{\boldsymbol{\xi}}(j+1) = \hat{\boldsymbol{\xi}}(j) + (\mathbf{J}^T(j)\mathbf{K}^{-1}\mathbf{J}(j))^{-1}\mathbf{J}^T(j)\mathbf{K}^{-1}(\tilde{\mathbf{f}} - \mathbf{f}(\hat{\boldsymbol{\xi}}(j)))$$
(15)

for j = 0, 1, ..., where $\mathbf{J}(j)$ is the Jacobian matrix of $\mathbf{f}(\boldsymbol{\xi})$ with respect to $\boldsymbol{\xi}$ evaluated at $\boldsymbol{\xi} = \hat{\boldsymbol{\xi}}(j)$:

$$\mathbf{J}(j) = \left[\mathbf{J}_1^T(j), \dots, \mathbf{J}_N^T(j)\right]^T.$$
 (16)

Here $\mathbf{J}_k(j) = [J_{1,k}(j), J_{2,k}(j)]$ with $J_{1,k}(j)$ and $J_{2,k}(j)$ given as

$$J_{1,k}(j) = \frac{V f_o((\hat{d}(j))^2 + h^2)}{c \left[(\hat{d}(j))^2 + h^2 + (\hat{x}_{\text{PCA}}(j) - x_k)^2 \right]^{3/2}}$$
$$J_{2,k}(j) = -\frac{V f_o \hat{d}(j) (\hat{x}_{\text{PCA}}(j) - x_k)}{c \left[(\hat{d}(j))^2 + h^2 + (\hat{x}_{\text{PCA}}(j) - x_k)^2 \right]^{3/2}}$$

where $\hat{x}_{PCA}(j)$ and $\hat{d}(j)$ are the first and second parameters of $\hat{\xi}(j)$. The GN iteration in (15) is initialized to the proposed closed-form solution in (8) and (13), and iterated for 10 times. Note that the ML algorithm has two optimum solutions due to the solution ambiguity (refer to Section 2 for a discussion on how to resolve this ambiguity).

For comparison purposes, the root mean squared errors (RMSE) of the estimates of x_{PCA} and d, obtained by the proposed algebraic solution and the iterative ML algorithm, are computed over 100,000 Monte Carlo simulation runs. The square roots of the CRLBs for x_{PCA} and d are also computed. Here the CRLBs for x_{PCA} and d are the first and second diagonal terms, respectively, of the CRLB matrix CRLB = $(\mathbf{J}_o^T \mathbf{K}^{-1} \mathbf{J}_o)^{-1}$, where \mathbf{J}_o has the same expression as $\mathbf{J}(j)$ in (16) but evaluated at the true emitter position.

Figs. 3 and 4 plot the RMSE performance of the proposed algebraic solution and the iterative ML algorithm for x_{PCA} and d, respectively, against the frequency measurement noise standard deviation $\sigma \in \{1, 2, ..., 10\}$ Hz. It is observed that the RMSEs of the proposed solutions for both x_{PCA} and d do not attain their corresponding CRLBs. However, the RMSE performance of the iterative ML algorithm, which is initialized by the proposed algebraic solution, achieves the CRLB as shown in Figs. 3 and 4. This observation confirms the effectiveness of the proposed closed-form solution for initialization purposes.

5. CONCLUSION

This paper has developed a new algebraic solution for singlepass satellite Doppler positioning based on exploitation of



Fig. 3. RMSE of x_{PCA} versus measurement noise standard deviation for the proposed solution and the ML algorithm.



Fig. 4. RMSE of *d* versus measurement noise standard deviation for the proposed solution and the ML algorithm.

trigonometric relationships between the PCA point, emittersubtrack distance and Doppler measurements collected by a LEO satellite in a rectilinear motion model. The algebraic solution is derived from linearization of nonlinear Doppler equations in the unknown PCA point and emitter-subtrack distance. The proposed algorithm is computationally inexpensive compared to other existing PCA-based estimation techniques requiring expensive nonlinear curve fitting procedures. Its ability to seed iterative ML estimator to achieve almost efficient performance was demonstrated via numerical simulations. Due to space limitation, the numerical comparison between the proposed algorithm with other existing PCA-based initialization techniques was not included and will be considered in future work.

6. REFERENCES

- Y. T. Chan and F. L. Jardine, "Target localization and tracking from Doppler-shift measurements," *IEEE J. Ocean. Eng.*, vol. 15, no. 3, pp. 251–257, July 1990.
- [2] A. Amar and A. J. Weiss, "Localization of narrowband radio emitters based on Doppler frequency shifts," *IEEE Trans. Signal Process.*, vol. 56, no. 11, pp. 5500–5508, Nov. 2008.
- [3] B. Ristic and A. Farina, "Target tracking via multi-static Doppler shifts," *IET Radar, Sonar Navigation*, vol. 7, no. 5, pp. 508–516, June 2013.
- [4] N. H. Nguyen and K. Dogancay, "Optimal sensor placement for Doppler shift target localization," in *Proc. 2015 IEEE Radar Conf. (RadarCon)*, Arlington VA, USA, May 2015, pp. 1677–1682.
- [5] D. Lindgren, G. Hendeby, and F. Gustafsson, "Distributed localization using acoustic Doppler," *Signal Process.*, vol. 107, no. 0, pp. 43–53, 2015.
- [6] K. C. Ho and Y. T. Chan, "Geolocation of a known altitude object from TDOA and FDOA measurements," *IEEE Trans. Aerosp. Electron. Syst.*, vol. 33, no. 3, pp. 770–783, July 1997.
- [7] K. C. Ho and W. Xu, "An accurate algebraic solution for moving source location using TDOA and FDOA measurements," *IEEE Trans. Signal Process.*, vol. 52, no. 9, pp. 2453–2463, Sept. 2004.
- [8] B. H. Lee, Y. T. Chan, F. Chan, H.-J. Du, and F. A. Dilkes, "Doppler frequency geolocation of uncooperative radars," in *Proc. IEEE Military Commun. Conf.* (*MILCOM*), Orlando FL, USA, Oct. 2007, pp. 1–6.
- [9] N. H. Nguyen and K. Dogancay, "Optimal sensortarget geometries for Doppler-shift target localization," in *Proc. European Signal Process. Conf. (EUSIPCO)*, Nice, France, Aug. 2015, pp. 180–184.
- [10] W. C. Scales and R. Swanson, "Air and sea rescue via satellite systems: Even experimental systems have helped survivors of air and sea accidents. Two different approaches are discussed.," *IEEE Spectr.*, vol. 21, no. 3, pp. 48–52, Mar. 1984.
- [11] N. Levanon and M. Ben-Zaken, "Random error in AR-GOS and SARSAT satellite positioning systems," *IEEE Trans. Aerosp. Electron. Syst.*, vol. 21, no. 6, pp. 783–790, Nov. 1985.
- [12] N. Levanon, "Theoretical bounds on random errors in satellite Doppler navigation," *IEEE Trans. Aerosp. Electron. Syst.*, vol. 20, no. 6, pp. 810–816, Nov. 1984.

- [13] J. V. King, "Cospas-Sarsat satellite system for search and rescue," in *Commerce in Space: Infrastructures, Technologies, and Applications*, P. Olla, Ed., chapter 4, pp. 69–87. IGI Global, Hershey, PA, 2008.
- [14] J. V. King, "Overview of the Cospas-Sarsat satellite system for search and rescue," *Proceedings of the Radio Club of America*, vol. 85, no. 1, pp. 8–13, 2013.
- [15] I. Ali, N. Al-Dhahir, and J. E. Hershey, "Doppler characterization for LEO satellites," *IEEE Trans. Commun.*, vol. 46, no. 3, pp. 309–313, Mar. 1998.
- [16] P. Klein, "Time and distance of closest approach detector of moving object," July 13, 1965, U.S. Patent 3,195,136.
- [17] P. E. Schmid, J. J. Lynn, and F. O. Vonbun, "Single pass Doppler positioning for search and rescue satellite missions," in *Proc. IEEE PLANS*, San Diego, CA, Nov. 1976, pp. 58–67.
- [18] P. Argentiero and J. Marint, "Ambiguity resolution for satellite Doppler positioning systems," *IEEE Trans. Aerosp. Electron. Syst.*, vol. 15, no. 3, pp. 439–447, May 1979.
- [19] T. Green, "Satellite Doppler data processing for platform navigation," *IEEE Trans. Geosci. Electron.*, vol. 13, no. 1, pp. 28–38, Jan. 1975.
- [20] P.E. Schmid and J.J. Lynn, "Satellite Doppler-data processing using a microcomputer," *IEEE Trans. Geosci. Electron.*, vol. 16, no. 4, pp. 340–349, Oct. 1978.
- [21] K. Aksnes, P. H. Andersen, and E. Haugen, "A precise multipass method for satellite Doppler positioning," *Celestial Mechanics*, vol. 44, no. 4, pp. 317–338, 1988.
- [22] J. W. Marini, "Initial position estimates for satelliteaided search and rescue," Tech. Rep., NASA-GSFC X-932-76-245, Oct. 1976.
- [23] L. Ljung and T. Soderstrom, *Theory and Practice of Recursive Identification*, MIT Press, Cambridge, MA, 1983.