MEMBRANE SHAPE AND BOUNDARY CONDITIONS ESTIMATION USING EIGENMODE DECOMPOSITION

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ABSTRACT

This paper investigates the problem of estimating the shape or the boundary impedance of a vibrating membrane from acoustic measurements in a limited sub-domain of the membrane. In acoustics, polygonal room shapes are usually estimated through room impulse response measurements. Impedance values of materials are, in turn, often calculated from the measurement of the acoustic reflection coefficients at the boundaries. In this work, we develop an alternative frequency-domain method to estimate the shape of a convex membrane with generalized Robin boundary conditions, from the measurement of its eigenmodes on a small portion of its surface. Reciprocally, we show that the same model allows to estimate the membrane borders' impedances when its shape is known.

Index Terms— Eigenmodes decomposition, shape estimation, complex impedance, modal interpolation

1. INTRODUCTION

Estimating structural parameters of vibrating structures from their eigenvalues is a class of problems coming from spectral geometry. It has first been popularized by the Weyl law, demonstrated in the 1910's [1]. This asymptotic theorem showed that the number of eigenvalues of the Laplacian on a bounded domain, below a certain frequency, could be linked to the surface and the perimeter of this domain. Later, the famous question "*Can one hear the shape of a drum ?*" asked in 1966 by Mark Kac [2] opened new research perspectives. In his article, Kac wondered if two membranes (drums) sharing the same set of eigenvalues also shared the same shape. After a few decades of research, this question was finally answered negatively in the 1990's, when Gordon and Webb [3] constructed membranes of different geometries, but sharing the same spectrum.

Recently, this problem has been revisited to estimate the shapes of polygonal rooms, from time domain acoustic measurements [4, 5, 6, 7]. Most of these methods are based on the measurement of room impulse responses between some acoustic sources and microphones, and on the use of the image source model [8, 9]. In [10, 11], Dokmanic *et al.* measured the direct times of arrival between a source s and some microphones m, and the times of arrival of the first echoes reflected off the walls. Each echo was interpreted as a direct signal coming from an image source whose position was symmetric to the real source with respect to one wall (see Figure 1). The main difficulty was to sort the echoes and associate them with the right wall, in order to estimate the positions of the image sources. For simple geometries, the room shape (or main dimensions) could then be geometrically deduced.

Nevertheless, these methods slightly deviate from the general framework of Kac's article, as they are designed for time domain measurements instead of eigenvalues. Moreover the image method cannot be easily applied when the room shape is curved.

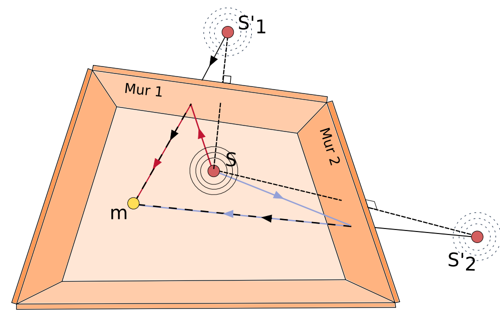


Fig. 1. Source image method applied to room geometry estimation (adapted from [11]). Here, the first echoes coming from the walls 1 and 2 (trajectories represented in red and blue), and measured by the microphone m, can be seen as the first arrivals (in black) of a field generated by the image sources s_1' and s_2', symmetric to the real source s with respect to each wall position.

In [12], we developed a new approach to estimate the shape of a convex membrane from measurements of its eigenmodes on a limited set of positions. Unlike Kac, who only considered the eigenvalues of the Laplacian on the domain, we used the complex-valued eigenmodes (amplitudes and phases) measured at different positions. Once the eigenmodes had been measured on a restrained portion of the membrane, we used eigenmode decomposition techniques in order to extrapolate the acoustic field and reveal the domain shape.

Another problem in spectral extraction of parameters is to estimate the value of the complex impedance on the boundaries. This problem is widespread in acoustics to measure materials properties for designing absorbers, diffusers, or walls, thanks to different methods like Kundt tubes or reverberant rooms [13, 14, 15]. Under semi-anechoic conditions, a simple technique consists in measuring the acoustic reflection coefficient between an incident pulse and its reflection by the considered material [16]. Recently, new approaches using Finite Difference Time Domain methods have been explored, like in [17, 18], in which the authors try to fit the parameters of an acoustic propagation model with time measurements to numerically estimate the value of the impedance on the boundaries of the discretized domain.

In this article, we show that the two problems of estimating the shape and the impedance of a domain can be embedded within a common framework thanks to eigenmode decomposition. After introducing the principles of wave-based eigenmode decomposition in a first section, we develop the following contributions:

- we present an extension of our previous work [12], in which we estimated the shape of a membrane with known Neumann boundary conditions, to known Robin boundary conditions ;
- furthermore, by adapting the problem to complex impedance on the membrane boundaries, we are able to solve a complementary problem: estimating the impedance on the membrane border when its shape is known ;
- numerical examples with two membranes of different shape, and several values of impedance, confirm the validity of our model ;
- results and extensions to this work are finally discussed.

2. ACOUSTIC MODEL

2.1. Eigenmode approximation

Let $\mathcal{D} \subset \mathbb{R}^d$ be a closed domain (here, we focus on $d = 2$). A wavefield $p_k(\mathbf{r})$ is an eigenmode of (the Laplacian on) $\mathcal{D}$ of wavenumber k if it is a solution to the homogeneous Helmholtz equation in $\mathcal{D}$:

$$(\Delta + k^2)p_k = 0 \quad \text{on } \mathcal{D}. \tag{1}$$

The generalized boundary conditions (BC) on $\partial\mathcal{D}$ can be modeled by Robin BC:

$$\frac{\partial p_k}{\partial n} + \gamma p_k = 0 \quad \text{on } \partial\mathcal{D}, \tag{2}$$

where $\partial/\partial n$ is the normal derivative and γ the complex impedance. Note that when $\gamma = 0$ (respectively $\gamma \to \infty$), the field is satisfying Neumann BC (resp. Dirichlet BC).

Assume now that $p_k(\mathbf{r})$ is partially observed on a set of M sampling points $\Omega = \{\mathbf{r}_m\}_{m=1}^M \in \mathcal{D}$, and let R be the approximate radius of Ω. Vekua theory states that a solution p_k to the Helmholtz equation (1) can be approximated by a linear combination of $L_k = 2kR + 1$ (at nearest integer) plane waves [20, 21]:

$$p_k(\mathbf{r}) \approx \sum_{\ell=1}^{L_k} \alpha_{k,\ell} e^{\jmath \mathbf{r}\cdot\mathbf{k}_\ell} \tag{3}$$

where $\mathbf{k}_\ell$ is the ℓ^{th} wave vector whose direction is sampled on the circle of radius k ($\|\mathbf{k}_\ell\|_2 = k$) and $\jmath$ is the imaginary unit. Equation (3) can be discretized as:

$$\mathbf{p}_k \approx \mathbf{W}_k \boldsymbol{\alpha}_k \tag{4}$$

where $\mathbf{W}_k$ is a dictionary of plane waves sampled at positions $\mathbf{r}_m$, and the vector $\boldsymbol{\alpha}$ contains the coefficients of $\mathbf{p}_k$'s plane wave decomposition at the same positions.

Several applications of Vekua theory have been proposed recently, for example to perform impulse response interpolation inside a measurement domain $\mathcal{D}$ [22], or to cancel reverberation coming from the boundaries for acoustic source localization [23]. In [19], we used eigenmode decomposition to perform array position calibration. In this application, the positions $\mathbf{r}_m$ of the measurements $p_k(\mathbf{r}_m)$ were unknown, and the calibration problem boiled down to finding the set of sensors positions $\mathbf{r}_m$ parameterizing $\mathbf{W}_k$ and minimizing the discrepancy between measurements and model, through the cost function:

$$\hat{\mathbf{r}} = \arg\min_{\mathbf{r}} \sum_k \|\mathbf{p} - \mathbf{W}_k(\mathbf{r})\mathbf{W}_k^\dagger(\mathbf{r})\mathbf{p}\|_2^2. \tag{5}$$

We later extended this principle to the problem of estimating the shape of a convex vibrating domain [12].

2.2. Virtual measurements on the boundary

In the case of membrane shape estimation, with known Robin BC impedance, one can imagine that the border $\partial\mathcal{D}$ of the membrane is discretized as a collection of N *virtual sampling points* $\{\mathbf{q}_n\}_{n=1}^N \in \partial\mathcal{D}$, with $\{\boldsymbol{\nu}_n\}_{n=1}^N$ the associated outgoing normal vectors on $\partial\mathcal{D}$. We denote $\mathbf{Q}$ the $d \times N$ matrix with columns $\mathbf{q}_n$, and $\boldsymbol{\mathcal{N}}$ the $d \times N$ matrix with normalized columns $\boldsymbol{\nu}_n$. This principle of "virtual measurements" was introduced in [24] for reverberation cancellation and source localization in a room. Suppose that acoustic sensors were placed on these positions, measuring both p and $\partial p/\partial n$. By definition of the Robin BC, these measurements would satisfy Eq. (2). To each virtual sampling point (of unknown position) is associated a (known, here null) virtual measurement of the field $\frac{\partial p_k}{\partial n} + \gamma p_k$ corresponding to the BC. Thanks to these virtual samples, the problem of shape estimation can then be reinterpreted as a position calibration problem in which the positions $\mathbf{Q}$ of the virtual sampling points on $\partial\mathcal{D}$ and their associated normals $\boldsymbol{\mathcal{N}}$ have to be estimated. This problem is illustrated on Figure 2. We first used this approach in [12] to estimate a membrane shape with Neumann BC.

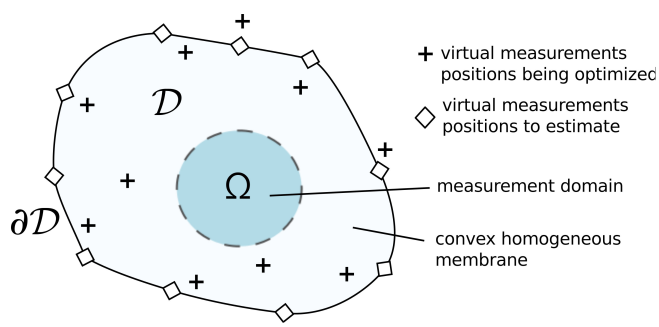


Fig. 2. Principle of shape estimation from the calibration of the position of virtual measurements $\mathbf{q}_n$, situated on the boundary $\partial\mathcal{D}$.

For this purpose, the observations of the eigenmodes made by the microphones (at known locations in Ω) were first gathered in the measurement vector $\mathbf{y}_k = (p_k(\mathbf{r}_m))_{1\le m\le M}$. We then defined an extended measurement vector:

$$\tilde{\mathbf{y}}_k = \begin{bmatrix} \mathbf{y}_k \\ \mathbf{0}_N \end{bmatrix} \tag{6}$$

where the real measurements $\mathbf{y}_k$ were concatenated with the N virtual measurements (the null vector $\mathbf{0}_N$) associated to the virtual sampling $\{\mathbf{q}_n\}_{n=1}^N \in \partial\mathcal{D}$ of the hypothesized boundary location.

Consequently, the plane wave dictionary $\mathbf{W}_k$ had to be extended to take into account the added virtual positions. In the following paragraph, we present a generalized model for Robin BC, in order to estimate either the shape of the membrane or its impedance in a larger set of situations.

2.3. Generalized plane waves dictionary and cost function

With an abuse of notation (the complex exponential is taken entrywise) we define

$$\mathbf{W}_k^D := e^{\jmath \mathbf{Q}^T \mathbf{K}_k} \tag{7}$$

the $N \times L_k$ matrix sampling the L_k considered plane waves at the N boundary locations associated to $\mathbf{Q}$ and ($\odot$ is the entrywise product)

$$\mathbf{W}_k^N := \frac{\partial}{\partial t} e^{\jmath(\mathbf{Q}+t\boldsymbol{\mathcal{N}})^T \mathbf{K}_k} = \jmath(\boldsymbol{\mathcal{N}}^T \mathbf{K}_k) \odot e^{\jmath \mathbf{Q}^T \mathbf{K}_k} \tag{8}$$

the $N \times L_k$ matrix sampling the normal derivative of the considered plane waves at the same locations. Finally, let $\boldsymbol{\gamma}_k \in \mathbb{C}^N$ be the vector of assumed impedances on the sampled boundary positions, and $\boldsymbol{\Gamma}_k = \text{diag}\{\boldsymbol{\gamma}_k\}$. The extension of the original plane waves dictionary $\mathbf{W}_k$ built upon the microphones positions and now taking into account the virtual sampling on the border and the complex impedances is:

$$\tilde{\mathbf{W}}_k := \tilde{\mathbf{W}}_k(\mathbf{Q}, \boldsymbol{\mathcal{N}}, \gamma_k) = \begin{bmatrix} \mathbf{W}_k \\ \boldsymbol{\Theta}_k \left(\mathbf{W}_k^N + \boldsymbol{\Gamma}_k \mathbf{W}_k^D\right) \end{bmatrix}. \tag{9}$$

A normalizing diagonal matrix $\boldsymbol{\Theta}_k = \text{diag}(\boldsymbol{\theta}_k)$, $\boldsymbol{\theta}_k \in \mathbb{C}^N$, is introduced in order to prevent the amplitude of the dictionary part corresponding to the virtual sampling to be too high when $\gamma \to \infty$, which would be interpreted as not having real measurements in the domain, thus no information on the field inside $\mathcal{D}$. It is chosen entrywise as

$$\theta = 1/(1+\gamma).$$

With this normalization, Neumann boundary conditions are achieved when $\theta\gamma \to 0$ and Dirichlet BC when $\theta\gamma \to 1$.

As with the model (5) used to perform array position calibration, when $\mathbf{Q}$ and $\boldsymbol{\mathcal{N}}$ match the true boundary location $\partial\mathcal{D}$ and the associated normal vectors, and $\boldsymbol{\gamma}_k$ matches the corresponding impedance, one must have $\tilde{\mathbf{y}}_k \approx \tilde{\mathbf{W}}_k \boldsymbol{\alpha}_k$. As a result, the match between $\mathbf{Q}, \boldsymbol{\mathcal{N}}$ and $\boldsymbol{\gamma} = \{\boldsymbol{\gamma}_k\}_k$ can be measured by the cost function:

$$\mathcal{J}(\mathbf{Q}, \boldsymbol{\mathcal{N}}, \gamma | \mathbf{y}) := \sum_k \|\tilde{\mathbf{y}}_k - \tilde{\mathbf{W}}_k \tilde{\mathbf{W}}_k^\dagger \tilde{\mathbf{y}}_k\|_2^2. \tag{10}$$

Minimizing the cost function (10) over $\mathbf{Q}$ and $\boldsymbol{\mathcal{N}}$ for known $\boldsymbol{\gamma}$ leads to shape estimation, whereas fixing the membrane shape allows to minimize (10) over $\boldsymbol{\gamma}$ to estimate the complex impedance. A joint optimization of these three parameters can be envisioned but would require developing appropriate optimization strategies, which is left to future work.

3. NUMERICAL EXPERIMENTS

In the following, two types of numerical experiments are conducted to validate our model. The method is first used to estimate the shapes of two membranes when the impedance on the boundary is known (for example in the case of a real drum, where we know that the normal velocity field is null on the boundary). In a second part, the shapes are known and discretized in a set of virtual positions. The model is then used to estimate the complex value of the impedance γ. In all these experiments, γ is chosen as constant on the boundary.

3.1. Membrane shape estimation (known boundary conditions)

For this first numerical validation, we consider two convex geometries of membranes. The first one, denoted $\mathcal{D}_1$ is a polygonal membrane whose dimensions are approximately 4 m × 3 m and whose impedance on the boundary is $\gamma_1 = 0$ (Neumann BC). The second one, denoted $\mathcal{D}_2$ is a truncated disc of radius 1 m. The value of the impedance on $\partial\mathcal{D}_2$ is $\gamma_2 = 4 + 12\jmath$. In both designs, a set of $M = 100$ sampling points are spread inside and on the borders of a circular zone Ω (of radius 0.7 m for $\mathcal{D}_1$, 0.25 m for $\mathcal{D}_1$) near the center of the membrane. The values of the eigenmodes at the measurement positions are simulated by the FreeFem++ solver implementation of the finite element method [25]. To construct the dictionary, the R parameter used to set the number L_k of plane waves is chosen slightly larger than the radius of the convex envelope of each membrane (assuming we have access to a rough estimation of the membrane dimensions, or by applying the Weyl law [1] to estimate its area and perimeter).

From equation (10), the boundary shape is estimated by solving the following minimization problem:

$$\{\hat{\mathbf{Q}}, \hat{\boldsymbol{\mathcal{N}}}\} = \arg\min_{\mathbf{Q}, \boldsymbol{\mathcal{N}}} \mathcal{J}(\mathbf{Q}, \boldsymbol{\mathcal{N}}, \boldsymbol{\gamma} = \mathbf{0} | \mathbf{y}). \tag{11}$$

To solve this non-convex optimization problem, we use the Matlab Signal Processing Toolbox with the Active Set method [26]. To facilitate the process, we initialize the virtual positions on a circle centered around the measurement domain Ω. For each virtual position of polar coordinates $\mathbf{q}_n = (r_n; \phi_n)$, only the r_n variable is optimized, the angle ϕ_n being constrained to its value on the initialized circle. We then start to solve the problem using only the eigenmode of lowest frequency. Indeed, the corresponding wavelength is the largest and leads to less local minima in the cost function, as shown previously in [19]. Once the algorithm has converged to a first estimate of the virtual samples' positions, we use this estimation to re-initialize the optimization step, but this time summing the cost function defined by (10) and (11) over one more eigenmode k to refine the estimation. This process is iterated until all eigenmodes have been used. For each case, the 50 first eigenmodes are used in the optimization.

Results are displayed on Figure 3. The membrane shapes $\mathcal{D}_1$ and $\mathcal{D}_2$ appear in yellow. The measurements positions, the initial virtual positions, the final estimated virtual positions, as well as an example of estimated virtual positions for 3 eigenmodes are represented on the same figure. The represented normals are the ingoing estimated normals (rows of $-\hat{\boldsymbol{\mathcal{N}}}$). Once all eigenmodes have been used, the envelope of the estimated positions is very close to the actual membrane border. We notice however that some estimated positions are located outside the membrane shape, suggesting that these points have been trapped into local minima during the minimization process. Furthermore, with the optimization constraint on the angles ϕ_n, it is logical that acute angles on the boundary are more difficult to estimate (see, for example, the top-right corner of the polygonal membrane $\mathcal{D}_1$). This problem could be addressed by adding new virtual positions and re-sampling the estimated boundary between two optimization steps, before adding a new eigenmode.

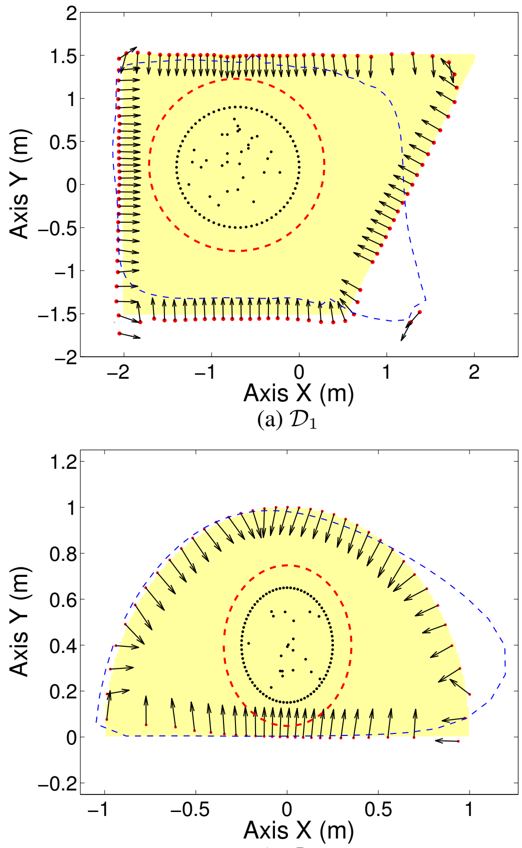
3.2. Boundary condition estimation (known membrane shape)

In the second series of experiments, we consider the reciprocal case where the membrane shape is known (thus $\mathbf{Q}$, $\boldsymbol{\mathcal{N}}$, $\mathbf{W}_k^N$ and $\mathbf{W}_k^D$ are known), but the value of the impedance γ at the boundary is unknown, and to be estimated. The problem is then to estimate its value. For the sake of simplicity, we first assume a constant impedance on $\partial\mathcal{D}$ and over the eigenmodes k.

The simulation setup is identical to the previous series of experiments. The eigenmodes of the two membranes $\mathcal{D}_1$ and $\mathcal{D}_2$ are computed at the measurements positions using the FreeFem++ software, for various values of γ. From equation (10), the minimization problem corresponding to this experiment is:

$$\hat{\gamma} = \arg\min_{\gamma} \mathcal{J}(\mathbf{Q}, \boldsymbol{\mathcal{N}}, \gamma | y). \tag{12}$$

In contrast to the former problem, the optimization is now performed using all the eigenmodes at once. The different values of γ and their estimations $\hat{\gamma}$ are listed in Table 1. To compare them, we



(a) $\mathcal{D}_1$

(b) $\mathcal{D}_2$

Fig. 3. Shape estimation of: (a) $\mathcal{D}_1$; (b) $\mathcal{D}_2$. The real shapes are represented in yellow. Black dots: positions of measurements drawn at random in region Omega (black dotted line). Red dotted line: initialization of the virtual measurements positions. Blue line: estimation of the positions at the 3^{rd} iteration. Red dots : final estimated positions of the virtual measurements (i.e. estimation of the shape). Arrows: estimated ingoing normals (rows of $-\hat{\mathcal{N}}$).

define an error on the estimated impedance by

$$\varepsilon_\gamma = \left| \frac{\hat{\gamma}}{1+\hat{\gamma}} - \frac{\gamma}{1+\gamma} \right|. \tag{13}$$

This definition is motivated by the behavior of high-valued impedances (Dirichlet BC). Indeed, when $\hat{\theta}\hat{\gamma} = \frac{\hat{\gamma}}{1+\hat{\gamma}} \approx 1$, it becomes difficult to distinguish nearby values of γ, as they tend to give the same eigenmode repartition in $\mathcal{D}$. This is for example the case with the last row of Table 1, for which we chose the impedance value to approximate a Dirichlet BC. As can be seen, the value of the estimated impedance seems to be strongly different from its correct value. Nevertheless, the estimated "normalized impedance" ($\hat{\gamma}/(1+\hat{\gamma}) \approx 0.999 + 0.004\jmath$) is very close to the ground truth normalized impedance ($\gamma/(1+\gamma) \approx 0.999$).

In a last experiment, we relax the assumption of a constant γ and now consider it only piecewise constant. In three dimensions this would for instance correspond to a room with walls, doors and windows made of different materials. We design a rectangular-shaped membrane $\mathcal{D}_3$, where each "wall" has a different impedance. The first two parallel walls, of length 2 m, share the same impedance $\gamma_1 = \gamma_3 = 0$ (Neumann BC) whereas the two other parallel walls, of width 1 m, share the impedance $\gamma_2 = \gamma_4 = 1000$ ($\approx$ Dirichlet BC). As the geometry of the membrane is known, it is easy to sample each wall i and constrain the obtained virtual positions to share a common impedance value γ_i. Impedances are estimated jointly and the obtained values are listed at the bottom rows of Table 1. We verify that the estimated "normalized" impedances still match the true ones with good accuracy.

Shape type	Real value γ	Estimated value $\hat{\gamma}$	Error ε_γ
$\mathcal{D}_1$	0	$0 + 0.0003\jmath$	$3 \cdot 10^{-4}$
	$23.5 - 8\jmath$	$23.1 - 7.97\jmath$	$6.1 \cdot 10^{-4}$
	$7.4 + 2.6\jmath$	$7.45 + 2.61\jmath$	$6.6 \cdot 10^{-4}$
	$12.7 - 10.3\jmath$	$12.83 - 10.37\jmath$	$4.9 \cdot 10^{-4}$
$\mathcal{D}_2$	$4 + 12\jmath$	$4.02 + 11.93\jmath$	$4.3 \cdot 10^{-4}$
	0	$0.0025 + 0.0003\jmath$	$2.5 \cdot 10^{-3}$
	$17 + 6.5\jmath$	$16.98 + 6.5\jmath$	$5.5 \cdot 10^{-5}$
	1000	$1440 + 6.25\jmath$	$3.1 \cdot 10^{-4}$
$\mathcal{D}_3$	$\gamma_1 = 0$	$\hat{\gamma}_1 = 0.05 + 0.01\jmath$	$4.8 \cdot 10^{-2}$
	$\gamma_2 = 2000$	$\hat{\gamma}_2 = 2801 - 0.07\jmath$	$1.4 \cdot 10^{-4}$
	$\gamma_3 = 0$	$\hat{\gamma}_3 = 0.05 - 0.01\jmath$	$4.8 \cdot 10^{-2}$
	$\gamma_4 = 2000$	$\hat{\gamma}_4 = 2782 - 0.42\jmath$	$1.4 \cdot 10^{-4}$

Table 1. Impedance estimation on $\partial\mathcal{D}$ (known geometry), for the membranes $\mathcal{D}_1$ and $\mathcal{D}_2$, for various values of γ, and for the membrane $\mathcal{D}_3$ with piecewise constant $\{\gamma_i\}_{i=1}^4$ on each wall i.

4. CONCLUSION AND FUTURE WORK

Using wave-based eigenmode decomposition methods, we have shown that it is possible to estimate the shape of a convex membrane by reinterpreting its boundary as a set of virtual sensors, of unknown positions but known measurements satisfying known boundary conditions. This re-interpretation recasts the acoustic shape estimation problem into a calibration problem in the frequency domain, in which the virtual positions have to be estimated. We generalized a previously proposed model to now handle the estimation of the shape of membranes with complex impedances. This new model also enables to estimate the boundary conditions (the impedance values on the boundary), for known membrane shapes.

Several procedures to improve the optimization step can be envisioned, such as a shape parameterization (for example with parametric families of curves) that would help sampling of the virtual positions on $\partial\mathcal{D}$. Expected extensions of this work are the estimation of frequency-dependant impedances over the frequencies, and experimental validation of these methods. Eventually, if the model itself can straightforwardly account for the third spatial dimension, such an extension would significantly weigh on the optimization step, as the number of variables (*i.e.* the number of virtual positions) would considerably increase. This will be the topic of further research.

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